Indian J. pure appl. Math., **34**(7) : 1035-1044, July 2003

TWO THEOREMS ON INVARIANT SUBMANIFOLDS OF PRODUCT RIEMANNIAN MANIFOLD

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(*Received* 25 *July* 20002; *accepted* 22 *March* 2002)

In this paper we discuss the distributions of a Riemannian almost-product structure in product Riemannian manifold. We show that an invariant submanifold of a product Riemannian manifold has a Riemannian almost-product structure, and investigate properties of integral manifold of distributions of Riemannian almost-product structure, i.e., pseudo-umbilical submanifold and curvature-invariant submanifold.

Key Words : Product Riemannian Manifold; Curvature-Invariant Submanifold; Pseudo-Umbilical Submanifold; Vertical and Horizontal Distributions.

1. INTRODUCTION

The geometry of a submanifold (M, g) of a locally product Riemannian manifold $(M_1 \times \overline{M}_2, \overline{g}_1 \otimes \overline{g}_2)$ was widely studied by many geometers. In particularly, Matsumoto K. has proved that (M, g) is a locally product Riemannian manifold of a Riemannian manifold (M_1, g_1) and a Riemannian manifold (M_2, g_2), if it is an invariant submanifold of $(M_1 \times \overline{M}_2, \overline{g}_1 \times \overline{g}_2)$ (see[4]). After then Senlin, Xu and Yilong, Ni, have updated theorem of Matsumoto and proved that $M_1 \subset \overline{M}_1$ and $M_2 \subset \overline{M}_2$. Moreover, they have proved that (M_1, g_1) and (M_2, g_2) are pseudo-umbilical Submanifolds of $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively, if (M, g) is a pseudo-umbilical submanifold of $(\overline{M}, \overline{g}) = (\overline{M}_1 \times \overline{M}_2, \overline{g}_1 \otimes \overline{g}_2)$. They have also demonstrated that M is isometric to the production of its two totally geodesic submanifolds (M_1, g_1) and (M_2, g_2) which are submanifolds of $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively (see[6]).

Furthermore, semi-invariant submanifolds of a locally product Riemannian manifolds were studied by Bejancu (see[1]).

Riemannian and Warped product structures are widely used be in geometry to construct new examples of Semi-Riemannian manifolds with interesting curvature properties. Product Riemannian metric tensor have also been useful in the study of several aspect of submanifold theory.

Product Riemannian manifold $(\overline{M}_1 \times \overline{M}_2, \overline{g})$ have been characterized by $\overline{M}_1$ and $\overline{M}_2$ are totally geodesic submanifolds of $(\overline{M}_1 \times \overline{M}_2, \overline{g})$.

In this paper, we have proved two theorems which have suplemented results of Matsumoto, Senlin, and Yilong. In the first theorem, we have proved that M is a curvature-invariant submanifold of product Riemannian manifold $(\overline{M}_1 \times M_1, \overline{g})$ if and only if M_1 and M_2 are curvature-invariant submanifolds of $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively. In the second theorem, we have demonstrated that M is a pseudo-umbilical submanifold of $(\overline{M}_1 \times \overline{M}_2, \overline{g})$, if M_1 and M_2 are the pseudo-umbilical submanifolds of $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively.

2. PRELIMINARIES

In this study, we use the same notations and terminologies as in[3], we use the fundamental Theorem 2.1 which was given in[6] by Senlin-Yilong.

We recall some necessary facts and formulas from the theory of submanifolds. For an arbitrary submanifold M of a Riemannian manifold $\overline{M}$, Gauss and Weingarten formulas are given by

$$\nabla_X Y = \nabla'_X Y + h(X, Y)$$

and

$$\nabla_X \xi = -A_\xi X + \nabla_X^{\perp} \xi,$$

respectively, where ∇ and ∇' are Levi-Civita connections on the Riemannian manifolds $\overline{M}$ and its submanifold M, respectively, X, Y are vector fields tangent to M, ξ is a normal vector field to M, $h : TM \times TM \to TM^{\perp}$ is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal vector bundle $TM^{\perp}$, and A_ξ is the shape operator of the second quadratic form for a normal vector ξ. From the formulas above it follows that

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle,$$

where the symbol $\langle\, ,\rangle$ means the Riemannian metric of $\overline{M}$.

