Indian J. pure appl. Math., **34**(7) : 1045-1059, July 2003
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EFFECT OF PERTURBATIONS IN THE CORIOLIS AND CENTERIFUGAL FORCES ON THE LOCATIONS AND STABILITY OF THE EQUILIBRIUM POINTS IN ROBE'S CIRCULAR PROBLEM WITH DENSITY PARAMETER HAVING ARBITRARY VALUE

P. P. HALLAN* AND NEELAM RANA**

**Department of Mathematics, Zakir Husain College, (University of Delhi), Jawahar Lal Nehru Marg, New Delhi* 110 002, *India*
***Department of Mathematics, University of Delhi, Delhi, India*

(*Received* 10 *June* 2002; *after final revision* 16 *December* 2002; *accepted* 17 *March* 2003)

When small perturbations ε and ε' are given to the coriolis and centrifugal forces respectively, the positions and linear stability of the equilibrium points in the Robe's (*Celest. Mech.* 16 (1977) 343) circular three body problem have been studied, when the density parameter K has arbitrary value. It is proved that on the line joining the center of the first primary and the second primary, there is an equilibrium point for all values of K except $K = 1 + 2\mu$, where μ is the mass parameter. And on this line, there is a second equilibrium point only when $K > 1$ provided it lies inside the first primary. When $K < 0$ and $K + \mu > 0$, there are two equilibrium points in the z - z plane, equidistant and forming triangles with the line joining the center of the first primary and the second primary provided they lie inside the first primary. If $K = (1 + \varepsilon')(1 - \mu)$, there are infinite number of equilibrium points in the x-y plane lying on a circle with center as the second primary and having radius $\left(1 - \frac{1}{3}\varepsilon'\right)$ provided the points lie inside the first primary. It is seen that the change in the coriolis force does not affect the locations of the equilibrium points. Further, it is proved that the circular and triangular equilibrium points are always unstable. And the equilibrium points collinear with the center of the first primary and the second primary are stable only when K and μ satisfy certain conditions.

Key Words : Restricted Three-Body Problem; Equilibrium Points; Coriolis Forces; Centrifugal Forces; Stability.

1. INTRODUCTION

In 1977, Robe discussed a new kind of restricted three body problem (Fig. 1) in which one of the primaries m_1^* is a rigid spherical shell m_1 of radius R filled with a homogeneous incompressible fluid of density ρ_1. The other primary is a mass point m_2 outside the shell and the third body m_3 is a small solid sphere of density ρ_3, inside the shell, with the assumption that the mass and radius of m_3 are infinitesimal. He has shown the existence of an equilibrium point with m_3 at the center of the shell, while m_2 describes a Keplerian orbit around it. Further he has discussed the linear stability of the equilibrium point in two cases. In the first case, m_2 describes a circular orbit around the shell and in the second case, the orbit is elliptic, but the shell is empty (i.e.$\rho_1 = 0$) or the densities ρ_1 and ρ_3 to be equal. In both instances, the domain of stability has been investigated for the whole range of parameters in the problem.

Szebehely (1967) considered the effect of small perturbation in the coriolis force on the stability of equilibrium points in the classical restricted problem, keeping the centrifugal force constant and proved that collinear points remain unstable and the range of stability of the triangular points increases or decreases depending upon whether the change ε in the coriolis force is positive or negative thereby establishing that coriolis force is a stabilizing force. In the same problem, Bhatnagar and Hallan (1978) studied the effect of the changes ε and ε' in the coriolis and centrifugal forces respectively on the stability of equilibrium points and concluded that the collinear points remain unstable and for the triangular points, the range of stability increases or decreases depending upon whether the point $(\varepsilon, \varepsilon')$ lies in one or the other of the two parts in which the line $36\varepsilon - 19\varepsilon' = 0$ divides the $(\varepsilon, \varepsilon')$ plane.

Shrivastava and Garain (1991) studied the effect of small perturbations in the coriolis and centrifugal forces on the location of the equilibrium point in the Robe's circular restricted three body problem when the densities ρ_1 and ρ_3 are equal and evaluated the concomitant shift in the location of the equilibrium point. Plastino and Plastino (1995) have discussed the linear stability of the equilibrium point in the Robe's problem by taking the shape of the fluid body as Roche's ellipsoid (Chandrasekhar, 1987). They have shown that the effect of the buoyancy force might be thought as being equivalent to a perturbation of the coriolis force and that the equilibrium point is always stable, when the density of the third body is greater than that of the surrounding medium. Giordano, Plastino and Plastino (1997) have studied the effect of drag force on the stability of equilibrium point, both in Robe's problem (1977) and the problem studied by Plastino and Plastino (1995). They have shown that in the Robe's problem, four regions of stability out of five are changed to instability, where as in the second case, regions of stability remain unchanged. Hallan and Rana (2001a) have studied the existence of all the equilibrium points, their location and linear stability in the Robe's problem. They have shown that the center of the first primary is always an equilibrium point, whatever be the values of the density parameter K, eccentricity parameter e and mass parameter μ, occurring in the problem. Apart from this equilibrium point (i) There is one more equilibrium point collinear with the center of the shell and the second primary only when $K > 1$, (ii) There are infinite number of equilibrium points lying on a circle of radius one and having center as the second primary only when $K = 1 - \mu$, (iii) There are two equilibirum points forming triangles with the center of the shell and the second primary only when $K < 0$ and $K + \mu > 0$ provided in all the cases (i), (ii) and (iii) the equilibrium points lie inside the shell. Further, they have shown that the circular points and the triangular points are always unstable. And the equilibrium point existing for $K > 1$ and lying on the line joining the center of the shell and second primary is stable only when

