Indian J. pure appl. Math., **34**(7) : 1025-1033, July 2003

MEROMORPHIC FUNCTIONS SHARING FOUR SMALL FUNCTIONS IM*

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(*Received* 8 *May* 2002; *accepted* 22 *November* 2002)

In this paper, we prove some results on two meromorphic functions $f(z)$ and $g(z)$ sharing four small functions IM^* when $f(z) \not\equiv g(z)$. It is a generalizations of a result obtained by Gundersen[2] and Mues[7].

Key Words : Meromorphic Function; Small Function; Uniqueness; Sharing Small Function

1. INTRODUCTION AND THE MAIN RESULTS

In this paper, we use the same symbols as given in Nevanlinna theory of meromorphic functions (see [5] & [6]). By $S(r, f)$ we denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to +\infty$, possibly outside a set of r of finite linear measure. We denote the set as E. It is not necessarily the same when it appears. If two meromorphic functions f and g have same a-point with same multiplicities (ignoring multiplicities), then we say f and g share the value a CM (IM). Let $f(z)$ be a non-constant meromorphic function in the complex plane and let $S(f)$ be the set of meromorphic functions $\beta(z)$ in the complex plane which satisfy

$$T(r, \beta) = S(r, f),$$

where $S(r, f)$ is any quantity satisfying

$$S(r,f) = o(T(r,f))$$

for $r \to \infty, r \in J$, mes $J = +\infty$. Such a meromorphic function $\beta(z)$ is said to be a small function of $f(z)$. Note that $S(f)$ is a field.

Let $h_1(z)$ and $h_2(z)$ be two non-constant meromorphic functions, and let $a(z)$ (or ∞) be the common small function of $h_1(z)$ and $h_2(z)$. We denote by $\overline{N}(r, h_1(z) = a(z) = h_2(z))$ (resp. $\overline{N}_E(r, h_1(z) = a(z) = h_2(z))$) the counting function of those common $a(z)$-points of $h_1(z)$ and $h_2(z)$, regardless of multiplicity (resp. with the same multiplicity). Each point counted only once.

(*i*) If $\overline{N}\left(r, \dfrac{1}{h_j(z) - a(z)}\right) - \overline{N}_E(r, h_1(z) = a(z) = h_2(z)) = S(r, h_j)$ $(j = 1, 2)$, then we say that $h_1(z)$ and $h_2(z)$ share small function $a(z)$ CM*;

*Project supported by the NSFC (No. 19871050) and RFDP (No. 98042209).

(*ii*) If $\overline{N}\left(r, \frac{1}{h_j(z)-a(z)}\right) - \overline{N}(r, h_1(z)=a(z)=h_2(z)) = S(r, h_j)$ $(j = 1, 2)$, then we say that $h_1(z)$ and $h_2(z)$ share small function $a(z)$ IM*

In 1989, Mues proved the following result.

bTheorem Ab[7] — *Let f and g be non-constant meromorphic functions and share four distinct values a_j (j = 1, 2, 3, 4) IM. If $f(z) \not\equiv g(z)$, then*

1. $T(r,f) = T(r,g) + S(r,f), T(r,g) = T(r,f) + S(r,g)$;

2. $\sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f-a_i}\right) = 2T(r,f) + S(r,f)$;

3. $\overline{N}\left(r, \frac{1}{f-b}\right) = T(r,f) + S(r,f), \overline{N}\left(r, \frac{1}{g-b}\right) = T(r,g) + S(r,g)$,

where $b \neq a_j$ $(j = 1, 2, 3, 4)$;

4. $N_0\left(r, \frac{1}{f'}\right) = S(r, f), N_0\left(r, \frac{1}{g'}\right) = S(r, g)$,

where $N_0\left(r, \frac{1}{f'}\right)$ is the counting function which count the zeros of f' but $f-a_j$ $(j = 1, 2, 3, 4)$, notation $N_0\left(r, \frac{1}{g'}\right)$ can be similarly defined;

5. $\sum_{j=1}^{4} \overline{N}^*(r, a_j) = S(r,f)$,

where $\overline{N}^*(r, a_j)$ is the counting function which count the multiple common zeros of $f-a_j$ and $g-a_j$, each point counted only once.

