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KAMENEV-TYPE OSCILLATION CRITERIA FOR SUBLINEAR DELAY DIFFERENCE EQUATIONS

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By means of Riccati transformation techniques, we establish some new oscillation criteria for second order nonlinear difference equation

$$\Delta(p_n(\Delta x_n)^{\gamma}) + q_n x_{n-\sigma}^{\beta} = 0. \quad n = 0, 1, 2, \ldots$$

when $0 < \beta < 1$ is a quotient of odd positive integers.

Key Words : Oscillation; Sublinear; Differential Equations

1. INTRODUCTION

In recent years, the asymptotic behaviour of second order difference equations has been the subject of investigations by many authors, see for example [1-3, 7-10, 12, 13, 15-20].

In this paper, we will be concerned with a class of second order sublinear delay difference equations of the form

$$\Delta(p_n(\Delta x_n)^{\gamma}) + q_n x_{n-\sigma}^{\beta} = 0, \quad n = 0, 1, 2, \qquad \ldots (1.1)$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers, γ is quotient of odd positive integers, $0 < \beta < 1$ is quotient of odd positive integers, σ is a fixed nonnegative integer, $\left\{p_n\right\}_{n=0}^{\infty}$ and $\left\{q_n\right\}_{n=0}^{\infty}$ are sequences of real numbers such that $p_n > 0, q_n \geq 0$ and $\{q_n\}$ has a positive subsequence, and for some $n_0 > 0$,

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_n}\right)^{\frac{1}{\gamma}} = \infty, \qquad \ldots (1.2)$$

or $$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_n}\right)^{\frac{1}{\gamma}} < \infty. \qquad \text{... (1.3)}$$

By a solution of (1.1) we mean a nontrivial sequence $\{x_n\}$ which is defined for $n \geq -\sigma$ and satisfies eq. (1.1) for n = 0, 1, 2 ... . Clearly if

$$x_n = A_n \text{ for } n = -\sigma, ..., -1, 0 \qquad \text{... (1.4)}$$

are given, then eq. (1.1) has a unique solution satisfying the initial condition (1.4). A solution $\{x_n\}$ of (1.1) is said to be oscillatory if for every $n_1 > 0$ there exists an $n \geq n_1$ such that $x_n x_{n+1} \leq 0$, otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory if all its solutions are oscillatory.

A number of dynamical behaviour of solutions of second order difference equations are possible, here we will only be concerned with conditions which are sufficient for all solutions of (1.1) to be oscillatory or tends to zero as $n \to \infty$.

Our concern is motivated by several recent papers, especially those by Szmanda[12], Liu *et al.*, Szafranski and Szmanda[13], Zhang and Jinlian Zhang[19] and Zhicheng and Yu[20], Most if the above papers considered the equations

$$\Delta^2 x_n + q_n f(x_{n-\sigma}) = 0, \qquad \text{... (1.5)}$$

or its generalization

$$\Delta (p_n \Delta x_n) + q_n f(x_{n-\sigma}) = 0. \qquad \text{... (1.6)}$$

In the continuous case, the differential equation

$$x''(t) + q(t) f(x(t)) = 0, \; t \geq t_0 \qquad \text{... (1.7)}$$

has been tackled by many authors, see the survey papers[6, 14] which give over 300 references. It is known that, due to Kamenev[5], the average function $A_\lambda(t)$ defined by

$$A_\lambda(t) = \frac{1}{t^\lambda} \int_{t_0}^{t} (t-s)^\lambda q(s)\, ds, \;\; \lambda \geq 1 \qquad \text{... (1.8)}$$

plays a crucial role in the oscillation of eq. (1.7).

