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ON REGULAR GENERALIZED FUZZY CLOSED SETS AND GENERALIZATIONS OF FUZZY CONTINUOUS FUNCTIONS

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In this paper we define and study another various generalizations of fuzzy continuous functions by using the concept of regular generalized fuzzy closed sets. A comparative study regarding the mutual interrelations among these functions along with those functions obtained by Balasubramanian and Sundram[4] is made. Finally, we have introduced and studied the notuons of *rgf*-connectedness, *rgf*-extremally disconnectedness and *rgf*-compactness.

Key Words : Fuzzy Topology; Regular Generalized Fuzzy Closed Set; Regular Generalized Fuzzy Continuous Functions; *rgf*-Connected Set; *rgf*-Extremally Disconnected Space; *rgf*-Compact Space.

1. INTRODUCTION AND PRELIMINARIES

Balasubramanian and Sundaram[4] defined generalized fuzzy closed sets in a fuzzy topological space (in short, fts) and introduced certain types of near-fuzzy continuous functions between fts's, i.e. generalized fuzzy continuous, fuzzy *gc*-irresolute, strongly *gf*-continuous and perfectly *gf*-continuous functions etc. They also intrduced the notions of *gf*-connectedness, *gf*-extremally disconnectedness and *gf*-compactness and studied properties of those notions under above-mentioned functions.

In this paper, we study another generalization of fuzzy continuous functions and their applications. Section 2 is devoted to regular generalized fuzzy closed sets and study their properties. In Section 3 we introduce regular generalized fuzzy continuous functions and their properties by using regular generalized fuzzy closure cl*. In section 4 we introduce fuzzy *rgc*-irresolute functions and study their properties, whereas in section 5 we introduce and study strongly *rgf*-continuous and perfectly *rgf*-continuous functions and investigate inter-relations among these functions and those functions defined by Balasubramanian and Sundaram[4]. In section 6 and 7, using the concept of regular generalized fuzzy closed (open) set, we introduce and study the notions of *rgf*-connectedness, *rgf*-extremally disconnectedness and *rgf*-compactness, respectively.

Throughout this paper, simply by X and Y we shall denote fts's (X, τ) and (Y, σ) and $f: X \to Y$ will mean that f is a function from (X, τ) to (Y, σ). A fuzzy point in X with support $x \in X$ and value $\alpha (0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy set A of X, cl (A), int (A) and $1 - A$ will denote the closure of A, the interior of A and the complement of A, respectively, whereas the constant fuzzy sets taking on the values 0 and 1 on X are denoted by 0_X and 1_X, respectively. A fuzzy set A in X is said to be q-coincident[8] with a fuzzy set B, denoted by AqB, if there exists $x \in X$ such that $A(x) + B(x) > 1$. It is known that $A \leq B$ if and only if A and $1 - B$ are not q-coincident, denoted by $A \not{q} (1 - B)$. A fuzzy set A in an fts X is called q-neighbourhood (in short, q-nbd) of x_α if there exists a fuzzy open set U such that $x_\alpha q U \leq A$. it is known that $x_\alpha \in$ cl (A) if and only if for every q-nbd U of x_α, UqA. A fuzzy set A of an fts X is called fuzzy regular open if A = int (cl (A)).

2. Regular Generalized Fuzzy Closed Sets in Fuzzy Topology

Definition 2.1[4] — A fuzzy set A in an fts X is called generalized fuzzy closed (in short, *gf*-closed) if cl $(A) \leq U$ whenever $A \leq U$ and U is fuzzy open. A fuzzy set A is called generalized fuzzy open (in short, *gf*-open) if its complement 1 A is *gf*-closed.

Definition 2.2 — A fuzzy set A in a fts X is called regular generalized fuzzy closed (in short, *rgf*-closed) if cl $(A) \leq U$ whenever $A \leq U$ and U is fuzzy regular open. A fuzzy set A is called regular generalized fuzzy open (in short, *rgf*-open) if its complement $1 - A$ is *rgf*-closed.

Remark 2.3 : Every fuzzy closed (resp. fuzzy open) set is *gf*-closed (resp. *gf*-open) and every *gf*-closed (resp. *gf*-open) set is *rgf*-closed (resp. *rgf*-open), but the converses are not true.

Example 2.4 — Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$, where $A_1(a) = 0.5$, $A_1(b) = 0.7, A_1(c) = 0.6; A_2(a) = 0.4, A_2(b) = 0.7, A_2(c) = 0.3$ Define fuzzy sets A_3 and A_4 in X as follows :

$$A_3(a) = 0.4, A_3(b) = 0.3, A_3(c) = 0.4; A_4(a) = 0.3, A_4(b) = 0.3, A_4(c) = 0.2$$

Then we have

(*i*) In $(X, \tau_1), A_3$ is *gf*-closed set but not fuzzy closed.