In this paper, we get the following results which are the generalizations of a result obtained by Gundersen[2] and Mues[7], i.e. 5 in Theorem A on small functions.

***Theorem* 1** —*Let f and g be non-constant meromorphic functions and share four distinct small functions a_j (j = 1, 2, 3, 4) IM*. If $f(z) \not\equiv g(z)$, then*

$$\sum_{j=1}^{4} \overline{N}^*(r, a_j) = S(r,f),$$

where $\overline{N}^*(r, a_j)$ is the counting function which count the multiple common zeros of $f-a_j$ and $g-a_j$, each point counted only once.

Let $f(z)$ and $g(z)$ be non-constant meromorphic functions and $a(z)$ be a small function of $f(z)$ and $g(z)$. We denote by $N_E(r, a)$ the reduced counting function of those common zeros of

$f(z) - a(z)$ and $g(z) - a(z)$ with the same multiplicities, and denote $N_E^{(1)}(r, a)$ the reduced counting function of those common simple zeros of $f(z) - a(z)$ and $g(z) - a(z)$. We have the following

Corollary — Let $f(z)$ and $g(z)$ be non-constant meromorphic functions and share four distinct small functions $a_j(z)$ $(j = 1, 2, 3, 4)$ IM*. If $f(z) \not\equiv$ (z), then for $j = 1, 2, 3, 4$ we have

$$N_E(r, a_j) = N_E^{1)}(r, a_j) + S(r, f).$$

2. Some Lemmas and Notations

Lemma 1[3] — Suppose that f, $a(z)$ and $b(z)$ are all meromorphic functions ($\not\equiv \infty$), $a(z)$ and $b(z)$ are distinct small functions of f. Set

$$L(f, a, b) := \begin{vmatrix} f & f' & 1 \\ a & a' & 1 \\ b & b' & 1 \end{vmatrix}$$

Then $$L(f, a, b) \not\equiv 0,$$

and $$m\left(r, \frac{L(f, a, b) f^k}{(f-a)(f-b)}\right) = S(r, f) \ (k = 0, 1).$$

Lemma 2 — Let f and g be two non-constant meromorphic functions sharing four distinct small functions $b_1(z), b_2(z), b_3(z)$ and ∞ IM*. Set

$$H := \frac{L(f, b_1, b_2)(f-g) L(g, b_2, b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} - \frac{L(g, b_1, b_2)(f-g) L(f, b_2, b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)}$$

Then we have $T(r, H) = S(r, f)$.

Proof : From Lemma 1 we get $m(r, H) = S(r, f)$. The zeros of $f - b_1$ and $f - b_3$ contribute to $N(r, H)$ only $S(r, f)$. Let z_∞ be a pole of f with multiplicity

p, a pole of g with multiplicity q, and $b_j(z_\infty)(b_j(z_\infty) - 1) \neq 0, \infty$ $(j = 1, 2, 3)$.

Without loss of generality, we suppose that $p \geq q$, then when $z \to z_\infty$, we have

$$\frac{L(f, b_1, b_2)(f-g) L(g, b_2, b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} \sim (b_2 - b_1)(b_3 - b_2)\left(1 - \frac{g}{f}\right)\frac{f' g'}{f g^2},$$

$$\frac{L(g, b_1, b_2)(f-g) L(f, b_2, b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)} \sim (b_2 - b_1)(b_3 - b_2)\left(1 - \frac{g}{f}\right)\frac{f' g'}{f g^2}.$$