Philos[11] further improves Kamenev's result by proving the following: Suppose there exist continuous functions $H, h : D \equiv \{(t, s) : t \geq s \geq t_0\} \to R$ such that

(*i*) $H(t, t) = 0, t \geq t_0$, (*ii*) $H(t, s) > 0, t > s \geq t_0$, and H has a continuous and nonpositive partial derivative on D with respect to the second variable and satisfies

$$-\frac{\partial H(t, s)}{\partial s} = h(t, s)\sqrt{H(t, s)} \geq 0. \qquad \text{... (1.9)}$$

Further, suppose that

$$\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[H(t, s)\, q(s) - \frac{1}{4} h^2(t, s) \right] ds = \infty. \qquad \text{... (1.10)}$$

Then every solution of eq. (1.7) oscillates.

By means of Riccati transformation techniques, we establish discrete Kamenev type oscillation criteria for the second-order sublinear difference eq. (1.1) when (1.2) holds. When (1.3) holds we present some conditions which are sufficient for all solutions to be oscillatory or converges to zero. Our results in the linear case, i.e. when $\beta = \gamma = 1$, and (1.2) holds include the results of Szmanda[12] in the nondelay case and Szafranski and Szmanda[13] in the delay case, improve the results of Wang and Yu[14] and Zhang and Jinlian Zhang[20] in linear case, and when (1.3) holds our results are essentially new.

2. MAIN RESULTS

First we consider the case when (1.2) holds and $\Delta p_n \geq 0$.

***Theorem* 2.1** — *Assume that (1.2) holds. Furthermore, assume that there exists a positive sequence $\{\rho_n\}_{n=0}^{\infty}$ such that for every $\alpha \geq 1$ and positive number M,*

$$\lim_{n \to \infty} \sup \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{(\rho_{i-\sigma} - \sigma)^{\frac{1}{\gamma}} \alpha^{1-\beta} (l+1-\sigma)^{1-\beta} (\Delta \rho_l)^2}{4 \beta (M)^{(\gamma-1)/\gamma} \rho_l} \right] = \infty \qquad \text{... (2.1)}$$

Then every solution of eq. (1.1) oscillates.

PROOF : Suppose to the contrary that $\{x_n\}$ is an eventually nonoscillatory solution of (1.1) such that $x_{n-\sigma} > 0$ for all $n \geq n_0 > 0$. We shall consider only this case, since the substitution $y_n = -x_n$ transforms eq. (1.1) into an equation of the same form. From eq. (1.1) we have

$$\Delta (p_n (\Delta x_n)^{\gamma}) = -q_n x_{n-\sigma}^{\beta} \leq 0, \quad n \geq n_0 \qquad \text{... (2.2)}$$

and so $\left\{ p_n (\Delta x_n)^{\gamma} \right\}$ is an eventually nonincreasing sequence. We first show that $p_n (\Delta x_n)^{\gamma} \geq 0$ for $n \geq n_0$. In fact, if there exists an integer $n_1 \geq n_0$ such that $p_{n_1} (\Delta x_{n_1})^{\gamma} = c < 0$, then (2.2) implies that $p_n (\Delta x_n)^{\gamma} \leq c$ for $n \geq n_1$ that is

$$\Delta x_n \leq \left(\frac{c}{p_n} \right)^{\frac{1}{\gamma}}$$

and, hence

$$x_n \leq x_{n_1} + c^{\frac{1}{\gamma}} \sum_{i=n_1}^{n-1} \left(\frac{1}{p_i} \right)^{\frac{1}{\gamma}} \to -\infty \text{ as } n \to \infty \qquad \text{... (2.3)}$$

which contradicts the fact that $x_n > 0$ for $n \geq n_0$, then $p_n (\Delta x_n)^\gamma \geq 0$. Also we claim that $\Delta^2 x_n \leq 0$. If not there exists $n_1 \geq n_0$ such that $\Delta^2 x_n > 0$ for $n \geq n_1$ and this implies that $\Delta x_{n+1} > \Delta x_n$, so that since $\Delta p_n \geq 0$, $p_{n+1} (\Delta x_{n+1})^\gamma > p_{n+1} (\Delta x_n)^\gamma \geq p_n (\Delta x_n)^\gamma$ and this contradicts the fact that $\{p_n (\Delta x_n)^\gamma\}$ is nonincreasing sequence, then $\Delta^2 x_n \leq 0$, and therefore we have

$$x_n > 0, \ \Delta x_n \geq 0 \text{ and } \Delta^2 x_n \leq 0 \text{ for } n \geq n_0. \quad \dots (2.4)$$