$$16\mu(K-1)^2 < [\mu + \sqrt{\mu(-4+4K+\mu)}]^3.$$

The results of stability of the equilibrium point, the center of the first primary, are same as that of those given by Robe (1977). Again, Hallan and Rana (2001b) studied the effect of small perturbations ε and ε' in the coriolis and centrifugal forces respectively on the location and linear stability of the equilibrium point in the Robe's circular problem with densities ρ_1 and ρ_3 equal that is K has the value zero. They proved that there is only one equilibrium point which lies on the line joining the center of the shell and the second primary and lies to the right or left of center of the shell according as ε' is positive or negative. Further, they have shown that the equilibrium point is stable for $\mu_c < \mu \le 1$ and unstable for $0 < \mu < \mu_c$ where $\mu_c = \frac{8}{9} + \frac{2}{3}\left(\frac{43}{25}\varepsilon' - \frac{10}{3}\varepsilon\right)$. Since the above paper (2001b) deals only with the case $K = 0$, so in the present study, we consider the effect of perturbations in the coriolis and centrifugal forces on the locations and linear stability of the equilibrium points in the Robe's circular problem when the parameter K can have any value.

2. Equations of Motion and Equilibrium Points

In the uniformly rotating dimensionless coordinate system Oxyz, the origin O being at the center of mass of the two primaries, Ox pointing towards m_2 and Oxy being the orbital plane of the finite bodies, the equations of motion of m_3 in Robe's circular problem are

$$\ddot{x} - 2\dot{y} - x = \frac{\partial V}{\partial x},$$

$$\ddot{y} + 2\dot{x} - y = \frac{\partial V}{\partial y},$$

$$\ddot{z} = \frac{\partial V}{\partial z},$$

where
$$V = \frac{\mu}{[(1-\mu-x)^2+y^2+z^2]^{1/2}} - \frac{K}{2}[(x+\mu)^2+y^2+z^2],$$

$$\mu = \frac{m_2}{m_1^* + m_2}, \quad 0<\mu<1, \; K = \frac{4}{3}\pi \frac{\rho_1 a^3}{m_1^* + m_2}\left(1 - \frac{\rho_1}{\rho_3}\right).$$

Here, a is the radius of the circular orbit which m_2 describes around m_1^*. Coordinates of the center of the shell and the second primary are respectively $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$.

Let the perturbations in the coriolis and centrifugal forces be expressed with the help of the parameters α and β, each having unperturbed value equal to unity. Then, the equations of motion become

$$\ddot{x} - 2\alpha\dot{y} = \frac{\partial \Omega}{\partial x},$$

$$\ddot{y} + 2\alpha\dot{x} = \frac{\partial \Omega}{\partial y},$$

$$\ddot{z} = \frac{\partial \Omega}{\partial z}, \qquad \text{... (1)}$$

where
$$\Omega = \frac{\beta}{2}(x^2+y^2) + \frac{\mu}{[(1-\mu-x)^2+y^2+z^2]^{1/2}} - \frac{K}{2}[(x+\mu)^2+y^2+z^2].$$

Take α and β as

$$\alpha = 1+\varepsilon, \quad |\varepsilon| \ll 1,$$

$$\beta = 1+\varepsilon', \quad |\varepsilon'| \ll 1.$$

Equilibrium points are given by the equations

$$\Omega_x = \beta x + \frac{\mu(1-\mu-x)}{[(1-\mu-x)^2+y^2+z^2]^{3/2}} - K(x+\mu) = 0,$$

$$\Omega_y = \beta y - \frac{\mu y}{[(1-\mu-x)^2+y^2+z^2]^{3/2}} - Ky = 0,$$

$$\Omega_z = \frac{-\mu z}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - Kz = 0.$$

That is, the positions of the equilibrium points are the solutions of the following systems of eqs. (2) to (5) :

(*i*) When $y = 0$, $z = 0$, we have

$$\beta x + \frac{\mu(1-\mu-x)}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - K(x+\mu) = 0. \quad \text{... (2)}$$

(*ii*) When $z = 0$, we have

$$\beta x + \frac{\mu(1-\mu-x)}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - K(x+\mu) = 0, \quad \text{... (3)}$$

$$\beta - \frac{\mu}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - K = 0.$$

(*iii*) When $y = 0$, we have

$$\beta x + \frac{\mu(1-\mu-x)}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - K(x+\mu) = 0$$

$$\frac{\mu}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - K(x+\mu) = 0, \quad \text{... (4)}$$

(*iv*) $$\beta x + \frac{\mu(1-\mu-x)}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - K(x+\mu) = 0$$

$$\beta - \frac{\mu}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} - K = 0$$

$$\frac{\mu}{[(1-\mu-x)^2 + y^2 + z^2]^{3/2}} + K = 0.$$

System of eq. (5) has no solution.