So we know H is analytic at z_∞. hence the poles of f contribute to $N(r, H)$ also $S(r, f)$. Notice that H can be written as :

$$H \equiv \frac{-(f-g)}{(f-b_1)(g-b_3)}\left[\left[(b_1 - b_2)\frac{f' - b_2'}{f - b_2} - (b_1' - b_2')\right]\left[(b_3 - b_2)\frac{g' - b_2'}{g - b_2} - (b_3' - b_2')\right]\right.$$

$$-\frac{f(g-1)}{g(f-1)}\left[(b_1-b_2)\frac{g'-b_2'}{g-b_2}-(b_1'-b_2')\right]\left[(b_3-b_2)\frac{f'-b_2'}{f-b_2}-(b_3'-b_2')\right]\Bigg]$$

From the formula above we know that the zeros of $f-b_2$ contribute to $N(r, H)$ also $S(r, f)$. So we have $N(r, H) = S(r, f)$. Hence $T(r, H) = S(r, f)$.

Lemma 3 — Let f and g be two non-constant meromorphic functions, $b_1(z)$, $b_2(z)$, $b_3(z)$ and ∞ be four distinct small functions of f and g. Set

$$H := \frac{L(f, b_1, b_2)(f-g)L(g, b_2, b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} - \frac{L(g, b_1, b_2)(f-g)L(f, b_2, b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)},$$

$$H^* := \frac{L(f, b_2, b_1)(f-g)L(g, b_1, b_3)}{(f-b_2)(f-b_1)(g-b_1)(g-b_3)} - \frac{L(g, b_2, b_1)(f-g)L(f, b_1, b_3)}{(g-b_2)(g-b_1)(f-b_1)(f-b_3)},$$

$$H^{**} := \frac{L(f, b_1, b_3)(f-g)L(g, b_3, b_2)}{(f-b_1)(f-b_3)(g-b_3)(g-b_2)} - \frac{L(g, b_1, b_3)(f-g)L(f, b_3, b_2)}{(g-b_1)(g-b_3)(f-b_3)(f-b_2)}.$$

Then we have $H \equiv -H^* \equiv -H^{**}$.

PROOF : We only prove $H \equiv -H^*$. The proof of $H \equiv -H^{**}$ is similar.

$$H \equiv (f-g)\left\{\frac{1}{b_2-b_1}\left(\frac{L(f, b_1, b_2)}{f-b_2} - \frac{L(f, b_1, b_2)}{f-b_1}\right)\frac{1}{b_3-b_2}\left(\frac{L(g, b_2, b_3)}{g-b_3} - \frac{L(g, b_2, b_3)}{g-b_2}\right)\right.$$

$$\left.-\frac{1}{b_2-b_1}\left(\frac{L(g, b_1, b_2)}{g-b_2} - \frac{L(g, b_1, b_2)}{g-b_1}\right)\frac{1}{b_3-b_2}\left(\frac{L(f, b_2, b_3)}{f-b_3} - \frac{L(f, b_2, b_3)}{f-b_2}\right)\right\}$$

$$\equiv (f, g)\left\{\left(\frac{f'-b_2'}{f-b_2} - \frac{f'-b_1'}{f-b_1}\right)\left(\frac{g'-b_3'}{g-b_3} - \frac{g'-b_2'}{g-b_2}\right) - \left(\frac{g'-b_2'}{g-b_2} - \frac{g'-b_1'}{g-b_1}\right)\left(\frac{f'-b_3'}{f-b_3} - \frac{f'-b_2'}{f-b_2}\right)\right\}$$

$$\equiv (f-g)\left\{\frac{f'-b_2'}{f-b_2}\frac{g'-b_3'}{g-b_3} - \frac{g'-b_2'}{g-b_2}\frac{f'-b_3'}{f-b_3} - \frac{f'-b_1'}{f-b_1}\frac{g'-b_3'}{g-b_3}\right.$$

$$\left.+\frac{g'-b_1'}{g-b_1}\frac{f'-b_3'}{f-b_3} + \frac{f'-b_1'}{f-b_1}\frac{g'-b_2'}{g-b_2} - \frac{g'-b_1'}{g-b_1}\frac{f'-b_2'}{f-b_2}\right\}.$$

By making an exchange of the positions of b_1, b_2, we have $H \equiv -H^*$. Then we completes the Proof of Lemma 3.