Define the sequence $\{w_n\}$ by

$$w_n = \rho_n \frac{p_n (\Delta x_n)^\gamma}{x^\beta_{n-\sigma}} \quad \dots (2.5)$$

then $w_n > 0$, and

$$\Delta w_n = p_{n+1} (\Delta x_{n+1}) \Delta \left[\frac{\rho_n}{x^\beta_{n-\sigma}} \right] + \frac{\rho_n \Delta (p_n (\Delta x_n)^\gamma)}{x^\beta_{n-\sigma}}. \quad \dots (2.6)$$

From (1.1) and (2.6), we have

$$\Delta w_n = -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n p_{n+1} (\Delta x_{n+1})^\gamma \Delta (x^\beta_{n-\sigma})}{x^\beta_{n+1-\sigma} x^\beta_{n-\sigma}} \quad \dots (2.7)$$

From (2.2) and (2.4), we get

$$p_{n-\sigma} (\Delta x_{n-\sigma})^\gamma \geq p_{n+1} (\Delta x_{n+1})^\gamma, \text{ and } x_{n+1-\sigma} \geq x_{n-\sigma}, \quad \dots (2.8)$$

and then from (2.7) and (2.8), we have

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n p_{n+1} (\Delta x_{n+1})^\gamma \Delta (x^\beta_{n-\sigma})}{\left(x^\beta_{n+1-\sigma} \right)^2} \quad \dots (2.9)$$

Now, by using the inequality (cf. [4, p. 39])

$$x^\beta - y^\beta \geq \beta x^{\beta-1} (x - y) \text{ for all } x \neq y > 0 \text{ and } 0 < \beta \leq 1.$$

Then, we have

$$\Delta (x^\beta_{n-\sigma}) = x^\beta_{n+1-\sigma} - x^\beta_{n-\sigma} \geq \beta (x_{n+1-\sigma})^{\beta-1} (x_{n+1-\sigma} - x_{n-\sigma})$$

$$= \beta (x_{n+1-\sigma})^{\beta-1} (\Delta x_{n-\sigma}). \quad \dots (2.10)$$

Substitute from (2.10) in (2.9), we have

$$\Delta w_n \le -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \rho_n p_{n+1} \frac{\beta(x_{n+1-\sigma})^{\beta-1}(\Delta x_{n-\sigma})(\Delta x_{n+1})^{\gamma}}{\left(x^{\beta}_{n+1-\sigma}\right)^2}. \quad \ldots (2.11)$$

From (2.8) and (2.11), we have

$$\Delta w_n \le -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta\rho_n(p_{n+1})^{\frac{1}{\alpha}} p_{n+1}}{(p_{n-\sigma})^{\frac{1}{\gamma}}(x_{n+1-\sigma})^{1-\beta}} \frac{(\Delta x_{n+1})^{\gamma+1}}{\left(x^{\beta}_{n+1-\sigma}\right)}$$

hence,

$$\Delta w_n \le -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1}$$

$$-\frac{\beta\rho_n(p_{n+1})^{\frac{1}{\gamma}-1}}{(\rho_{n+1})^2(p_{n-\sigma})^{\frac{1}{\gamma}}(x_{n+1-\alpha})^{1-\beta}} \frac{(p_{n+1})^2(\rho_{n+1})^2(\Delta x_{n+1})^{2\gamma}}{\left(x^{\beta}_{n+1-\sigma}\right)^2(\Delta x_{n+1})^{\gamma-1}} \quad \ldots (2.12)$$