(*a*) Linear equilibrium points

In this case $y = 0$, $z = 0$ and we have to consider eq. (2). Then the x-coordinates of the equilibrium points are the roots of the equation

$$\beta x + \frac{\mu}{(1-\mu-x)^2} - K(x+\mu) = 0. \quad \text{... (6)}$$

(*i*) When $\beta = 1$, eq. (6) has at most two roots (Hallan and Rana, 2001a) namely

$x_{11} = -\mu$, for any value of K,

$$x_{21} = \frac{-2 + \mu + 2k - 2K\mu - \sqrt{\mu(-4 + 4K + \mu)}}{2(K-1)},$$

only for $K > 1$, provided $(x_{21}+\mu)^2 < R^2$, where

$$R=\left[\frac{3\,(m_1^* - m_1)}{4\,\pi\,\rho_1}\right]^{1/3}$$

The third root of the eq. (6) is greater than $1-\mu$, so it is rejected.

(ii) When $\beta = 1+\varepsilon'$, eq. (6) has the roots

$$x_1 = x_{11} + p'$$

$$x_2 = x_{21} + q', \quad (K > 1)$$

where p', q' are very small.

Substituting the values of x_1 and x_2 in eq. (6) and rejecting second and higher powers of p', q' and ε' we get

$$p' = p\,\varepsilon', \quad q' = q\,\varepsilon',$$

where

$$p = \frac{\mu}{1+2\,\mu-K}, \quad (1+2\,\mu-K \neq 0)$$

$$q = \frac{[2-\mu-2K+2\,K\,\mu+\sqrt{\mu\,(-4+4K+\mu)}]\,[\mu+\sqrt{\mu\,(-4+4K+\mu)}]^3}{2\,(K-1)^2\,[16\,\mu\,(K-1)^2-(\mu+\sqrt{(-4+4K+\mu)})^3]}.$$

Consider the equilibrium point $(x_1, 0, 0)$. When there are no perturbations, this point is $(x_{11}, 0, 0)$ which is the center of the shell. The equilibrium point $(x_1, 0, 0)$ lies to the right or left of the center of the shell according as ε' is negative or positive, when the point (μ, K) lies in the region I (Fig. 2). And when point (μ, K) lies in the region II (Fig. 2), this equilibrium point lies to the right or left of the center of the shell according as ε' is positive or negative.

Consider the equilibrium point $(x_2, 0, 0)$. When there are no perturbations, this point is $(x_{21}, 0, 0)$. The equilibrium point $(x_2, 0, 0)$ lies to the right or left of the equilibrium point $(x_{21}, 0, 0)$ according as q and ε' are of same signs or of opposite signs.

(b) Equilibrium points in (x, y) plane

In this case $z = 0$ and we have to consider the system of eq. (3). Then the x, y coordinates of the equilibrium points are the roots of the equations

$$\beta\,x + \frac{\mu\,(1-\mu-x)}{[(1-\mu-x)^2+y^2]^{3/2}} - K\,(x+\mu) = 0, \qquad \dots (6a)$$

$$\beta - \frac{\mu}{[(1-\mu-x)^2+y^2]^{3/2}} - K = 0. \qquad \dots (6b)$$

(i) When $\beta = 1$, this system has a solution if $K = 1-\mu$. Then we have from eqs. (6a) and (6b),

$$(1-\mu-x)^2+y^2=1. \qquad \text{... (6c)}$$

This represents a circle of radius 1 with centre as the second primary. This is true for all points lying on the straight line AB in Fig. 3. Only those points lying within the spherical shell are equilibrium points.

(*ii*) When $\beta=1+\varepsilon'$, eqs. (6*a*) and (6*b*) have a solution only when $K=1-\mu+\varepsilon'(1-\mu)$. With this value of K, the solution of eqs. (6*a*) and (6*b*) are the points lying on the circle $(1-\mu-x)^2+y^2=\left(1-\frac{1}{3}\varepsilon'\right)^2$, $z=0$. As before only those points which lie inside the spherical shell are equilibrium points, which are infinite in number. Thus, after perturbation ε' in the centrifugal force, line AB becomes $A'B$ or $A''B$ (Fig. 3) according as ε' is positive or negative. Also, radius of the circle increases or decreases according as the value of ε' is negative or positive.

(*c*) Equilibrium points in (x, z) plane

In this case $y=0$ and we have to consider the system of eq. (4). Then the coordinates of the equilibrium points are the roots of the equations

$$\beta x+\frac{\mu(1-\mu-x)}{(1-\mu-x)^2+z^2]^{3/2}}-K(x+\mu)=0 \qquad \text{... (6d)}$$