Symbols $\overline{R}$ and R are the Riemannian curvature tensors of Levi-Civita connections ∇ and ∇' on $\overline{M}$ and M, respectively. The key role the theory of submanifolds is played by the following equations of Gauss, Codazzi and Ricci :

$$\langle \overline{R}(X, Y) Z, W \rangle = \langle R(X, Y) Z, W \rangle + \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle$$

$$(\overline{R}(X, Y) Z)^{\perp} = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z),$$

$$\langle \overline{R}(X, Y) \zeta, \eta \rangle = \langle \overline{R}^{\perp}(X, Y) \xi, \eta \rangle - \langle | A_\xi, A_\eta | X, Y \rangle,$$

where the vectors X, Y, Z, W are tangent to M, the vectors ξ and η are orthogonal to M, and the derivative ∇h is defined by

$$(\nabla_X h)(Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla'_X Y, Z) - h(\nabla'_X Z, Y).$$

M is called a curvature-invariant submanifold if it has

$$(\overline{R}(X, Y) Z)^{\perp} = 0,$$

which is equivalent to

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z)$$

for all $X, Y, Z \in TM$.

If the ambient space $\overline{M}$ is a space of constant sectional curvature c, the equations of Gauss, Codazzi and Ricci reduce to

$$\overline{K}(X, Y, Z, W) = c \{ \overline{g}(X, W) \overline{g}(Y, Z) - \overline{g}(X, Z) \overline{g}(Y, W) \}$$

$$+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z),$$

and $$K\perp (X, Y, \xi, \eta) = g([A_\xi, A_\eta](X, Y),$$

respectively, where $\overline{K}$ is Riemannian-Christoffel curvature tensor of $\overline{M}^2$.

Definition 2.1 — For a submanifold $M \subseteq \overline{M}$ the mean-curvature vector field H is defined by the formula

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_i\}$ is a local orthonormal basis in TM. If a submanifold $M \subseteq \overline{M}$ having one of the conditions

$$h = 0, \langle h(X, Y) H \rangle = \lambda \langle X, Y \rangle, H = 0, \lambda \in C^\infty(M, R),$$

then it is called totally geodesic, pseudo-umbilical and minimal, respectively[2].

Let M be a totally umbilical submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$. M has also constant sectional curvature $c + \| H \|^2$, if $\overline{M}$ has constant sectional curvature c. See[2]

A Riemannian manifold M with Riemannian metric $\langle , \rangle$ and tensor field S of type (1, 1) on TM such that

$$S^2 = I, \langle X, Y \rangle = \langle SX, SY \rangle$$

is called a Riemannian almost-product structure.

Since $S^2 = I$, the eigenspace of S for the eigenvalues +1 and −1 are called the vertical distribution and horizontal distributions of M, respectively.

Now let $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$ be Riemannian manifolds, with the dimension n_1 and n_2, respectively. $\overline{M}_1 \times \overline{M}_2$ is the Riemannian product of $\overline{M}_1$ and $\overline{M}_2$. We denote by P and Q the projection mappings of $T(\overline{M}_1 \times \overline{M}_2)$ to $T\overline{M}_1$ and $T\overline{M}_2$, respectively. Then we have

$$P + Q = I, P^2 = P, Q^2 = Q, PQ = QP = 0.$$

If we put $J = P - Q$, then we can easily see that $J^2 = I$, where I denotes the identity transformation of $T(\overline{M}_1 \times \overline{M}_2)$.

We define a Riemannian metric of $\overline{M}_1 \times \overline{M}_2$ by

$$\overline{g}(X, Y) = \overline{g}_1(PX, PY) + \overline{g}_2(QX, QY)$$

for all $X, Y \in T(\overline{M}_1 \times \overline{M}_2)$. It follows that

$$\overline{g}(JX, Y) = \overline{g}(X, JY)$$

or $$\overline{g}(JX, JX) = \overline{g}(X, Y)$$

From the definition of $\bar{g}$, we can get the $\bar{M}_1$ and $\bar{M}_2$ are all totally geodesic submanifolds of product Riemannian manifold $\bar{M}_1 \times \bar{M}_2$[6].