From (2.4), we conclude that

$$x_n \le x_{n_0} + \Delta x_{n_0}(n-n_0), \qquad n \ge n_0$$

and consequently there exists a $n_1 \ge n_0$ and appropriate constant $\alpha \ge 1$ such that

$$x_n \le \alpha n \qquad \text{for} \qquad n \ge n_1$$

and this implies that

$$x_{n+1-\sigma} \le \alpha(n+1-\sigma) \quad \text{for} \quad n \ge n_2 = n_1 + \sigma - 1$$

and, hence

$$\frac{1}{(x_{x+1-\sigma})^{1-\beta}} \ge \frac{1}{(\alpha(n+1-\sigma))^{1-\beta}}. \quad \ldots (2.13)$$

Since $\{p_n(\Delta x_n)^{\gamma}\}$ is a positive and nonincreasing sequence, there exists a $n_2 \ge n_1$ sufficiently large such that $p_n(\Delta x_n)^{\gamma} \le 1/M$ for some positive constant M and $n \ge n_2$, and hence by (2.2) we have $p_{n+1}(\Delta x_{n+1})^{\gamma} \le 1/M$, so that

$$\frac{1}{(\Delta x_{n+1})^{\gamma-1}} \ge (Mp_{n+1})^{(\gamma-1)/\gamma} \quad \ldots (2.14)$$

then from (2.5), (2.12), (2.13) and (2.14) we have

$$\Delta w_n \le -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1}$$

$$-\frac{\beta\rho_n (M)^{(\gamma-1)/\gamma}}{(\rho_{n+1})^2 (p_{n-\sigma})^{\frac{1}{\gamma}} \alpha^{1-\beta} (n+1-\sigma)^{1-\beta}} w_{n+1}^2. \quad \ldots (2.15)$$

Hence,

$$\Delta w_n \leq -\rho_n q_n + \frac{(p_{n-\sigma})^{\frac{1}{\gamma}} \alpha^{1-\beta} (n+1-\sigma)^{1-\beta} (\Delta \rho_n)^2}{4\beta (M)^{(\gamma-1/\gamma} \rho_n}$$

$$-\left[\frac{\sqrt{\beta (M)^{(\gamma-1)/\gamma} \rho_n}}{\rho_{n+1}\sqrt{(\alpha(n+1-\sigma))^{1-\beta} p_{n-\sigma}}} w_{n+1} - \frac{\sqrt{\alpha^{1-\beta}(n+1-\sigma)^{1-\beta}(p_{n-\sigma})^{\frac{1}{\gamma}}}\,\Delta \rho_n}{2\sqrt{\beta (M)^{(\gamma-1)/\gamma} \rho_n}}\right]^2$$

$$< -\left[\rho_n q_n - \frac{\alpha^{1-\beta}(n+1-\sigma)^{1-\beta}(p_{n-\sigma})^{\frac{1}{\gamma}}(\Delta \rho_n)^2}{4\beta (M)^{(\gamma-1)/\gamma}\rho_n}\right]$$

Then, we have

$$\Delta w_n < -\left[\rho_n q_n - \frac{(p_{n-\sigma})^{\frac{1}{\gamma}} \alpha^{1-\beta} (n+1-\sigma)^{1-\beta} (\Delta \rho_n)^2}{4\beta (M)^{(\gamma-1/\gamma}\rho_n}\right]. \quad .. (2.16)$$

Summing (2.15) from n_2 to n, we obtain

$$-w_{n2} < w_{n+1} - w_{n_2} < -\sum_{l=n_2}^{n} \left[\rho_l q_l - \frac{(pl-\sigma)^{\frac{1}{\gamma}} \alpha^{1-\beta} (l+1-\sigma)^{1-\beta} (\Delta \rho_l)^2}{4\beta (M)^{(\gamma-1)/\gamma}\rho_l}\right]$$

which yields

$$\sum_{l=n_2}^{n} \left[\frac{\rho_l q_l - (p_{l-\sigma})^{\frac{1}{\gamma}} \alpha^{1-\beta} (l+1-\sigma)^{1-\beta} (\Delta \rho_l)^2}{4\beta (M)^{(\gamma-1)/\gamma}\rho_l}\right] < c_1$$

for all large n, and this is contrary to (2.1). The proof is complete.