(*ii*) In $(X, \tau_2), A_4$ is *rgf*-closed set but not *gf*-closed.

Theorem 2.5 — *If A and B are rgf-closed sets, then $A \vee B$ is rgf-closed.*

Proof : Let $A \vee B \leq U$ and U be fuzzy regular open. Then $A, B \leq U$ and thus cl (A), cl (B) $\leq U$. Hence cl $(A \vee B) \leq U$ from the fact that cl $(A) \vee$ cl (B) = cl $(A \vee B)$. □

However, the intersection of two *rgf*-closed sets is not *rgf*-closed as the following example shows.

Example 2.6 — Let $X = \{a, b, c\}$, and $\tau = \{0_X, 1_X, A_1\}$, where $A_1(a) = 0.4, A_1(b) = 0.3$, $A_1(c) = 0.5$. Define fuzzy sets A_2 and A_3 in X as follows :

$$A_2(a) = 0.3,\ A_2(b) = 0.9,\ A_2(c) = 0.3;\ A_3(a) = 0.7, A_3(b) = 0.2, A_3(c) = 0.8.$$

Then A_2 and A_3 are *rgf*-closed sets but $A_2 \wedge A_3$ is not *rgf*-closed.

Theorem 2.7 — *If A is rgf-closed set and $A \leq B \leq cl(A)$, then B is rgf-closed.*

Proof : Let U be a fuzzy regular open set such that $B \leq U$. Since $A \leq B, A \leq U$ and A is *rgf*-closed, cl $(A) \leq U$. But cl $(B) \leq cl(A)$ since $B \leq cl(A)$ and also $cl(B) \leq U$. Hence B is *rgf*-closed.

□

Theorem 2.8 — *A fuzzy set A is rgf open if and only if $F \leq$ int (A) whenever F is fuzzy regular closed and $F \leq A$.*

Proof : Let A be a *rgf*-open set and F be a fuzzy regular closed set such that $F \leq A$. Then $1 - F$ is fuzzy regular open and $1 - A \leq 1 - F$. Since $1 - A$ is *rgf*-closed, 1–int (A) = cl $(1 - A) \leq 1 - F$ which implies $F \leq$ int (A).

Conversely, suppose that A is fuzzy set such that $F \leq$ int (A) whenever F is fuzzy regular closed and $F \leq A$. We claim that $1 - A$ is *rgf*-closed set. So let $1 - A \leq U$ where U is fuzzy regular open. Now since $1 - A \leq U$, $1 - U \leq A$. Hence by assumption we have $1 - U \leq$ int (A), i.e. $1-$ int $(A) \leq U$. Hence cl $(1 - A) \leq U$, which implies that $1 - A$ is *rgf*-closed. □

***Theorem* 2.9** — *If A and B are rgf-open sets with $A \wedge cl(B) = B \wedge cl(A) = 0_X$, then $A \vee B$ is rgf-open.*

PROOF : Let F be a fuzzy regular closed set such that $F \leq A \vee B$, Then $F \wedge cl(A) \leq A$ since $B \wedge cl(A) = 0_X$, and hence by Theorem 2.5, $(F \wedge cl(A)) \leq$ int (A). Similarly, $(F \wedge cl(B)) \leq int(B)$. Now we have

$$F = F \wedge (A \vee B) \leq (F \wedge cl(A)) \vee (F \wedge cl(B))$$

$$\leq \text{int } (a) \vee \text{ int } (B) \leq \text{ int } (A \vee B).$$

Hence $F \leq$ int $(A \vee B)$ and hence by Theorem 2.5, $A \vee B$ is *rgf*-open. □

***Theorem* 2.10** — *If int $(A) \leq B \leq A$ and A is rgf-open set, then B is rgf-open.*

Definition 2.11 — A function $f: X \to Y$ is called fuzzy regular continuous (in short, *fr*-continuous) if the inverse image of every fuzzy closed set in Y is fuzzy regular closed in X.

Clearly, every *fr*-continuous function is fuzzy continuous.

***Theorem* 2.12** — *If A is a rgf-closed set in X and if $f: X \to Y$ is fr-continuous and fuzzy closed, then f(A) is gf-closed in Y.*

PROOF : Let B be a fuzzy open set in Y such that $f(A) \leq B$. Then $A \leq f^{-1}(B)$. Since A is *rgf*-closed and $f^{-1}(B)$ is fuzzy regular open, cl $(A) \leq f^{-1}(B)$, i.e., $f(cl(A)) \leq B$. Also f is fuzzy closed, f $cl(A))$ is fuzzy closed and cl $(f(A)) \leq cl(f(cl(A)) = f(cl(A)) \leq B$. hence f (A) is *gf*-closed. □

However, under fuzzy closed and *fr*-continuous function, the image of *rgf*-open set need be not *gf*-open.