By $\bar{\nabla}$ we denote the Levi-Civita connection on $\bar{M}_1 \times \bar{M}_2$, it can be seen in[6] that

$$\bar{\nabla} P = \bar{\nabla} Q = \bar{\nabla} J = 0.$$

Now let M be a submanifold $\bar{M}_1 \times \bar{M}_2$ and B the differential of the imbedding i of M into $\bar{M}_1 \times \bar{M}_2$, i.e., $B = i_*$. Let X be a tangent vector field of M. Then we can write JBX in the following way

$$JBX = (JBX)^T + (JBX)^{\perp} = BfX + \xi,$$

where f is a tensor field type (1, 1) on TM, and $\xi = (JBX)^{\perp} \in TM^{\perp}$.

Definition 2.2 — M is said to be an invariant submanifold of product Riemannian manifold if $JBX = (JBX)^T = BfX$ always holds[6]

In the rest of this article, we assume that the submanifold M is invariant.

Now, let M be invariant submanifold of $\bar{M}_1 \times \bar{M}_2$, and $\langle\, ,\rangle$ is the Riemannian metric of submanifold M induced by $\bar{g}$. Then

$$\langle X, Y \rangle = i^* \bar{g}(X, Y) = \bar{g}(BX, BY) = \bar{g}(JBX, JBY)$$

$$= \bar{g}(BfX, BfY) = \langle fX, fY \rangle,$$

and $$BX = J^2 BX = JBfX = Bf^2 X \Rightarrow X = f^2 X \Rightarrow f^2 = I.$$

Thus f is a Riemannian almost-product structure on M. By T_1 and T_2 we denote vertical and horizontal distributions of M, respectively. Then we have

$$T_1 = \{ X \in TM \mid fX = X \}$$

and $$T_2 = \{ X \in TM \mid fX = -X \}.$$

From $f^2 = I$ we know that the eigenvalues of f is ± 1. So $TM = T_1 \oplus T_2$.

Gauss and Weingarten formulas are given by

$$\bar{\nabla}_{BX} BY = B \nabla_X Y + h(X, Y)$$

$$\bar{\nabla}_{BX} \xi = -B A_{\xi} X + \nabla_X^{\perp} \xi$$

for all $X, Y \in TM$, $\xi \in TM^{\perp}$, where h is the second fundamental form of M, A_{ξ} is the Weingarten endomorphism associated with ξ, and $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are the connections on $T(\bar{M}_1 \times \bar{M}_2)$ TM and $TM^{\perp}$, respectively.

Using the Gauss formula and $\bar{\nabla} J = 0$, we get $\nabla f = 0$. Thus distributions T_1 and T_2 are parallel and involutive[6].

Theorem 2.1 — *(Senlin X and Yilong N, [6]) Let M be an invariant submanifold of a product Riemannian manifold $(\bar{M}_1 \times \bar{M}_2, \bar{g})$. By M_1 and M_2 we denote the integral manifolds of distributions the vertical and the horizontal, respectively. Then M_1 and M_2 are totally geodesic* *submanifolds of M. Moreover* M_1 *and* M_2 *are submanifolds Riemannian manifolds of* $(\overline{M}_1, \overline{g}_1)$ *and* $(\overline{M}_2, \overline{g}_2)$, *respectively.*

3. Curvature-Invariant Submanifolds of a Product Riemannian Manifold

In this section, we will prove the main theorems of this article

***Theorem* 3.1** —*Let M be an invariant submanifold of a product Riemannian manifold* $(\overline{M}_1 \times \overline{M}_2, \overline{g})$. *By* M_1 *and* M_2 *we denote integral manifolds of vertical and horizontal distributions of M, respectively. Then M is a curvature-invariant submanifold of product Riemannian manifold* $(\overline{M}_1 \times \overline{M}_2, \overline{g})$ *if and only if* M_1 *and* M_2 *are the curvature-invariant submanifolds of Riemannian manifolds* $(\overline{M}_1, \overline{g}_1)$ *and* $(\overline{M}_2, \overline{g}_2)$, *respectively.*