Remark 21.1 : Note that from Theorem 2.1, we can obtain different conditions for oscillation of all solutions of eq. (1.1) when (1.2) holds by different choices of $\{\rho_n\}_{n=0}^{\infty}$.

Remark 2.2 : When $\gamma = \beta = 1$, eq. (1.1) reduces to the linear delay difference equation

$$\Delta(p_n \Delta x_n) + q_n x_{n-\sigma} = 0, \; n = 0, 1, 2 \ldots \quad \ldots (2.17)$$

and the condition (2.1) in Theorem 2.1 reduces to

$$\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left[\rho_l q_l-\frac{(p_{l-\sigma})(\Delta\rho_l)^2}{4\rho_l}\right]=\infty. \qquad \dots (2.18)$$

Then Theorem 2.1 and Corollary 1 in[13] are the same in the case when $\gamma=\beta=1$. Also when $\sigma=0$ and $p_n=1$ and $\gamma=\beta=1$ Theorem 2.1 and Corollary 3 in[12] are the same.

In the following theorem, we provide new sufficient conditions for oscillation of eq. (1.1). This result is the discrete analogy of Philos[11] type condition for oscillation of second order differential equations.

Theorem 2.2 — *Assume that (1.2) holds. Let $\{\rho_n\}_{n=0}^{\infty}$ be a positive sequence. Furthermore, we assume that there exists a double sequence $\{H_{m,n}: m\ge n\ge 0\}$ such that (i) $H_{m,m}=0$ for $m\ge 0$, (ii) $H_{m,n}>0$ for $m>n\ge 0$, (iii) $\Delta_2 H_{m,n}=H_{m,n+1}-H_{m,n}\le 0$ for $m\ge n\ge 0$. If*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1}\left[H_{m,n}\rho_n q_n-\frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(h_{m,n}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m\,n}}\right)^2\right]=\infty \qquad \dots (2.19)$$

where

$$h_{m,n}=-\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},\quad m\ge n\ge 0,\ \bar{\rho}_n=\beta(M)^{(\gamma-1)/\gamma}\rho_n/\alpha^{1-\beta}(n+1-\sigma)^{1-\beta}(p_{n-\sigma})^{1/\gamma}$$

Then every solution of eq. (1.1) oscillates.

PROOF : We proceed as in Theorem 2.1, we assume that eq. (1.1) has a nonoscillatory solution, say $x_{n-\sigma}>0$ for all $n\ge n_0$. From the proof of Theorem 2.1 we obtain (2.15) for all $n\ge n_2$. From (2.15) we have for $n\ge n_2$,

$$\Delta w_n\le-\rho_n q_n+\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\frac{\bar{\rho}_n}{(\rho_{n+1})^2}w_{n+1}^2 \qquad \dots (2.20)$$

or

$$\rho_n q_n\le-\Delta w_n+\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}-\frac{\bar{\rho}_n}{(\rho_{n+1})^2}w_{n+1}^2. \qquad \dots (2.21)$$

Therefore, we have

$$\sum_{n=k}^{m-1}H_{m,n}\rho_n q_n\le-\sum_{m=k}^{m-1}H_{m,n}\Delta w_n+\sum_{n=k}^{m-1}H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}w_{n+1}$$

$$-\sum_{n=k}^{m-1}H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2}w_{n+1}^2 \qquad \dots (2.22)$$

which yields, after summing by parts the first term in the right hand side,

$$\sum_{n=k}^{m-1} H_{m,n}\rho_n q_n$$

$$\leq H_{m,k}\ w_k + \sum_{n=k}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=k}^{m-1}$$

$$H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=k}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2$$

$$= H_{m,k} w_k - \sum_{n=k}^{m-1} h_{m,n}\sqrt{H_{m,n}}\, w_{n+1} + \sum_{n=k}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1}$$

$$-\sum_{n=k}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 = H_{m,k} w_k$$

$$-\sum_{n=k}^{m-1}\left[w_{n+1} + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}}\,\bar{\rho}_n}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}} H_{m,n}\right)\right]^2$$

$$+\frac{1}{4}\sum_{n+k}^{m-1}\frac{(\rho_{n+1})^2}{\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2$$