Example 2.13 — Let $X = \{a\}$, $Y = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A\}$ and $\tau_2 = \{0_Y 1_Y, B_1, B_2\}$ where A, B_1 and B_2 is fuzzy sets defined by $A(a) = 0.5$; $B_1(a) = 1, B_1(b) = 0.5, B_1(c) = 1$; $B_2(a) = 1$, $B_2(b) = 0$, $B_2(c) = 1$. Define a function $f: (X, \tau_1) \to (Y, \tau_2)$ by $f(a) = b$. Clearly, f is *fr*-continuous and fuzzy closed. Now we will show that image of *rgf*-open set is not *gf*-open. Consider a fuzzy set A_1 in X defined by $A_1(a) = 0.8$. Then A_1 is *rgf*-open in (X, τ_1) but $f(A_1)$ is not *gf*-open in (Y, τ_2).

Definition 2.14 — A fuzzy set A in an fts X is called *rgf-q*-neighbourhood (in short, *rgf-q*-nbd) of a fuzzy point x_α if there is a *rgf*-open set U such that $x_\alpha q\, U \leq A$.

***Theorem* 2.15** — *Let X be an fts. Then A is rgf-open if and only if for each fuzzy point x_α with $x_\alpha q\, A$, A is rgf-q-nbd of x_α*

3. Regular Generalized Fuzzy Continuous Functions and their Properties

Definition 3.1[4] — A function $f: X \to Y$ is called generalized fuzzy continuous (in short, *gf*-continuous) if the inverse image of every fuzzy closed set in Y is *gf*-closed in X.

Definition 3.2 — A function $f: X \to Y$ is called regular generalized fuzzy continuous (in short, *rgf*-continuous) if the inverse image of every fuzzy closed set in Y is *rgf*-closed in X.

Every fuzzy continuous function is *gf*-continuous and every *gf*-continuous function is *rgf*-continuous. However, the converses are not true as Example 3.3 in Balasubramanian and Sundaram[3] and the following example show.

Example 3.3 — Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$, where $A_1(a) = 0.4, A_1(b) = 0.7, A_1(c) = 0.3; A_2(a) = 0.7, A_2(b) = 0.7, A_2(c) = 0.8$. Let $f: (X, \tau_1) \to (X, \tau_2)$ be the identity. Then f is *rgf*-continuous but not *gf*-continuous since $f^{-1}(1 - A_2)$ is not *gf*-closed in (X, τ) for fuzzy closed set $1 - A_2$ in (X, τ_2).

Balasubramanian and Sundaram[4] defined the generalized fuzzy closure operator cl^* to obtain some properties of *gf*-continuity. So, in similar way, we define the regular generalized fuzzy closure operatr cl* for any fuzzy set A in (X, τ) as follows :

$$cl_*(A) = \wedge \{B \mid A \leq B \text{ and } B \text{ is } rgf\text{-closed}\}.$$

Theorem 3.4 — *Let A be a fuzzy set in X and x_α be a fuzzy point in X. Then $x_\alpha \in cl_*(A)$ if and only if for each rgf-q-nbd U of x_α, UqA.*

The following are the properties of *rgf*-continuous functions.

Theorem 3.5 — *Let $f: (X, \tau) \to (Y, \sigma)$ be a function.*

(i) The following statements are equivalent :

(a) f is rgf-continuous.

(b) The inverse image of each fuzzy open set in Y is rgf-open in X.

(ii) If $f: (X, \tau) \to (Y, \sigma)$ is rgf-continuous, then $f(cl_*(A)) \leq cl(f(A))$ for any fuzzy set A in X.

(iii) The following statement are equivalent :

(a) For each fuzzy point x_α in X and each fuzzy open q-nbd of $f(x_\alpha)$, there exists a rgf-open q-nbd U of x_α such that $f(U) \leq V$.

(b) For each fuzzy set A in X, $f(cl_*(A)) \leq cl(f(A))$.

(c) For each fuzzy set B in Y, $cl_*(f^{-1}(B)) \leq f^{-1}(cl(B))$.

(d) The function $f: (X, \tau_*) \to (Y, \sigma)$ is fuzzy continuous.

PROOF : (*i*) It is clear from definitions.

(*ii*) Let A be any fuzzy set in X. Then $A \leq f^{-1}(f(A)) \leq f^{-1}(cl(f(A)))$. Since f is *rgf-continuous*, $cl_*(A) \leq f^{-1}(cl(f(A))$. Hence $f(cl_*(A)) \leq cl(f(A))$.