Lemma 4 — Let f and g be two non-constant meromorphic functions, $b_1(z)$, $b_2(z)$, $b_3(z)$ and ∞ be four distinct small functions of f and g. Set

$$H := \frac{L(f, b_1, b_2)(f-g)L(g, b_2, b_3)}{(f-b_1)(f-b_2)(g-b_2)(g-b_3)} - \frac{L(g, b_1, b_2)(f-g)L(f, b_2, b_3)}{(g-b_1)(g-b_2)(f-b_2)(f-b_3)},$$

then $H \equiv 0 \Leftrightarrow \tilde{H} \equiv 0$,

where
$$\tilde{H} := \frac{L(F, \tilde{b}_1, \tilde{b}_2)(F-G)L(G, \tilde{b}_2, \tilde{b}_3)}{(F-\tilde{b}_1)(F-\tilde{b}_2)(G-\tilde{b}_2)(G-\tilde{b}_3)} - \frac{L(G, \tilde{b}_1, \tilde{b}_2)(F-G)L(F, \tilde{b}_2, \tilde{b}_3)}{(G-\tilde{b}_1)(G-\tilde{b}_2)(F-\tilde{b}_2(F-\tilde{b}_3)}$$

$$F = \frac{1}{f-b_1} + b_1, G = \frac{1}{g-b_1} + b_1, \tilde{b}_1 = b_1, \tilde{b}_2 = \frac{1}{b_2 - b_1} + b_1, \tilde{b}_3 = \frac{1}{b_3 - b_1} + b_1.$$

PROOF : Without loss of generality, we suppose that $b_1(z) = 0, b_2(z) = 1, b_3(z) = b(z)$, $(b(z) \not\equiv \infty, 0.1)$. From Lemma 3 we know that

$$-H \equiv H^* = \frac{L(f, 1, 0)(f-g)L(g, 0, b)}{(f-1)fg(g-b)} - \frac{L(g, 1, 0)(f-g)L(f, 0, b)}{(g-1)gf(f-b)}.$$

So
$$F = \frac{1}{f}, G = \frac{1}{g}, \tilde{b}_1 = 0, \tilde{b}_2 = 1, \tilde{b}_3 = b^{-1}.$$

$$-\tilde{H} \equiv \tilde{H}^* = \frac{L(F, 1, 0)(F-G)L(G, 0, b^{-1})}{(F-1)FG(G-b^{-1})} - \frac{L(G, 1, 0)(F-G)L(F, 0, b^{-1})}{(G-1)GF(F-b^{-1})}$$

$$= \frac{L(f^{-1}, 1, 0)(f^{-1} - g^{-1})L(g^{-1}, 0, b^{-1})}{(f^{-1}-1)f^{-1}g^{-1}(g^{-1}-b^{-1})} - \frac{L(g^{-1}, 1, 0)(f^{-1} - g^{-1})L(f^{-1}, 0, b^{-1})}{(g^{-1}-1)g^{-1}f^{-1}(f^{-1}-b^{-1})}$$

$$= \frac{\frac{f'}{f^2}\frac{g-f}{fg}\begin{vmatrix} \frac{1}{g} & \frac{g'}{g^2} \\ \frac{1}{b} & \frac{b'}{b^2} \end{vmatrix}}{\frac{1-f}{f}\frac{1}{f}\frac{1}{g}\frac{b-g}{bg}} - \frac{\frac{g'}{g^2}\frac{g-f}{fg}\begin{vmatrix} \frac{1}{f} & \frac{f'}{f^2} \\ \frac{1}{b} & \frac{b'}{b^2} \end{vmatrix}}{\frac{1-g}{g}\frac{1}{g}\frac{1}{f}\frac{b-f}{bf}}$$