Then,

$$\sum_{n=n_1}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]$$

$$< H_{m,n_1} w_{n_1} \leq H_{m,0} w_{n_1}$$

which implies that

$$\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\bar{\rho}_n}\left(h_{m,n} - \frac{\nabla\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]$$

$$< H_{m,0}\left(w_{n_1} + \sum_{n=0}^{n_1-1}\rho_n q_n\right)$$

Hence $$\lim_{m \to \infty} \sup \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \left[H_{m,n} \rho_n q_n - \frac{(\rho_{n+1})^2}{4 \bar{\rho}_n} \left(h_{m,n} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right]$$

$$< \left(w_{n_1} + \sum_{n=0}^{n_1 - 1} \rho_n q_n \right) < \infty.$$

and this contradicts (2.19). Then every solution of eq. (1.1) oscillates.

Remark 2.3 : By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\left.\begin{aligned} & H_{m,n} = (m-n)^{\lambda}, && \lambda \geq 1, m \geq n \geq 0, \\ & H_{m,n} = \log \left(\frac{m+1}{n+1} \right)^{\lambda}, && \lambda \geq 1, m \geq n \geq 0, \\ & H_{m,n} = (m-n)^{\lambda} && \lambda > 2, m \geq n \geq 0, \end{aligned}\right\}$$

where $$(m-n)^{(\lambda)} = (m-n)(m-n+1) \ldots (m-n+\lambda-1), \text{ and}$$

$$\Delta_2 (m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-1)^{(\lambda)} = -\lambda (m-n)^{(\lambda-1)}.$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n > \geq 0$. Hence we have the following results.

Corollary 2.1 — Assume that all the assumptions of Theorem 2.2 hold, except the condition (2.19) is replaced by

$$\lim_{n \to \infty} \sup \frac{1}{m^{\lambda}} \sum_{n=0}^{m} \left[(m-n)^{\lambda} \rho_n q_n - \frac{\lambda^2 \rho_{n+1}^2 (m-n)^{\lambda-2}}{4 \bar{\rho}_n} \right] = \infty \qquad \text{... (2.23)}$$

Then, every solution of eq. (1.1) oscillates.

Corollary 2.2 — Assume that all the assumptions of Theorem 2.2 hold, except the condition (2.19) is replaced by

$$\lim_{n \to \infty} \sup \frac{1}{(\log (m+1))^{\lambda}} \sum_{n=0}^{m-1} \left[\left(\log \frac{m+1}{n+1} \right)^{\lambda} \rho_n q_n \right.$$

$$\left. - \frac{\rho_{n+1}^2}{4 \bar{\rho}_n} \left(\frac{\lambda}{n+1} \left(\frac{m+1}{n+1} \right)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{\left(\log \frac{m+1}{n+1} \right)^{\lambda}} \right)^2 \right] = \infty. \qquad \text{... (2.24}$$

Then, every solution of eq. (1.1) oscillates.

Corollary 2.3 — Assume that all the assumptions of Theorem 2.2 hold, except the condition

(2.19) is replaced by

$$\lim_{m\to\infty} \sup \frac{1}{m^{(\lambda)}} \sum_{n=0}^{m-1} (m-n)^{(\lambda)} \left[\rho_n q_n - \frac{p_{n+1}^2}{4\rho_n} \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta p_n}{p_\infty+1} \right)^2 \right]$$

Next, we consider the case when (1.3) holds and $\Delta p_n \geq 0$.