$$\frac{\mu}{[(1-\mu-x)^2+z^2]^{3/2}}+K=0$$

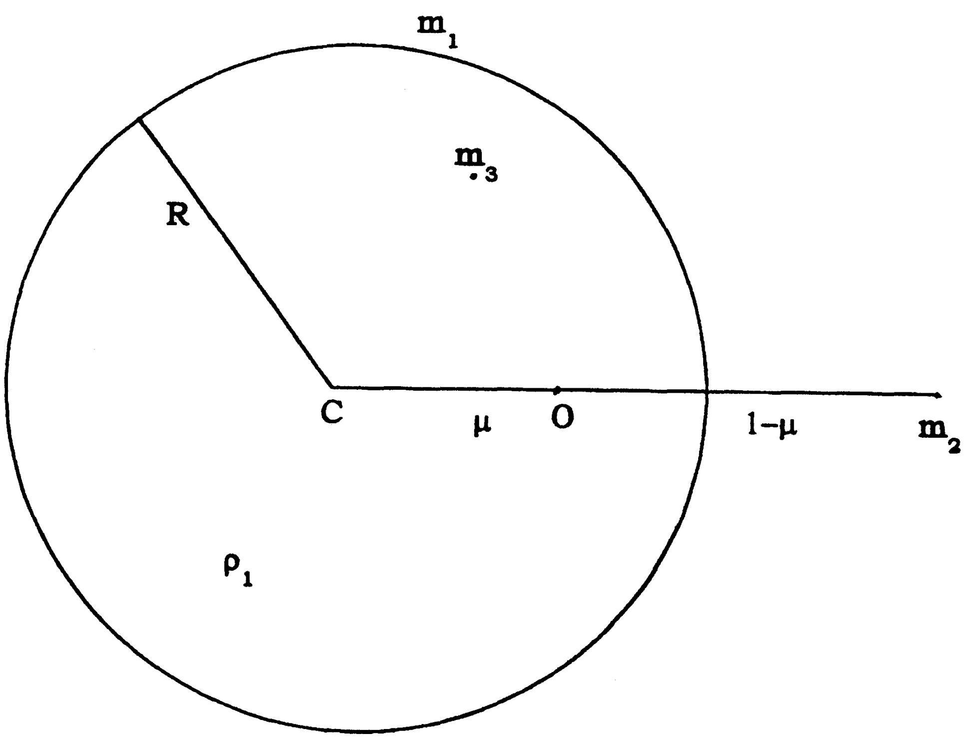

FIG. 1. Robe's restricted three body problem.

or

$$\frac{1}{[(1-\mu-x)^2+z^2]^{3/2}}=-\frac{K}{\mu} \quad \text{... (6e)}$$

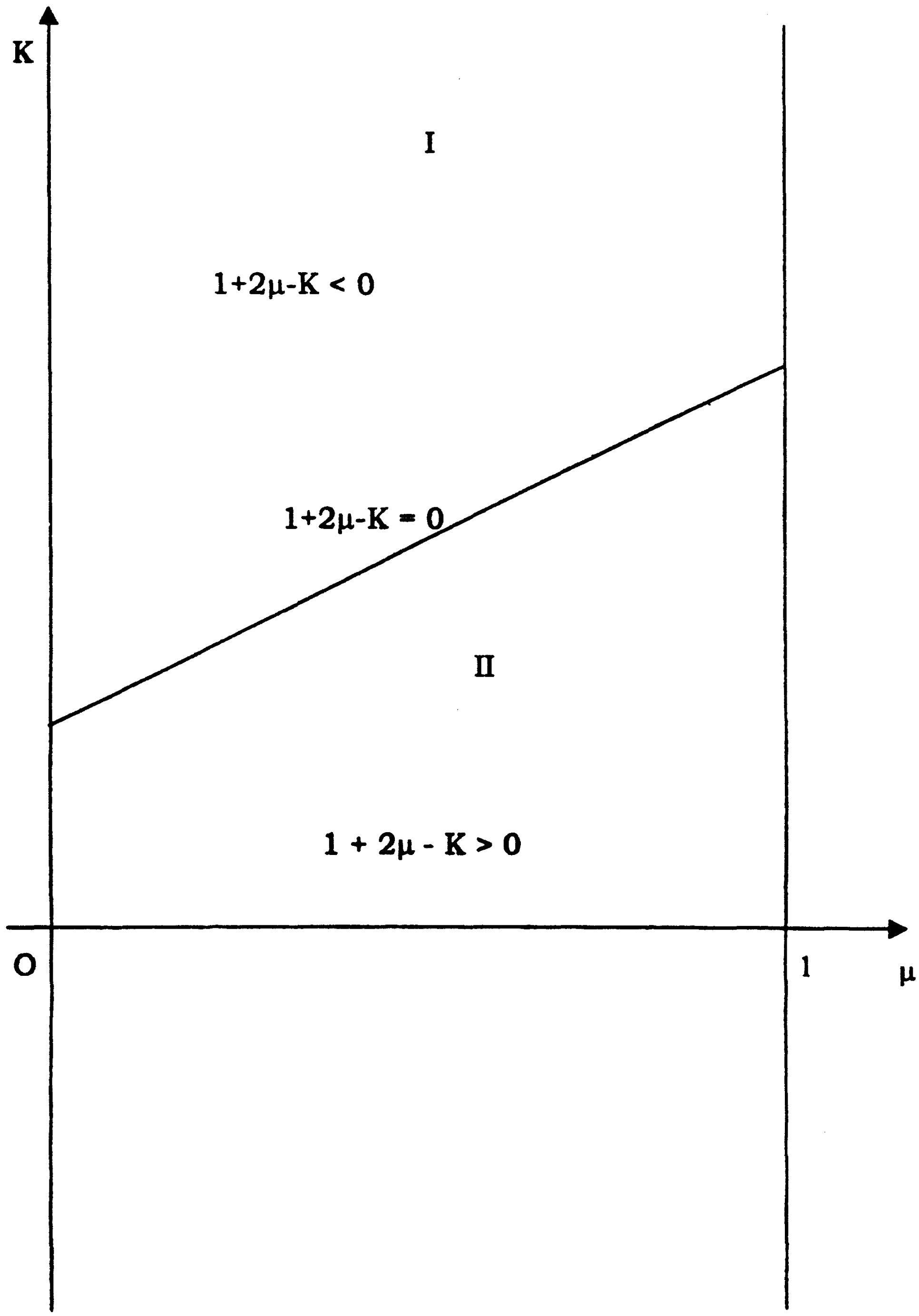

FIG. 2. Regions showing the location of the equilibrium point (x_1, 0, 0) in the parameter plane $\mu - K$.