$$= \frac{1}{b}\left\{ \frac{-f'(f-g)\begin{vmatrix} g & g' \\ b & b' \end{vmatrix}}{(f-1)fg(g-b)} - \frac{-g'(f-g)\begin{vmatrix} f & f' \\ b & b' \end{vmatrix}}{(g-1)gf(f-b)} \right\}$$

$$= \frac{1}{b}H^* = -\frac{1}{b}H.$$

Noting that $b(z) \equiv 0$, so we have

$$H \equiv 0 \Leftrightarrow \tilde{H} \equiv 0.$$

3. PROOF OF THEOREM 1 AND COROLLARY

Without loss of generality, we suppose that $a_1(z) = 0, a_2(z) = 1, a_3(z) = a(z), a_4(z) = \infty$, $(a(z) \not\equiv \infty, 0.1)$, otherwise, a quasi-Möbius transformation will do.

In what follows we set

$$S_0 = \{ z \mid a(z) = 0, \text{ or } 1, \text{ or } \infty \}.$$

If $a(z) \equiv$ constant, from 5 in Theorem A we know that the result of Theorem 2 is valid. In the rest of this section, we assume that $a(z) \not\equiv$ constant.

Set

$$H := \frac{L(f,0,1)(f-g)L(g,1,a)}{f(f-1)(g-1)(g-a)} - \frac{L(g,0,1)(f-g)L(f,1,a)}{g(g-1)(f-1)(f-a)}.$$

From Lemma 2 we have

$$T(r,H) = S(r,f).$$

Case 1 — $H \not\equiv 0$.

Let z_∞ be a pole of f with multiplicity $p\,(\geq 2)$, a pole of g with multiplicity $q\,(\geq 2)$, and $z_\infty \notin S_0$. Without loss of generality, we suppose that $p \geq q \geq 2$, then when $z \to z_\infty$, we have

$$\frac{L(f,0,1)(f-g)L(g,1,a)}{(f)(f-1)(g-1)(g-a)} \sim (a-1)\left(1-\frac{g}{f}\right)\frac{f'g'}{fg^2},$$

$$\frac{L(g,0,1)(f-g)L(f,1,a)}{(g)(g-1)(f-1)(f-a)} \sim (a-1)\left(1-\frac{g}{f}\right)\frac{f'g'}{fg^2}.$$

By simple computation, we get $H(z_\infty) = 0$.

Let z_0 be a zero of f with multiplicity $p\,(\geq 2)$, a zero of g with multiplicity $q\,(\geq 2)$, and $z_0 \notin S_0$. From the proof of Lemma 3 we know that H can be written as

$$H \equiv (f-g)\left\{\frac{f'}{f-1}\frac{g'-a'}{g-a} - \frac{g'}{g-1}\frac{f'-a'}{f-a} - \frac{f'}{f}\frac{g'-a'}{g-a}\right.$$

$$\left. + \frac{g'}{g}\frac{f'-a'}{f-a} + \frac{f'}{f}\frac{g'}{g-1} - \frac{g'}{g}\frac{f'}{f-1}\right\}.$$

Clearly, z_0 is a pole of

$$\left[\frac{f'}{f-1}\frac{g'-a'}{g-a} - \frac{g'}{g-1}\frac{f'-a'}{f-a} - \frac{f'}{f}\frac{g'-a'}{g-a} + \frac{g'}{g}\frac{f'-a'}{f-a} + \frac{f'}{f}\frac{g'}{g-1} - \frac{g'}{g}\frac{f'}{f-1}\right]$$

with multiplicity at most 1, but z_0 is a zero of $(f-g)$ with multiplicity at least 2. So we have $H(z_0) = 0$.