PROOF : By $\overline{R}$ we denote the Riemannian curvature tensor of product Riemannian manifold $\overline{M}_1 \times \overline{M}_2$. Then we can easily see that $\overline{R}(JX, JY) = \overline{R}(X, Y)$ for all $X, Y \in T(\overline{M}_1 \times \overline{M}_2)$. So we have

$$\overline{R}(X_1, X_2) = 0$$

and
$$\overline{R}(X, Y) = \overline{R}(X_1, Y_1) + \overline{R}(X_2, Y_2)$$

for all $X_1, Y_1 \in T\overline{M}_1, X_2, Y_2 \in T\overline{M}_2$ and $X = X_1 + X_2, Y = Y_1 + Y_2 \in T(\overline{M}_1 \times \overline{M}_2)$. From $P^2 = P$, $Q^2 = Q$, $P + Q = I$, $\overline{\nabla} P = \overline{\nabla} Q = 0$ and using the properties of the Riemannian curvature tensor $\overline{R}$ and first Bianchi, it can be proved by direct calculations that

$$\begin{aligned}
\overline{R}(X, Y)Z &= \overline{R}(X_1, Y_1)Z + \overline{R}(X_2, Y_2)Z \\
&= P\overline{R}(X_1, Y_1)Z + Q\overline{R}\ (X_1, Y_1)Z + P\overline{R}(X_2, Y_2)Z = Q\overline{R}(X_2, Y_2)Z \\
&= P\overline{R}(X_1, Y_1)Z + Q\overline{R}(X_2, Y_2)Z \\
&\quad - P(\overline{R}(ZX_1)Y_1 + \overline{R}(Y_1, Z)X_1) + Q(\overline{R}(Z, X_1)Y_1 - \overline{R}(Y_1, Z)X_1) \\
&= \overline{R}(X_1, Y_1)PZ + \overline{R}(X_2, Y_2)QZ \\
&= \overline{R}_1(X_{1,} Y_1)Z_1 + \overline{R}_2(X_2, Y_2)Z_2,
\end{aligned}$$

where $\overline{R}_1$ and $\overline{R}_2$ are the Riemannian curvature tensors of the Riemannian manifolds $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively, $X_1, Y_1, Z_1 \in TM_1, X_2, Y_2, Z_2 \in T\overline{M}_2$ and $X = X_1 + X_2, Y = Y_1 + Y_2, Z = Z_1 + Z_2$.

Now we consider the following imbeddings :

$$i : \overline{M}_1 \to \overline{M}_1 \times \overline{M}_2, i_* = B$$

$$i_1 : M_1 \to \overline{M}_1, i_{1*} = B_1$$

$$i_2 : M_2 - \overline{M}_{2,} i_{2*} = B_2.$$

Thus we have

$$\overline{R}(BX, BY)BZ = \overline{R}_1(B_1X_1, B_1Y_1)B_1Z_1 + \overline{R}_2(B_2X_2, B_2Y_2)B_2Z_2 \qquad \text{... (III.1)}$$

for all $X_1, Y_1, Z_1 \in TM_1, X_2, Y_2, Z_2 \in TM_2$ and $X = X_1 + X_2,\ Y = Y_1 + Y_2, Z = Z_1 + Z_2$.

Using the Gauss and Weingarten formulas, by direct calculations, it is well known that

$$\bar{R}(BX, BY)\, BZ = BR'(X, Y)\, Z - BA_{h(Y,Z)} X + BA_{h(X,Z)} Y$$

$$+ (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z),$$

where R', h and A are the Riemannian curvature tensor, the second fundamental form and the shape operator of invariant submanifold M of $\bar{M}_1 \times \bar{M}_2$, respectively.