(*iii*) (*a*) $\Rightarrow$ (b) Let $y_\alpha \in f(cl_*(A))$ and V be any fuzzy open q-nbd of y_α. Then there exists a $x \in X$ such that $f(x)_\alpha = y_\alpha$ and $x_\alpha \in cl_{*(}(A)$, and by (a) there exists a *rgf*-open *q*-nbd U of x_α such that $f(U) \leq V$. Since $x_\alpha \in cl_*(A)$, UqA ad hence $V qf(A)$. Hence $y_\alpha = f(x_\alpha) \in cl(f(A))$.

(*b*) $\Rightarrow$ (*a*) Let $x_\alpha \in X$ and V be any fuzzy open *q*-nbd of $f(x_\alpha)$. Put $A = f^{-1}(1 - V)$. Then $x_\alpha \notin A$. Since $f(cl_*(A)) \leq cl(f(A)) \leq 1 - V$, $cl_*(A) \leq f^{-1}(1 - V) = A$, which implies $cl_*(A) = A$. Since $x_\alpha \notin cl_*(A)$, there exists a *rgf*-open *q*-nbd U of x_α such that $U \bar{q} A$ and hence $f(U) \leq f(1 - A) \leq V$.

(*b*) $\Leftrightarrow$ (*c*) Straightforward

(*b*) $\Leftrightarrow$ (*d*) Since the fuzzy closure of A in (X, τ_*) coincide with the set $cl_*(A)$, the equivalence is easily proved. □

The converse of Theorem 3.5 (ii) need not be true as seen from the following example.

Example 3.6 — Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$ where A_1, A_2 are fuzzy sets in X defined by $A_1(a) = 0.6$ $A_1(b) = 0.7$, $A_1(c) = 0.6$; $A_2(a) = 0.3$, $A_2(b) = 0.7$, $A_2(c) = 0.3$. Consider a function $f: (X, \tau_1) \to (X, \tau_2)$ defined by $f(a) = f(b) = f(c) = b$. Then for any fuzzy set A, $f(cl_*(A)) \leq cl(f(A))$, but f is not *rgf*-continuous. (Since $1 - A_2$ is a fuzzy closed in (X, τ_2) but $f^{-1}(1 - A_2)$ is not *rgf*-closed in (X, τ_1)).

Definition 3.7[4] — An fts X is said to be fuzzy $T_{1/2}$ if every *gf*-closed set in X is fuzzy closed in X.

Definition 3.8 — An fts X is said to be fuzzy regular $T_{1/2}$ if every *rgf*-closed set in X is fuzzy regular closed in X.

Every fuzzy regular-$T_{1/2}$ space is fuzzy $T_{1/2}$ but the converse need not be true as seen from the follwing example.

Example 3.9 — Let $X = \{a, b\}$ and $\tau = \{0_x, 1_x, A\}$ where A is fuzzy set in X defined by $0 \leq A(a) \leq 1$, $0 \leq A(b) < \frac{1}{2}$. Then (X, τ) is fuzzy $T_{1/2}$, but not fuzzy regular-$T_{1/2}$.

Theorem 3.10 — *Let $f: X \to Y$ and $g: Y \to Z$ be functions.*

(i) If f and g are rgf-continuous and Y is fuzzy regular-$T_{1/2}$, then the composition $g \circ f$: $X \to Z$ is also rgf-continuous.

(ii) If f is rgf-continuous and g are fuzzy continuous, then the composition $g \circ f$ if rgf-continuous.

The following Example shows that the composition of any two *rgf*-continuous functions need not be *rgf*-conitnuous.

Example 3.11 — Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$, $\tau_2 = \{0_X, 1_X, A_2\}$ and $\tau_3 = \{0_X, 1_X, A_3\}$ where A_1, A_2 and A_3 are fuzzy sets in X defined as follows :

$$A_1(a) = 0.4, A_1(b) = 0.3, A_1(c) = 0.4;$$

$$A_2(a) = A_2(b) = A_2(c) = 0.5;$$

$A_3(a) = 0.7, A_3(b) = 0.8, A_3(c) = 0.7.$

Let $f : (X, \tau) \to (X, \tau_2)$ be a function defined by $f(a) = f(b) = f(c) = b$ and $g : (X, \tau_2) \to (X, \tau_3)$ be the identity. Then f and g are *rgf*-continuous but $g \circ f$ is not *rgf*-continuous; for $1 - A_3$ is fuzzy closed in $(X, \tau_3), f^{-1}(g^{-1}(1 - A_3))$ is not *rgf*-closed in (X, τ_1). Hence $g \circ f$ is not *rgf*-continuous.

Remark 3.12 : For $f: X \to Y$, when X is fuzzy regular-$T_{1/2}$, then *rgf*-continuity, *gf*-continuity, fuzzy continuity and *fr*-continuity are equivalent.