Similarly, if z_1 (resp. z_a) is a zero of $f-1$ (resp. $f-a$) with multiplicity $p\,(\geq 2)$, a zero of $g-1$ (resp. $g-a$) with multiplicity $q\,(\geq 2)$, and $z_1 \notin S_0$ (resp. $z_a \notin S_0$), from Lemma 3 we also have $H(z_1) = 0$ (resp. $H(z_a) = 0$).

From the discussion above, we have

$$\sum_{j=1}^{4} \overline{N}^*(r,a_j) \leq N\left(r,\frac{1}{H}\right)$$

$$\leq T(r,H) + O(1) = S(r,f)$$

Case 2 — $H \equiv 0$.

Let $\overline{N}_D^*(r, a_j)$ (resp. $\overline{N}_E^*(r, a_j)$) denote the counting function which count the multiple common zeros of $f - a_j$ and $g - a_j$ with different multiplicities (resp. the same multiplicity), each point counted only once. Obviously, we have

$$\overline{N}^*(r, a_j) = \overline{N}_D^*(r, a_j) + \overline{N}_E^*(r, a_j).$$

We may assume that $\overline{N}^*(r, 0) \neq S(r, f)$, noting Lemma 3 and Lemma 4, other cases can be considered similarly. Notice that

$$\overline{N}^*(r, 0) = \overline{N}_D^*(r, 0) + \overline{N}_E^*(r, 0),$$

so we only need to consider the following two subcases.

Subcase 1 — $\overline{N}_D^*(r, 0) \neq S(r, f)$.

From $H \equiv 0$ we have

$$\frac{L(f, 0, 1)\, L(g, 1, a)}{f(g-a)} = \frac{L(g, 0, 1)\, L(f, 1, a)}{g(f-a)}.$$

That is
$$\frac{f'}{f}\left[a' - (a-1)\frac{g'-a'}{g-a}\right] = \frac{g'}{g}\left[a' - (a-1)\frac{f'-a'}{f-a}\right]. \qquad \dots (*)$$

Let z_0 be a common zero of f and g with multiplicity p and q respectively, $z_0 \notin S_0$ and $p > q \geq 2$. Then the left-hand side of (*) has a simple pole at z_0 with residue $p\dfrac{a'(z_0)}{a(z_0)}$. On the other hand, the right-hand side of (*) also has a simple pole at z_0, however the residue is $q\dfrac{a'(z_0)}{a(z_0)}$, a contradiction.

Subcase 2 — $\overline{N}_E^*(r, 0) \neq S(r, f)$.

From (*) we have

$$\frac{f'}{f}\,\frac{a'(g-a) - (a-1)(g'-a')}{g-a} = \frac{g'}{g}\,\frac{a'(f-a) - (a-1)(f'-a')}{f-a},$$

So we get

$$\frac{f-a}{f} f' [a'(g-a) - (a-1)(g'-a')] = \frac{g-a}{g} g' \ [a'(f-a) - (a-1)(f'-a')].$$

hence we obtain

$$\left(f' - a\frac{f'}{f}\right)[a'(g-a) - (a-1)(g'-a')]$$
$$= \left(g' - a\frac{g'}{g}\right)[a'(f-a) - (a-1)(f'-a')]. \qquad \dots (**)$$

Let z_0 be a common zero of f and g with the same multiplicity p, $z_0 \notin S_0$ and $p \geq 2$. Without loss of generality, we assume that $p = 2$, other cases $p > 2$ can be considered similarly. Then f and g can be written in a neighbourhood of z_0.