(*i*) When $\beta = 1$, substitution of eq. (6*e*) in eq. (6*d*) gives the value of x as $x_{31} = K$. Eq. (6*e*), then gives the values of z as $\pm\, z_{31} = \sqrt{b_1^2 - a_1^2}$

and
$$a_1 = (1 - \mu - K),\ b_1 = \left(-\frac{\mu}{K}\right)^{1/3},$$

provided $K < 0$ and $K + \mu > 0$ (Hallan and Rana (2001a))

These equilibrium points $(x_{31}, 0, \pm z_{31})$ make triangles with the centre of the shell and the second primary and they will lie within the spherical shell only if $(K+\mu)^2 + z_{31}^2 < R^2$.

(*ii*) When $\beta = 1 + \varepsilon'$, eq. (6*d*) gives the value of x as

$$x_3 = \frac{K}{1+\varepsilon'} = K(1-\varepsilon') = x_{31}(1-\varepsilon').$$

Then substitution in eq. (6*e*) gives the value of $\pm z_3$ whereas

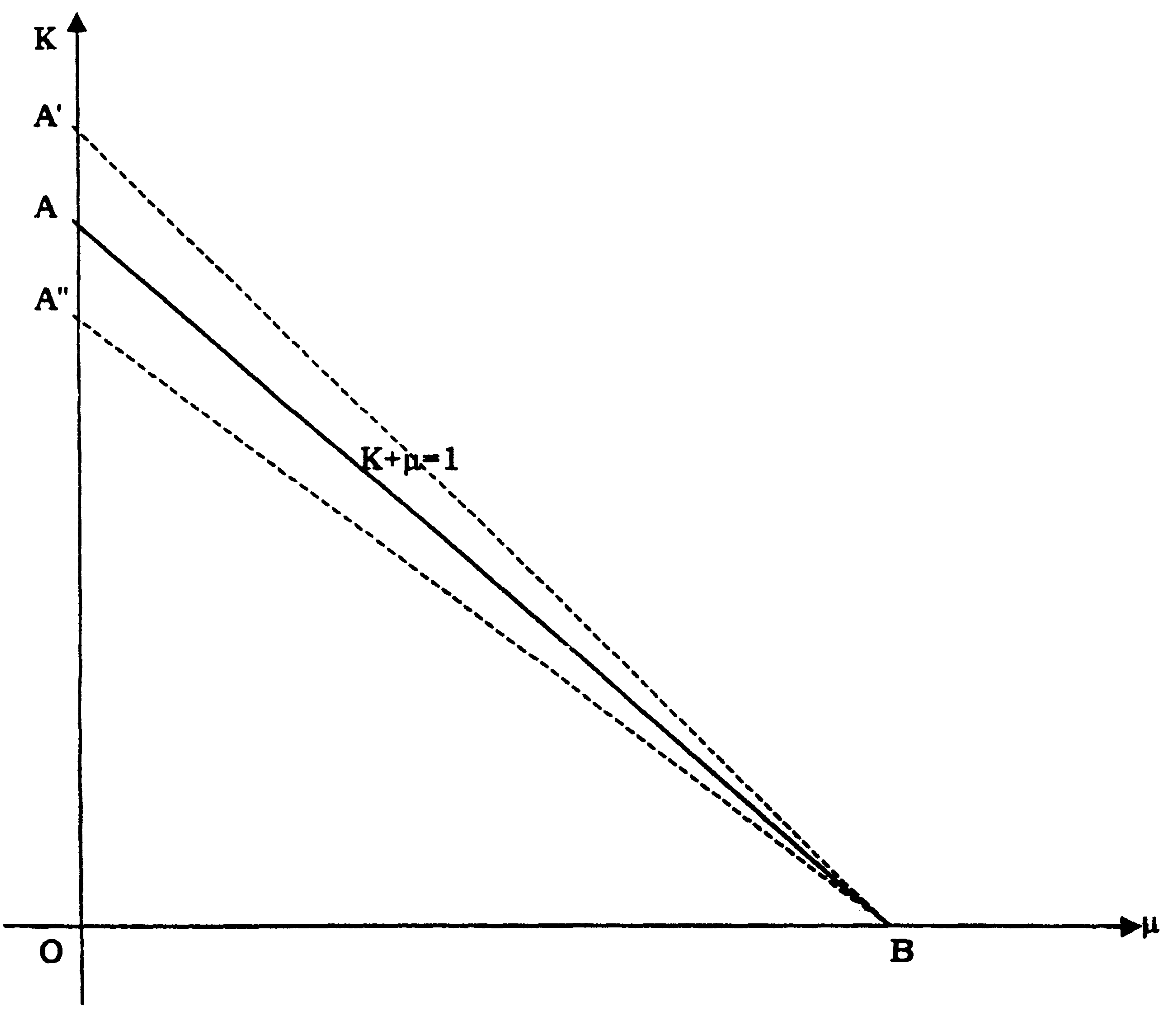

FIG. 3. Line showing the locations of the circular equilibrium points in the parameter plane $\mu - K$.

$$z_3^2 = \left(-\frac{\mu}{K}\right)^{2/3} - (1-\mu-K+K\varepsilon')^2$$

$$= b_1^2 - (a_1 + K\varepsilon')^2$$

$$= z_{31}^2\left(1 - \frac{2K\varepsilon' a_1}{z_{31}^2}\right)$$

So, $$z_3 = z_{31} - \frac{K\varepsilon' a_1}{z_{31}}.$$

These equilibrium points $(x_3, 0, \pm z_3)$ making triangles with the center of the shell and the second primary move towards or away from the center of the shell according as the perturbation ε' is negative or positive. And these points are equidistant from the line joining the center of shell and the second primary.