4. Fuzzy *RGC*-Irresolute Functions and their Properties

Definition 4.1[4]— A function $f: X \to Y$ is called fuzzy *gc*-irresolute if the inverse image of every *gf*-closed set in Y is *gf*-closed in X.

Definition 4.2 — A function $f: X \to Y$ is called fuzzy *rgc*-irresolute if the inverse image of every *rgf*-closed set in Y is *rgf*-closed X.

Every fuzzy *rgc*-irresolute function s *rgf*-continuous but the converse is not true (see Example 4.3). And the following Examples 4.4 and 4.5 show that fuzzy *gc*-irresolute function and fuzzy *gc*-irresolute function are, in general, independent.

Example 4.3 Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$, where A_1 and A_2 are fuzzy sets in X defined by $A_1(a) = 0.7, A_1(b) = 0.8, A_1(c) = 0.7$; $A_2(a) = A_2(b) = A_2(c) = 0.5$. Let $f: (X, \tau_1) \to (X, \tau_2)$ be the identity. Then f is *rgf*-continuous but not fuzzy *rgc*-irresolute; for a fuzzy set A_3 in X defined by $A_3(a) = 0.3, A_3(b) = 0.1, A_3(c) = 0.3$ is *rgf*-closed in $(X, \tau_2), f^{-1}(A_3)$ is not *rgf*-closed in (X, τ_1).

Example 4.4 — Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$ where A_1 and A_2 are fuzzy sets in X defined by $A_1(a) = A_1(c) = 0, A_1(b) = 0.5; A_2(a) = A_2(c) = 0, A_2(b) = 1$. Let $f: (X, \tau_1) \to (X, \tau_2)$ be the identity. Then f is fuzzy *gc*-irresolute but not fuzzy *rgc*-irresolute; for A_1 is *rgf*-closed in $(X, \tau_2), f^{-1}(A_1)$ is not *rgf*-closed in (X, τ_1). Hence f is not fuzzy *rgc*-irresolute.

Example 4.5 — Let $X = \{a, b, c\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$ where A_1 and A_2 are fuzzy sets defined by $A_1(a) = A_1(c) = 0$, $A_1(b) = 1; A_2(a) = A_2(c) = 0.7$, $A_2(b) = 0.5$. Let $f: (X, \tau_1) \to (X, \tau_2)$ be the identity. Then f is fuzzy *rgc*-irresolute but not fuzzy *gc*-irresolute; for a fuzzy set A_3 in X defined by $A_3(a) = A_3(c) = 0, A_3(b) = 0.5$ is *gf*-closed in $(X, \tau_2), f^{-1}(A_3)$ is not *gf*-closed in (X, τ_1).

The following are the properties of fuzzy *rgc*-irresolute functions.

***Theorem* 4.6** — *Let $f: (X, \tau) \to (Y, \sigma)$ be a function.*

(i) The following statements are equivalent:

(a) f is fuzzy rgc-irresolute.

(b) The inverse image of every rgf-open set in Y is rgf-open in X.

(ii) If $f: (X, \tau) \to (Y, \sigma)$ is fuzzy rgc-irresolute, then $f(cl_*(A)) \leq cl_*(f(A))$ for any fuzzy set A in X.

(iii) The following statement are equivalent:

(a) For each fuzzy point x_α in X and each rgf-open q-nbd of $f(x_\alpha)$, there exists a rgf-open q-nbd U of x_α such that $f(U) \leq V$.

(b) For each fuzzy set A in X, $f(cl_*(A)) \leq cl_*(f(A))$.

(c) For each fuuzzy set B in Y, $cl_*(f^{-1}(B)) \leq f^{-1}(cl_*(B))$.

(d) The function $f: (X, \tau_*) \to (Y, \sigma_*)$ is fuzzy continuous.

PROOF : It is similar to that of Theorem 3.5.

***Theorem* 4.7** — *Let $f: X \to Y$ and $g: Y \to Z$ be functions.*

(i) If f and g are fuzzy rgc-irresolute, then the composition $g \circ f$ is fuzzy rgc-irresolute.

(ii) If f is fuzzy rgc-irresolute and g are rgf-continuous, then the composition $g \circ f$ is rgf-continuous.

5. STRONGLY *rgf*-CONTINUOUS AND PERFECTLY *rgf*-CONTINUOUS FUNCTIONS

Definition 5.1[4] — A function $f: X \to Y$ is called

(*i*) perfectly fuzzy continuous if the inverse image of every fuzzy open set in Y is both fuzzy open and fuzzy closed in X,

(*ii*) strongly *gf*-continuous if the inverse image of every *gf*-open set in Y is fuzzy open in X.

(*iii*) perfectly *gf*-continuous if the inverse image of every *gf*-open set in Y is both fuzzy open and fuzzy closed in X.