$$f = a_1 (z - z_0)^2 + a_2 (z - z_0)^3 + a_3 (z - z_0)^4 + \dots, a_1 \neq 0, \quad \dots (1)$$

$$g = b_1 (z - z_0)^2 + b_2 (z - z_0)^3 + b_3 (z - z_0)^4 + \dots, b_1 \neq 0, \quad \dots (2)$$

$$f' = 2a_1 (z - z_0) + 3a_2 (z - z_0)^2 + 4a_3 (z - z_0)^3 + \dots, \quad \dots (3)$$

$$g' = 2b_1 (z - z_0) + 3b_2 (z - z_0)^2 + 4b_3 (z - z_0)^3 + \dots, \quad \dots (4)$$

$$\frac{f'}{f} = \frac{2}{z - z_0} + \frac{a_2}{a_1} + \left(\frac{2a_3}{a_1} - \frac{a_2^2}{a_1^2} \right)(z - z_0) + \dots, \quad \dots (5)$$

$$\frac{g'}{g} = \frac{2}{z - z_0} + \frac{b_2}{b_1} + \left(\frac{2b_3}{b_1} - \frac{b_2^2}{b_1^2} \right)(z - z_0) + \dots . \quad \dots (6)$$

Now we write $a(z)$ in a neighbourhood of z_0.

$$a = c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + c_3 (z - z_0)^3 + \dots, c_0 \neq 0, 1, \quad \dots (7)$$

$$a' = c_1 + 2c_2 (z - z_0) + 3c_3 (z - z_0)^2 + \dots . \quad \dots (8)$$

Substituting (1)–(8) to (**), we get

$$(2a_1 (z - z_0) + 3a_2 (z - z_0)^2 + \dots - (c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + c_3 (z - z_0)^3 + \dots)$$

$$\left(\frac{2}{z - z_0} + \frac{a_2}{a_1} + \left(\frac{2a_3}{a_1} - \frac{a_2^2}{a_1^2} \right)(z - z_0) + \dots) \right) [(c_1 + 2c_2 (z - z_0) + 3c_3 (z - z_0)^2 + \dots)$$

$$(- c_0 - c_1 (z - z_0) - c_2 (z - z_0)^2 - c_3 (z - z_0)^3 - \dots$$

$$+ b_1 (z - z_0)^2 + b_2 (z - z_0)^3 + b_3 (z - z_0)^4 + \dots)$$

$$- (c_0 - 1 + c_1 (z - z_0) + c_2 (z - z_0)^2 + c_3 (z - z_0)^3 + \dots)$$

$$(- c_1 - 2c_2 (z - z_0) - 3c_3 (z - z_0)^2 - \dots$$

$$+ 2b_1 (z - z_0 + 3b_2 (z - z_0)^2 + 4b_3 (z - z_0)^3 + \dots)]$$

$$= (2b_1 (z - z_0) + 3b_2 (z - z_0)^2 + \dots - (c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + c_3 (z - z_0)^3 + \dots)$$

$$\left(\frac{2}{z-z_0}+\frac{b_2}{b_1}+\left(\frac{2b_3}{b_1}-\frac{b_2^2}{b_1^2}\right)(z-z_0)+\dots\right)[(c_1+2c_2(z-z_0)+3c_3(z-z_0)^2+\dots)$$

$$(-c_0-c_1(z-z_0)-c_2(z-z_0)^2-c_3(z-z_0)^3-\dots$$

$$+a_1(z-z_0)^2+a_2(z-z_0)^3+a_3(z-z_0)^4+\dots)$$

$$-(c_0-1)+c_1(z-z_0)+c_2(z-z_0)^2+c_3(z-z_0)^3+\dots)$$

$$(-c_1-2c_2(z-z_0)-3c_3(z-z_0)^2-\dots$$

$$+\ 2a_1(z-z_0)+3a_2(z-z_0)^2+4a_3(z-z_0)^3+\dots]\dots \quad (***)$$

Comparing the coefficients of both sidesa (* * *) we obtain $a_j=b_j, j=1,2,3,$ ..... Hence we have $f(z)\equiv g(z)$., a contradiction. This completes the proof of Theorem 1.

From Theorem 1, we can get the Corollary easily.

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