(*d*) When $K = 0$ and $\beta = 1$ we have only one equilibrium point at $x = 1-\mu$ i.e. the centre of the shell. When $\beta = 1+\varepsilon'$, we again have only one equilibrium point at $\left(-\mu + \frac{\mu\varepsilon'}{1+2\mu}, 0, 0\right)$ which confirms the result of Hallan and Rana (2001b).

From the above discussion, it is concluded that the locations of the equilibrium points are not affected by the perturbation in the coriolis force.

3. STABILITY

Let the third body of infinitesimal mass be displaced to $(x_0+\xi, y_0+\eta, z_0+\zeta)$, where ξ, η, ζ are very small and (x_0, y_0, z_0) is an equilibrium point. The equations of motion in the linearized form become

$$\ddot{\xi} - 2(1+\varepsilon)\dot{\eta} = \Omega_{xx}^0\,\xi + \Omega_{xy}^0\,\eta + \Omega_{xz}^0\,\zeta$$

$$\ddot{\eta} + 2(1+\varepsilon)\dot{\xi} = \Omega_{yx}^0\,\xi + \Omega_{yy}^0\,\eta + \Omega_{yz}^0\,\zeta,$$

$$\ddot{\zeta} = \Omega_{zx}^0\,\xi + \Omega_{zy}^0\,\eta + \Omega_{zz}^0\,\zeta, \qquad \ldots (7)$$

where the superscript 0 denotes that the second derivatives are to be evaluated at the point (x_0, y_0, z_0). We have

$$\Omega_{xx} = \beta - K - \frac{\mu[-2(1-\mu-x)^2 + y^2 + z^2]}{[(1-\mu-x)^2 + y^2 + z^2]^{5/2}}$$

$$\Omega_{yy} = \beta - K - \frac{\mu[(1-\mu-x)^2 - 2y^2 + z^2]}{(1-\mu-x)^2 + y^2 + z^2]^{5/2}}$$

$$\Omega_{zz} = -K - \frac{\mu[(1-\mu-x)^2 + y^2 - 2z^2]}{(1-\mu-x)^2 + y^2 + z^2]^{5/2}}$$

$$\Omega_{xy} = \frac{-3\,\mu\,(1-\mu-x)\,y}{[(1-\mu-x)^2+y^2+z^2]^{5/2}} = \Omega_{yx}$$

$$\Omega_{xz} = \frac{-3\,\mu\,(1-\mu-x)\,z}{[(1-\mu-x)^2+y^2+z^2]^{5/2}} = \Omega_{zx}$$

$$\Omega_{yz} = \frac{3\,\mu\,y\,z}{[(1-\mu-x)^2+y^2+z^2]^{5/2}} = \Omega_{zy}$$

(*i*) Stability of the points collinear with the center of the shell and the second primary

(*a*) Equilibrium Point $(x_1,\ 0,\ 0)$. At this point

$$\Omega^0_{xx} = 1 - K + 2\,\mu + \varepsilon'\,(1 + 6\,\mu\,A_0),$$

$$\Omega^0_{yy} = 1 - K - \mu + \varepsilon'\,(1 - 3\,\mu\,A_0),$$

$$\Omega^0_{zz} = -\,\mu - K - 3\,\mu\,A_0\,\varepsilon',$$

$$\Omega^0_{xy} = \Omega^0_{yz} = \Omega^0_{zx} = 0,$$

where $$A_0 = \frac{\mu}{1 + 2\,\mu - K}.$$

Then, the variational equations (7) become

$$\ddot{\xi} - 2\,(1+\varepsilon)\,\dot{\eta} = [1 - K + 2\,\mu + \varepsilon'\,(1 + 6\,\mu\,A_0)]\,\xi, \qquad \dots (8)$$

$$\ddot{\eta} + 2\,(1+\varepsilon)\,\dot{\xi} = [1 - K - \mu + \varepsilon'\,(1 - 3\,\mu\,A_0)]\,\eta, \qquad \dots (9)$$

$$\ddot{\zeta} = -\,[\mu + K + 3\,\mu\,A_0\,\varepsilon']\,\zeta. \qquad \dots (10)$$

The eq. (10) shows that the motion parallel to z-axis is stable when $K + \mu + 3\,\mu\,A_0\,\varepsilon' > 0$, that is when m_3 is denser than the fluid $(\rho_3 > \rho_1)$. The characteristic equation corresponding to the system of eq. (8) and (9) is

$$\lambda^4 + p_1\,\lambda^2 + q_1 = 0,$$

where $$p_1 = 2 + 2K - \mu + 8\,\varepsilon - \varepsilon'\,(2 + 3\,\mu\,A_0),$$

$$q_1 = (\mu + K - 1)\,(K - 1 - 2\,\mu) + \varepsilon'\,(2 - 2K + \mu + 3\,\mu\,A_0 - 3\,\mu\,K\,A_0 - 12\,\mu^2\,A_0).$$