By R_i', A_i and h_i, we denote Riemannian curvature tensor, the shape operator and the second fundamental form of submanifolds M_i of $\bar{M}_i$ for $i = 1, 2$, respectively. Then we have

$$\bar{R}_1(B_1 X_1, B_1 Y_1)\, B_1 Z_1 = B_1 R_1'(X_1, Y_1)\, Z_1 - B_1 A_{1\, h_1(X_1, Z_1)} Y_1$$

$$+ (\nabla_{X_1} h_1)(Y_1, Z_1) - (\nabla_{Y_1} h_1)(X_1, Z_1). \qquad \text{... (III.2)}$$

In the same way, we get

$$\bar{R}_2(B_2 X_2, B_2 Y_2)\, B_2 Z_2 = B_2 R_2'(X_2, Y_2)\, Z_2 - B_2 A_{2h_2(X_2, Z_2)} Y_2$$

$$+ (\nabla_{X_2} h_2)(Y_2, Z_2) - (\nabla_{Y_2} h_2)(X_2, Z_2) \qquad \text{... (III.3)}$$

where $X_i, Y_i, Z_i \in TM_i$ and R_i' is the Riemannian curvature tensor of the Levi-Civita connection ∇'^i on M_i for $i = 1, 2$.

From the eqs. (III.1), (III.3) and (III.3) we have

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) + (\nabla_{X_1} h_1)(Y_1, Z_1) - (\nabla_{Y_1} h_1)(X_1, Z_1)$$

$$+ (\nabla_{X_2} h_2)(Y_2, Z_2) - (\nabla_{Y_2} h_2)(X_2, Z_2).$$

If M is the curvature-invariant submanifold of $\bar{M} \times \bar{M}_2$, then

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = 0,$$

that is, $$(\nabla_{X_1} h_1)(Y_1, Z_1) - (\nabla_{Y_1} h_1)(X_1, Z_1) + (\nabla_{X_2} h_2)(Y_2, Z_2) - (\nabla_{Y_2} h_2)(X_2, Z_2) = 0,$$

which implies that

$$(\nabla_{X_1} h_1)(Y_1, Z_1) - (\nabla_{Y_1} h_1)(X_1, Z_1) = 0$$

and $$(\nabla_{X_2} h_2)(Y_2, Z_2) - (\nabla_{Y_2} h_2)(X_2, Z_2) = 0.$$

Thus M_1 and M_2 are the curvature-invariant submanifolds of Riemannian manifolds $\bar{M}_1$ and $\bar{M}_2$, respectively.

The converse is obvious. □

***Theorem* 3.2** —*Let M be an invariant submanifold of a product Riemannnian manifold $(\overline{M}_1 \times \overline{M}_2, \overline{g})$. By M_1 and M_2 we denote integral manifolds of vertical and horizontal distributions of M. Then M is the pseudo-umbilical submanifold of a product Riemannian manifold $(\overline{M}_1 \times \overline{M}_2, \overline{g})$ if and only if M_1 and M_2 and pseudo-umbilical submanifolds of Riemannian manifolds $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively.*

PROOF : We only have to prove the sufficient condition. Because the necessary condition was proved in[6] by Senlin and Yilong.

Now let M_1 and M_2 be pseudo-umbilical submanifolds of a Riemannian manifolds $(\overline{M}_1, \overline{g}_1)$ and $(\overline{M}_2, \overline{g}_2)$, respectively. Then there exist functions $\lambda \in C^{\infty}(M_1, R)$ and $\mu \in C^{\infty}(M_2, R)$ such that

$$\overline{g}_1 (h_1 (X_1, Y_1), H_1) = \lambda \langle X_1, Y_1 \rangle \qquad \text{... (III.4)}$$

and

$$\overline{g} (h_2 (X_2, Y_2), H_2) = \mu \langle X_2, Y_2 \rangle, \qquad \text{... (III.5)}$$

where the symbol $\langle , \rangle$ means the Riemannian metric of its submaifold $M \subset \overline{M}_1 \times \overline{M}_2$, H_1 and H_2 denote the mean curvature vector of M_1 and M_2 in $\overline{M}_1$ and $\overline{M}_2$, h_1 and h_2 denote the second fundamental forms of M_1 and M_2 in $\overline{M}_1$ and $\overline{M}_2$,respectively.