Definition 5.2 — A function $f: X \to Y$ is called

(*i*) strongly *rgf*-continuous if the inverse image of every *rgf*-open set in Y is fuzzy open in X,

(*ii*) perfectly *rgf*-continuous if the inverse image of every *rgf*-open set in Y is both fuzzy open and fuzzy closed in X.

Remark 5.3 : When Y is fuzzy regular-$T_{1/2}$, strongly *rgf*-continuity, strongly *gf*-continuity and fuzzy continuity are equivalent concepts, and also perfectly *rgf*-continuity, perfectly *gf*-continuity and perfectly fuzzy continuity are equivalent.

***Theorem* 5.4** — *Strong rgf-continuity ⇒ strong gf-continuity ⇒ fuzzy continuity.*

The converses of Theorem 5.4 are not true as Example 5.7 in Balasubramanian and Sundaram[3] and the following example show.

Example 5.5 — Let $X = \{a, b\}$ and $\tau_1 = \{0_X, 1_X, A\}$ where A is fuzzy set in X defined by $0 \leq A(a) \leq 1, 0.5 \leq A(b) \leq 1$. Let $f: (X, \tau_1) \to (X, \tau_1)$ be the identity. Then f is strongly *gf*-continuous but not strongly *rgf*-continuous.

***Theorem* 5.6** — *A function $f: X \to Y$ is strongly rgf-continuous if and only if the inverse image of every rgf-closed set in Y is fuzzy closed in X.*

***Theorem* 5.7** — *Let $f: X \to Y$, $g: Y \to Z$ be functions. If f is strongly rgf-continuous and g is rgf-continuous, then $g \circ f$ is fuzzy continuous.*

***Theorem* 5.8** — *Perfect rgf-continuity $\Rightarrow$ perfect gf-continuity, and perfect rgf-continuity $\Rightarrow$ strong rgf-continuity.*

The converses of Theorem 5.8 are not true.

Example 5.9 — Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A_1, A_2\}$ and $\tau_2 = \{0_X, 1_X, A_1\}$ where A_1 and A_2 are fuzzy sets in X defined by $A_1(a) = 1, 0.5 \le A_1(b) \le 1; A_2(a) = 0, 0 \le A_2(b) \le 0.5$. Let $f; (X, \tau_1) \to (X, \tau_2)$ be the identity. Then f is perfectly *gf*-continuous but not perfectly *rgf*-continuous.

Example 5.10 — Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A_1\}$ and $\tau_2 = \{0_X, 1_X, A_2\}$ where A_1 and A_2 are fuzzy sets in X defined by $0 \le A_1(a) \le 1, 0 \le A_1(b) \le \frac{1}{2}$; $0 \le A_2(a) \le \frac{1}{2}$, $0 \le A_2(b) \le 1$. Define $f: (X, \tau_1) \to (X, \tau_2)$ by $f(a) = b$ and $f(b) = a$. Then f is strongly *rgf*-continuous but not perfectly *rgf*-continuous.

***Theorem* 5.11** — *A function $f: X \to Y$ is perfectly rgf-continuous if and only if the inverse image of rgf-closed set in Y is both fuzzy open and fuzzy closed in X.*

Regarding the results above-mentioned so far, we have the table of implications as shown in Table 1.

TABLE 1

$\Rightarrow$	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*
a	1	1	1	1	0	0	0	0	0	0	0
b	0	1	1	1	0	0	0	0	0	0	0
c	0	0	1	1	0	0	0	0	0	0	0
d	0	0	0	1	0	0	0	0	0	0	0
e	0	0	1	1	1	0	0	0	0	0	0
f	0	0	0	1	0	1	0	0	0	0	0
g	1	1	1	1	0	0	1	0	0	0	0
h	0	1	1	1	1	0	0	1	0	0	0
i	1	1	1	1	1	0	1	1	1	0	0
j	0	1	1	1	1	1	0	1	0	1	0
k	1	1	1	1	1	1	1	1	1	1	1

In above table, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *j* and *k* denote fuzzy regular continuity, fuzzy continuity, *gf*-continuity, *rgf*-continuity, fuzzy *gc*-irresolute, fuzzy *rgc*-irresolute, perfect fuzzy continuity, strong *f*-continuity, perfect *gf*-continuity, strong *rgf*-continuity and perfect *rgf*-continuity, respectively. Also 1 denotes 'implies' and 0 denotes 'does not imply'.

6. *rgf*-CONNECTEDNESS AND THEIR PROPERTIES

Definition 6.1[4] — An fts X is said to be *gf*-connected if the only fuzzy sets which are both *gf*-open and *gf*-closed and 0_X and 1_X.

Definition 6.2 — An fts X is said to be *rgf*-connected if the only fuzzy sets which are both *rgf*-open and *rgf*-closed are 0_X and 1_X.