This equation is quadratic in λ^2. its roots are

$$\lambda^2 = \frac{-\,p_1 \pm \sqrt{D_1}}{2},$$

where $$D_1 = p_1^2 - 4\,q_1$$

$$= 16K - 8\mu + 9\mu^2 + 16\varepsilon(2 + 2K - \mu) - 2\varepsilon'(8 + 12\mu A_0 - 27\mu^2 A_0).$$

The equilibrium point $(x_1, 0, 0)$ is stable if

$$p_1 > 0, q_1 > 0 \text{ and } D_1 > 0.$$

If we take $K = 0$, then the equilibrium point $(x_1, 0, 0)$ becomes $\left(-\mu + \frac{\mu\varepsilon'}{1+2\mu}, 0, 0\right)$ and this point is stable when $D > 0$, where

$$D = \mu(9\mu - 8) + 16\varepsilon(2 - \mu) - 2\varepsilon'(8 + 12\mu A_0' - 27\mu A_0'),$$

$$A_0' = \frac{\mu}{1 + 2\mu}.$$

As shown in Hallan and Rana (2001b), $D > 0$ for $\mu_c < \mu < 1$, where

$$\mu_c = \frac{8}{9} + \frac{2}{3}\left(\frac{43}{25}\varepsilon' - \frac{10}{3}\varepsilon\right).$$

Hence, when $K = 0$, the equilibrium point $\left(-\mu + \frac{\mu\varepsilon'}{1+2\mu}, 0, 0\right)$ is stable when $\mu_c < \mu < 1$. This result is in confirmity with that given by Hallan and Rana (2001 b).

When there are no perturbations, the equilibrium point $(x_1, 0, 0)$ becomes $(-\mu, 0, 0)$ and it is stable if $p_2 > 0, q_2 > 0, D_2 > 0$, where

$$p_2 = 2 - \mu + 2K,$$

$$q_2 = (\mu + K - 1)(K - 1 - 2\mu),$$

$$D_2 = 16K - 8\mu + 9\mu^2,$$

which confirms the results of Robe (1977).

(*b*) Equilibrium Point $(x_2, 0, 0)$. At this point

$$\Omega^0_{xx} = 1 - K + 4A_1 + \varepsilon'\alpha_1,$$

$$\Omega^0_{yy} = 1 - K - 2A_1 + \varepsilon'\alpha_2,$$

$$\Omega^0_{zz} = -K - 2A_1 - \varepsilon'\alpha_2,$$

$$\Omega^0_{xy} = \Omega^0_{yz} = \Omega^0_{zx} = 0,$$

where $$A_1 = \frac{4\mu(K-1)^3}{[\mu + \sqrt{\mu(-4 + 4K + \mu)}]^3},$$

$$\alpha_1 = 1 + \frac{24\,(K-1)\,A_1}{[\mu + \sqrt{\mu\,(-4+4K+\mu)}]^q},$$

$$\alpha_2 = 1 - \frac{12\,(K-1)\,A_1}{[\mu + \sqrt{\mu\,(-4+4K+\mu)}]^q}.$$

The variational equations (7) become

$$\ddot{\xi} - (1+\varepsilon\,\dot{\eta} = [1 - K + 4A_1 + \varepsilon'\,\alpha_1]\,\xi,$$

$$\ddot{\eta} + 2\,(1+\varepsilon)\,\dot{\xi} = [1 - K - 2A_1 + \varepsilon'\,\alpha_2]\,\eta,$$

$$\ddot{\zeta} = -\,[K + 2A_1 + \varepsilon'\,\alpha_2]\,\zeta.$$

Last equation shows that the motion parallel to z-axis is always stable. The characteristic equation corresponding to the first two equations is

$$\lambda^4 + [2 + 2K - 2A_1 + 8\,\varepsilon - \varepsilon'\,(\alpha_1 + \alpha_2)\,]\,\lambda^2 + (1 - K + 4A_1 + \varepsilon'\,\alpha_1)$$

$$(1 - K - 2A_1 + \varepsilon'\,\alpha_2) = 0.$$

This equation is quadratic in λ^2. Its roots are

$$\lambda^2_{1,2} = \frac{-2 - 2K + 2A_1 - 8\,\varepsilon + \varepsilon'\,(\alpha_1 + \alpha_2) \pm \sqrt{\Delta}}{2},$$

where $$\Delta = 4\,[4\,(K - A_1) + 9A_1^2 + 8\,\varepsilon\,(1 + K - A_1) - \varepsilon'\,\{\alpha_1\,(2 - 3A_1) + \alpha_2\,(2 + 3A_1)\}]$$

The equlibirum point is stable if roots are real and negative that is, $\Delta > 0$, $\lambda_1^2 + \lambda_2^2 < 0$ and $\lambda_1^2\,\lambda_2^2 > 0$. Hence, equilibrium point $(x_2, 0, 0)$ is stable when

$$16\,\mu\,(K-1)^3 < (K - 1 - \varepsilon'\,\alpha_1)\,(\mu + \sqrt{\mu\,(-4+4K+\mu)})^3.$$