We choose a local field of adapted orthonormal basis

$$\left\{ e_1, \ldots, e_a, e_{a+1}, \ldots, e_{n_1}, e^1, \ldots, e^b, e^{b+1}, \ldots, e^{n_2} \right\}$$

of $T(\overline{M}_1 \times \overline{M}_2)$ so that when restricted locally to orthonormal basis over M, $\{e_1, \ldots, e_a\}$ are tangent vectors to M_1, $\{e^1, \ldots, e^b\}$ are tangent vectors to M_2, and $\left\{ e_{a+1}, \ldots, e_{n_1}, e^{b+1} \ldots, e^{n_2} \right\}$ are normal vectors to M.

From (III.4), for orthonormal basis $\{e_1, \ldots, e_a\}$ of TM_1 we write

$$\overline{g}\left(\sum_{i=1}^{a} h_1 (e_i, e_i), H_1 \right) = \lambda \sum_{i=1}^{a} \langle e_i, e_i \rangle$$

$$\overline{g}_1 (aH_1, H_1) = \lambda a,$$

i.e., $\lambda = \overline{g}_1 (H_1, H_1)$. Similarly, from (III.5), for orthonormal basis $\left\{ e^1, \ldots, e^b \right\}$ of TM_2, we have

$$\overline{g}_2 \left(\sum_{j=1}^{b} h_2 (e^j, e^j), H_2 \right) = \mu \sum_{j=1}^{b} \langle e^j, e^j \rangle$$

$$\overline{g}_2 (bH_2, H_2) = \mu b,$$

i.e.,

$$\mu = \overline{g}_2 (H_2, H_2).$$

H is the mean curvature vector of M in $\overline{M}_1 \times \overline{M}_2$, it was proved in [6] that

$$H = \frac{a}{m} H_1 + \frac{b}{m} H_2, \ a \| H_1 \|^2 = b \| H_2 \|^2$$

and $$h(X, Y) = h_1(X_1, Y_1) + h_2(X_2, Y_2)$$

where $m = \dim M$, $a = \dim M_1$, $b = \dim M_2, X_i, Y_i \in TM_i$ for $i = 1, 2$ and $X = X_1 + X_2, Y = Y_1 + Y_2$. Thus making use of projection mappings

$$P : T(\overline{M}_1 \times \overline{M}_2) \to T\overline{M}_1$$

and $$Q : T(\overline{M}_1 \times \overline{M}_2) \to T\overline{M}_2,$$

we get $$Ph(X, Y) = h_1(X_1, Y_1), \ Qh(X, Y) = h_2(X_2, Y_2)$$

and $$PH = \frac{a}{m} H_1, \ QH = \frac{b}{m} H_2$$

So we have

$$\overline{g}_1(Ph, PH) = \frac{m}{a} \overline{g}_1(PH, PH) \langle PX, PY \rangle \quad \text{... (III.6)}$$

and $$\overline{g}_2(Qh, QH) = \frac{m}{b} \overline{g}_2(QH, QH) \langle QX, QY \rangle. \quad \text{... (III.7)}$$

If we add equations (III.6) and (III.7), then we get

$$\overline{g}_1(Ph(X, Y), PH) + \overline{g}_2(Qh(X, Y), QH) = \frac{m}{a} \overline{g}_1(PH, PH) \langle PX, PY \rangle$$

$$+ \frac{m}{b} \overline{g}_1(QH, QH) \langle QX, QY \rangle.$$

On the other hand, we derive

$$\overline{g}(H, H) = \overline{g}\left(\frac{a}{m} H_1 + \frac{b}{m} H_2, \frac{a}{m} H_1 + \frac{b}{m} H_2 \right)$$

$$= \frac{a^2}{m^2} \overline{g}(H_1, H_1) + \frac{b^2}{m^2} \overline{g}(H_2, H_2)$$

$$= \frac{a^2}{m^2} \overline{g}(H_1, H_1) + \frac{a, b}{m^2} \overline{g}(H_1, H_1)$$

$$= \frac{a}{m^2}(a + b) \overline{g}(H_1, H_1) = \frac{a}{m} \overline{g}_1(H_1, H_2).$$