***Theorem* 6.3** — *Every rgf-connected space is gf-connected and every gf-connected and every gf-connected space is fuzzy connected*[7].

PROOF : It proved in [4; Theorem 7.2] that every *gf*-connected space is fuzzy connected. So we proved that every *rgf*-connected space is *gf*-connected. Let X be a *rgf*-connected space and suppose that X is not *gf*-connected. Then there exists a proper fuzzy set A ($A \neq 0_X, A \neq 1_X$) such that A is both *gf*-open and *gf*-closed. Since *gf*-open set is *rgf*-open, X is not *rgf*-connected – a contradiction. □

However, the converses are not true as Example 7.3 in Balasubramanian and Sundaram[4] and the following example show.

Example 6.4 — Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A\}$ where A is a fuzzy set in X defined by $\frac{1}{2} \leq A \leq 1$, $A(b) = 1$. Then (X, τ) is *gf*-connected but not *rgf*-connected; For any fuzzy set B in X, B is *rgf*-open and *rgf*-closed in (X, τ). Hence (X, τ) is not *rgf*-connected

***Theorem* 6.5** — *For fuzzy regular-$T_{1/2}$ space X, the following are equivalent :*

(i) X is rgf-connected

(ii) X is gf-connected

(iii) X is fuzzy connected.

PROOF : *(i)* ⇒ *(ii)* ⇒ *(iii)* follow from Theorem 6.3

(iii) ⇒ *(i)* — Assume that X is fuzzy regular-$T_{1/2}$ and fuzzy connected. If possible, let X be not *rgf*-connected, then there exists a proper fuzzy set A such that A is both *rgf*-open and *rgf*-closed. Since X is fuzzy regular-$T_{1/2}$, A is fuzzy open and fuzzy closed, which implies that X is not fuzzy connected - a contradiction. □

***Theorem* 6.6** — *If $f: X \to Y$ is rgf-continuous surjection and X is rgf-connected, then Y is fuzzy connected.*

***Theorem* 6.7** — *If $f: X \to Y$ is fuzzy rgc-irresolute surjection and X is rgf-connected, then Y is rgf-connected.*

***Theorem* 6.8** — *If $f: X \to Y$ is perfectly rgf-continuous surjection and X is fuzzy connected, then Y is rgf-connected.*

***Theorem* 6.9** — *An fts X is rgf-connected if and only if it has no non-zero rgf-open sets A and B such that $A + B = 1$.*

Corollary 6.10 — An fts X is *rgf*-connected if and only if it has no non-zero *rgf*-open sets A and B such that $A + B = 1$, cl $(A) + B = A +$ cl $(B) = 1$.

Now, we define the regular generalized fuzzy interior operator int_* for any fuzzy set A in an fts (X, τ) as follows: $\text{int}_* (A) = \vee \{B \mid B \leq A$ and B is *rgf*-open$\}$. It is easy to see that for any fuzzy set A in X, $1 - cl_* (A) = \text{int}^* (1 - A)$.

Definition 6.11 — A *rgf*-open set A is called regular *rgf*-open if $A = \text{int}_*\ (\text{cl}_*\ (A))$. The fuzzy complement of regular *rgf*-open set is called regular *rgf*-closed.

Definition 6.12 — An fts X is called *rgf*-super connected if there is no proper regular *rgf*-open set in X.

Theorem 6.13 — *In an fts X, the following are equivalent :*

(i) X is rgf-super connected.

(ii) For every non-zero rgf-open set A, $cl_*\ (A) = 1$.

(iii) For every rgf-closed set A with $A \neq 1$, $int_*\ (A) = 0$.

(iv) X does not have non-zero rgf-open sets A and B such that $A + B \leq 1$.

(v) X does not have non-zero fuzzy sets A and B such that $cl_*\ (A) + B = A + cl_*\ (B) = 1$.

Definition 6.14 — An fts X is said to be *rgf*-strongly connected if it has no non-zero *rgf*-closed sets A and B such that $A + B \leq 1$.

Theorem 6.15 — *An fts X is rgf-strongly connected if and only if it has no non-zero rgf-open sets A and B such that* $A \neq 1, B \neq 1$ *and* $A + B \leq 1$.

Remark 6.16 : Every *rgf*-super connected space is *gf*-super connected, and every *rgf*-strong connected space *gf*-strong connected.

The converses of Remark 6.16 are not true. Also the following examples show that *gf*-super connectedness and *gf*-strong connectedness are independent.

Example 6.17 — let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A\}$ where A is fuzzy set defined by $\frac{2}{3} \leq A\ (a) \leq 1$ and $A(b) = 0$. Then (X, τ) is *gf*-super connected but it is neither *gf*-strongly connected nor *rgf*-super connected.