***Theorem* 2.3** — *Assume that (1.3) holds. Furthermore, we assume that there exist positive seuences $\{\rho_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ such that (2.1) holds and*

$$\Delta \beta_n \leq 0, \Delta (p_n \Delta \beta_n) \leq 0, \sum_{n=n_0}^{\infty} \beta_{n+1} q_n = \infty \text{ and}$$

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_n \beta_n} \sum_{i=n_0}^{n-1} \beta_{i+1} q_i \right)^{\frac{1}{\gamma}} = \infty \qquad \text{... (2.25)}$$

for some $n_0 > 0$. Then every solution of eq. (1.1) oscillates or $\lim_{n\to\infty} x_n = 0$.

PROOF : Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_{n-\sigma} > 0$ for all $n \geq n_0$. We shall consider only this case, since the substitution $y_n = -x_n$ transforms eq. (1.1) into an equation of the same form. From eq. (1.1) we have

$$\Delta (p_n (\Delta x_n)^{\gamma}) = - q_n x_{n-\sigma}^{\beta} \leq 0, \quad n \geq n_0 \qquad \text{... (2.26)}$$

and so $\{p_n \Delta x_n\}$ is an eventually nonincreasing sequence, and then from (2.24) there exist two possible cases of Δx_n.

In the case when $\{\Delta x_n\}$ is eventually positive, we may follow the proof of Theorem 2.1 and obtain a contradiction.

If $\{\Delta x_n\}$ is eventually negative. Then $\lim_{n\to\infty} x_n = b \geq 0$. We assert that b = 0. If not then $x_{n-\sigma}^{\beta} \to b^{\beta} > 0$ as $n \to \infty$, and hence there exists $n_2 \geq n_1$ such that $x_{n-\sigma}^{\beta} \geq b^{\beta}$. Therefore from (2.26) we have

$$\Delta (p_n (\Delta x_n)^{\gamma}) \leq - q_n b^{\beta}$$

Define the sequence $u_n = \beta_n (p_n (\Delta x_n)^{\gamma})$ for $n \geq n_2$. Then we have

$$\Delta u_n \leq - b^{\beta} \beta_{n+1} q_n + \Delta \beta_n (p_n (\Delta x_n)^{\gamma}).$$

Summing the last inequality from n_2 to $n-1$, we have

$$u_n \le u_{n_2} - b^{\beta} \sum_{s=n_2}^{n-1} \beta_{s+1} q_s + \sum_{s=n_2}^{n-1} (p_s \Delta \beta_s)(\Delta x_s)^{\gamma}$$

and then

$$u_n \le u_{n_2} - b^{\beta} \sum_{s=n_2}^{n-1} \beta_{s+1} q_s + p_s \Delta \beta_s (\Delta x_s)^{\gamma} \Big|_{s=n_2}^{n}$$

$$- \sum_{s=n_2}^{n-1} \Delta (p_s \Delta \beta_s)(\Delta x_{s+1})^{\gamma}.$$

In view of (2.25) we have

$$u_n \le M - b^{\beta} \sum_{s=n_2}^{n-1} \beta_{s+1} q_s$$

where $M = u_{n_2} - p_{n_2} \Delta \beta_{n_2} (\Delta x_{n_2})^{\gamma}$. In view of (2.25), since $\sum_{n=n_0}^{\infty} \beta_{n+1} q_n = \infty$ it is possible to choose integer n_3 to n we obtain

$$x_{n+1} \le x_{n_3} - \left(\frac{b^{\beta}}{2}\right)^{\frac{1}{\gamma}} \sum_{s=n_3}^{n} \left(\frac{1}{p_s \beta_s} \sum_{i=n_2}^{s-1} \beta_{i+1} q_i\right)^{\frac{1}{\gamma}}.$$

Condition (2.25) implies that $\{x_n\}$ is eventually negative, which is a contradiction. The proof is complete.