In the same way, we have $\overline{g}(H, H) = \frac{b}{m} \overline{g}_2(H_2, H_2)$. Thus from the eqs. (II.6) and (II.7) we have

$$\overline{g}_1(Ph(X, Y), PH) + \overline{g}_2(Qh(X, Y), QH) = \frac{m}{a} \cdot \frac{a^2}{m^2} \overline{g}_1(H_1, H_1) \langle PX, PY \rangle$$

$$+ \frac{m}{b} \cdot \frac{b^2}{m^2} \bar{g}_2 (H_2, H_2) \langle QX, QY \rangle$$

$$\bar{g} (h (X, Y), H) = \frac{a}{m} \bar{g}_1 (H_1, H_1) \langle PX, PY \rangle$$

$$+ \frac{b}{m} \bar{g}_2 (H_2, H_2) \langle QX\, QY \rangle$$

$$= \bar{g} (H, H) \langle X_1, Y_1 \rangle + \bar{g} (H, H) \langle X_2, Y_2 \rangle$$

$$= \bar{g} (H, H) \langle X, Y \rangle,$$

which implies that M is a pseudo-umbilical submanifold of product Riemannian manifold $\overline{M}_1 \times \overline{M}_2$. This completes the proof of the theorem. □

Proposition 3.3 — Let M be an invariant submanifold of a product Riemannian manifold $(\overline{M}_1 \times \overline{M}_2, \bar{g})$. If $\overline{M}_1 \times \overline{M}_2$ has constant sectional curvature and M is a totally umbilical submanifold of $\overline{M}_1 \times \overline{M}_2$, then M has to be totally geodesic submanifold of $\overline{M}_1 \times \overline{M}_2$.

PROOF : Since $\overline{M}_1$ and $\overline{M}_2$ are totally geodesic submaniolds of $\overline{M}_1 \times \overline{M}_2$, $\overline{M}_1$ and $\overline{M}_2$ have also constant sectional curvature, if $\overline{M}_1 \times \overline{M}_2$ has constant sectional curvature. Moreover, we derive $\overline{M}_1 \times \overline{M}_2$,, $\overline{M}_1$ and $\overline{M}_2$ have constant sectional curvature the same c. We have

$$h (X, Y) = g (X, Y) H \qquad \text{... (III.8)}$$

for all $X, Y \in TM$ because M is totally umbilical submanifold of a $(\overline{M}_1 \times \overline{M}_2, \bar{g})$, In this case, M has also constant sectional curvature $c + \| H \|^2$.

We take $X = X_1, Y = Y_1 \in TM_1$ in eq. (III.8) and using the projection mapping $P : T(\overline{M}_1 \times M_2) \to T\overline{M}_1$ we obtain

$$h_1 (X_1, Y_1) = g (X_1, Y_1) \frac{a}{m} H_1.$$

In the same way, we take $X = X_2, Y = Y_2 \in TM_2$ in eq. (III.8) and using the projection mapping $Q : T(\overline{M}_1 \times M_2) \to T\overline{M}_2$ we get

$$h_2 (X_2, Y_2) = g (X_2, Y_2) \frac{b}{m} H_2.$$

Thus we derive M_1 and M_2 have constant sectional curvatures $c + \frac{a^2}{m^2} \| H_1 \|^2$ and $c + \frac{b^2}{m^2} \| H_2 \|^2$, respectively. Since M_1 and M_2 are totally geodesic submanifolds of M, M, M_1 and M_2 have the same constant sectional curvatures, that is,

$$c + \| H \|^2 + c + \frac{a^2}{m^2} \| H_1 \|^2 = c + \frac{b^2}{m^2} \| H_2 \|^2.$$

It follows that $H=H_1=H_2=0$. Since M is a total umbilical submanifold of $\overline{M}_1 \times \overline{M}_2, M$ is a totally geodesic submanifold of $\overline{M}_1 \times \overline{M}_2$. This completes the proof of the proposition.

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