Example 6.18 — Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, A_1, A_2\}$ where A_1 and A_2 are fuzzy sets defined by $A_1\ (a) = \frac{2}{3}, A_1\ (b) = 1; 0 \leq A_2\ (a) < \frac{1}{3}, A_2\ (b) = 0$. Then (X, τ) is *gf*-strongly connected but it is neither *gf*-super connected nor *rgf*-strongly connected.

7. *rgf*-EXTREMALLY DISCONNECTEDNESS AND *rgf*-COMPACTNESS

Definition 7.1 — An fts X is said to be regular generalized fuzzy extremally disconnected (in short, *rgf*-extremally disconnected) if $cl_*\ (A)$ is *rgf*-open, whenever A is *rgf*-open.

Theorem 7.2 — *For any fts X, then following are equivalent :*

(i) X is rgf-extremally disconnected.

(ii) For each rgf-closed set A, int (A) is rgf-closed.*

(iii) For each rgf-open set A, $cl_*\ (A) + cl_*(1 - cl_*\ (A)) = 1$.

(iv) For each pair of rgf-open set A, ,B with $cl_*\ (A) + B = 1$, $cl_*\ (A) + cl_*\ (B) = 1$.

PROOF (*i*) $\Rightarrow$ (*ii*) : Let A be any *rgf*-closed set. Then $1 - A$ is *rgf*-open and so by (*i*) cl_* $(1 - A) = 1 - \text{int}_* (A)$ is *rgf*-open, which implies that $\text{int}_* (A)$ is *rgf*-closed.

(*ii*) $\Rightarrow$ (*iii*) Let A is *rgf*-open set. Since $1 - \text{cl}_* (A) = \text{int}_* (1 - A)$, we have

$$cl_* (A) + cl_* (1 - cl_* (A)) = cl_* (A) + cl_* (int_* (1 - A)).$$

Since A is *rgf*-open, $1 - A$ is *rgf*-closed and so by (ii) $\text{int}_* (1 - A)$ is *rgf*-closed, i.e. $\text{cl}_* (\text{int}_* (1 - A)) = \text{int}_* (1 - A)$. Thus, we get

$$cl_* (A) = cl_* (1 - cl_* (A)) = cl_* (A) + int_* (1 - A) = cl_* (A) + 1 - cl_* (A) = 1.$$

(*iii*) $\Rightarrow$ (*iv*) Let A and B be any *rgf*-open ses such that $\text{cl}_* (A) + B = 1$. Then by (iii) we have

$$cl_* (A) + cl_* (1 - cl_* (A) = 1 = cl_* (A) + B.$$

This implies that $B = cl_* (1 - cl_* (A))$. But from hypothesis $B = - cl_* (A)$ and thus $cl_* (B) = cl_* (1 - cl_* (A))$. Hence $B = cl_* (B)$, and conseqiuently, $cl_* (A) + cl_* (B) = 1$.

(*iv*) $\Rightarrow$ (*i*) Let A be any *rgf*-open set. Put $B = 1 - cl_* (A)$. Then by (iv) we have $cl_* (A) + cl_* (B) = 1$, i.e. $cl_* (B) = 1 - cl_* (A)$. Then we get $B = cl_* (B)$ and thus B is *rgf*-closed. Hence $cl_* (A) = 1 - cl_* (B)$ is *rgf*-open. □

Definition 7.3 — A collection $\{A_\lambda\}_{\lambda \in \Lambda}$ of *rgf*-open sets in X is called *rgf*-open cover of a fuzzy set B in X if $B \leq \vee_{\lambda \in \Lambda} A_\lambda$.

Definition 7.4 — An fts X is called *rgf*-compact if every *rgf*-open cover of X has á finite subcover.

Definition 7.5 — A fuzzy set B in X is said to be *rgf*-compact relative to X (which we shall call a *rgf*-compact set) if for every collection $\{A_\lambda\}_{\lambda \in \Lambda}$ of *rgf* open sets of X such that $B \leq \vee_{\lambda \in \Lambda} A_\lambda$, there exists a finite subset Λ_0 of Λ such that $B \leq \vee_{\lambda \in \Lambda} A_\lambda$.

Theorem 7.6 — *Let X be a rgf-compact fts and A be a rgf-closed set in X. Then A is rgf-compact set.*

Theorem 7.7 — *(i) $f: X \to Y$ is rgf-continuous and X is rgf-compact, then f (X) is a fuzzy compact set.*

(ii) If $f: X \to Y$ is fuzzy rgc-irresolute and A is rgf-compact set of X, then f (A) is rgf-compact set in Y.

(iii) If $f: X \to Y$ is strongly rgf-continuous and X is fuzzy compact, then f (X) is a rgf-compact set in Y.

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