Remark 2.4 : From Theorem 2.3, we can obtain different conditions for oscillation of all solutions of eq. (1.1) when (1.3) holds by different choices of $\left\{ \rho_n \right\}_{n=0}^{\infty}$

***Theorem* 2.4** — *Assume that (1.3) holds, and let* $\left\{ \rho_n \right\}_{n=0}^{\infty}$ *and* $\left\{ \beta_n \right\}_{n=0}^{\infty}$ *be two positive seqiuences such that (2.25) holds. Furthermore, we assume that there exists a double sequence* $\left\{ H_{m,n} : m \ge n \ge 0 \right\}$ *as defined in Theorem 2.2 and (2.19) holds. Then every solution of eq. (1.1) oscillates or* $\lim_{n \to \infty} x_n = 0$.

PROOF : Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_{n-\sigma} > 0$ for all $n \ge n_0$.

If $\{\Delta x_n\}$ is eventually positive, we are then back to the case where (2.4) holds. Thus the proof of Theorem 2.2 goes through, and we may have a contradiction.

If $\{\Delta x_n\}$ is eventually negative the proof similar to the first part of Theorem 2.3 and hence is omitted.

Remark 2.5 : By choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1) when (1.3) holds. Let us consider the double sequence $\{H_{m,n}\}$ be defined as in Remark 2.3, for example we have the following results.

Corollary 2.4 — Assume that all the assumptions of Theorem 2.4 hold, except the condition (2.19) is replaced by (2.23). Then, every solution of eq. (1.1) oscillates or $\lim_{n\to\infty} x_n = 0$.

Corollary 2.5 — Assume that all the assumptions of Theorem 2.4 hold, except the condition (2.19) is replaced by (2.24). Then, every solution of eq. (1.1) oscillates or

$$\lim_{n\to\infty} x_n = 0.$$

Remark 2.6 : Our results improve the results of Zhang and Jinlin Zhang[20] and Wang and Yu[14] in linear case.

REFERENCES

1. R. P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, Second Edition, Revised and Expanded, Marcel Dekker, New York, 2000.
2. R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, Klwer Academic Publishers, 1997.
3. S. S. Cheng, *Funkcialaj Ekv.*, **37** (1994), 531-35.
4. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, 2nd edn. Cambridge Univ. Press 1952.
5. I. V. Kamenev, *Math. Zemetki*, (1978), 249-51 (in Russian)
6. A. G. Kartsatos, *Lecture Notes in Pune and Applied Mathematics*, Marcel Dekker, **28** (1977), 17-72.
7. B. Liu and S. S. Cheng, *J. Math. Anal. Appl.*, **204** (1996), 482-93.
8. Z. R. Liu, W. D. Chen and Y. H. Yu, *Kyungpook Math. J.*, **39** (1999), 127-32.
9. M. Peng, Q. Xu, L. Huang, W. Ge, *Comp. Math. Appl.*, **37** (1999), 9-18.
10. M. Peng, W. Ge and Q. Xu, *Appl. Math. Comp.*, **114** (2000), 103-14.
11. Ch. G. Philos, *Arch. Math.*, **53** (1989), 483-92.
12. B. Szmanda, *Anal. Pol. Math.*, **XLIII** (1983), 225-35.
13. Z. Szafranski and B. Szamanda, *Appl. Math. Comp.*, **83** (1997), 43-52.
14. J. S. W. Wong, *Funk. Ekv.*, **11** (1968), 207-34.
15. Z. Wang and J. Yu, *Fankcialaj Ekv.*, **34** (1991), 313-19.
16. P. J. Y. Wong and R. P. Agarwal, *Fankcialaj Ekv.*, **39** (1996), 491-517.
17. B. G. Zhang and G. D. Chen, *J. Math. Anal. Appl.*, **199** (1996), 872-41.
18. Z. Zhang and P. Bi, *J. Math. Anal. Appl.*, **255** (2001), 349-57.
19. G. Zhang and S. S. Cheng, *Pan Amer. Math. J.*, (to appear).
20. Z. Wang and J. Yu, *Fankcialaj Ekv.*, **34** (1991), 313-19.