(*ii*) Stability of triangular points

Triangular points are $(x_3, 0, \pm z_3)$. At these points

$$\Omega^0_{xx} = 1 - \frac{3a_1^2\,K}{b_1^2} + \varepsilon'\left(1 - \frac{6a_1\,K^2}{b_1^2}\right),$$

$$\Omega^0_{yy} = 1 + \varepsilon',$$

$$\Omega^0_{zz} = \frac{3K}{b_1^2}\,(a_1^2 - b_1^2) + \frac{6a_1\,K^2\,\varepsilon'}{b_1^2}$$

$$\Omega^0_{xz} = \pm\left[\frac{3a_1 K\sqrt{b_1^2 - a_1^2}}{b_1^2} + \frac{3K^2 \varepsilon'}{b_1^2}\left(\sqrt{b_1^2 - a_1^2} - \frac{a_1^2}{\sqrt{b_1^2 - a_1^2}}\right)\right],$$

$$\Omega^0_{xy} = \Omega^0_{yz} = 0.$$

The characteristic equation becomes

$$f(\lambda) = 0,$$

where

$$f(\lambda) = \lambda^6 + (2 + 3K + 8\,\varepsilon - 2\,\varepsilon')\,\lambda^4 + \left[\frac{1}{b_1^2}(b_1^2 + 6b_1^2 K - 9a_1^2 K) + \frac{24\,K\,\varepsilon}{b_1^2}(b_1^2 - a_1^2) + \varepsilon'\left(1 - 6K + \frac{3a_1 K}{b_1^2}(a_1 - 6K)\right)\right]\lambda^2 + \frac{3K}{b_1^2}(b_1^2 - a_1^2)$$

$$+ \frac{6K\,\varepsilon'}{b_1^2}(b_1^2 - a_1^2 - 2a_1 K).$$

Now $f(\lambda) \to \infty$ as $\lambda \to \infty$

and

$$f(0) = \frac{3K}{b_1^2}(b_1^2 - a_1^2) + \frac{6K\,\varepsilon'}{b_1^2}(b_1^2 - a_1^2 - 2a_1 K) < 0.$$

Thus, there is at least one positive root of the characteristic equation and consequently, both the equilibrium points are unstable.

(*iii*) Stability of Circular Points

These points exist only for $K = 1 - \mu + \varepsilon'(1 - \mu)$ and lie on the circle $(1 - \mu - x)^2 + y^2 = \left(1 - \frac{1}{3}\varepsilon'\right)^2$, $z = 0$. Coordinates of any point on the circle are of the form

$$\left(1 - \mu - \left(1 - \frac{1}{3}\varepsilon'\right)\cos\phi, \left(1 - \frac{1}{3}\varepsilon'\right)\sin\phi, 0\right).$$

At this point

$$\Omega^0_{xx} = (1 + \varepsilon')\,3\,\mu\cos^2\phi,$$

$$\Omega^0_{yy} = (1 + \varepsilon')\,3\,\mu\sin^2\phi,$$

$$\Omega^0_{zz} = -(1 + \varepsilon'),$$

$$\Omega^0_{xy} = -(1 + \varepsilon')\,3\,\mu\cos\phi\sin\phi,$$

$$\Omega^0_{yz} = \Omega^0_{xz} = 0.$$

Then, the variational eqs. (7) become

$$\ddot{\xi} - 2(1+\varepsilon)\,\dot{\eta} = [(1+\varepsilon')\,3\,\mu\cos^2\phi]\,\xi + [-(1+\varepsilon')\,3\,\mu\cos\phi\sin\phi]\,\eta,$$

$$\ddot{\eta} + 2(1+\varepsilon)\,\dot{\xi} = [-(1+\varepsilon')\,3\,\mu\cos\phi\sin\phi]\,\xi + [3\,\mu\,(1+\varepsilon')\sin^2\phi]\,\eta,$$

$$\ddot{\zeta} = -(1+\varepsilon')\,\zeta.$$

Last equations shows that the motion parallel to z-axis is always stable. The characteristic equation corresponding to the other two equations is

$$\lambda^2\left\{\lambda^2 - (3\,\mu - 4 - 8\,\varepsilon + 3\,\mu\,\varepsilon')\right\} = 0.$$

Two of the roots of this equation are equal to zero, so, the solution of the equations of motion will contain secular terms. Hence, the equilibrium points are unstable.

4. Conclusion

When there are small perturbations ε and ε' in the coriolis and centrifugal forces respectively in the Robe's circular problem with density parameter K having arbitrary value, the number of equilibrium points are same as that in the problem with no perturbations (Hallan and Rana, 2001a), but positions of the equilibrium points have changed. The change in the coriolis force does not affect the positions of the equilibrium points. It is seen that there are two equilibrium points collinear with the line joining the center of the shell and the second primary, one of them is $(x_{11} + p\,\varepsilon', 0, 0)$, which exists for all values of K except $K = 1 + 2\,\mu$. This point lies to the right or left of the center of the shell according as ε' is negative (positive) or positive (negative) and the point (μ, K) lies in region I (II) (Fig. 2). The other equilibrium point is $(x_{21} + q\,\varepsilon', 0, 0)$, which exists only for $K > 1$ provided it lies within the spherical shell. This point shifts from its position $(x_{21}, 0, 0)$, when there are no perturbations and lies to the left or right of this point according as q and ε' are of opposite signs or of same signs. For $K < 0$ and $K + \mu > 0$, there are two equilibrium point in the x-z plane, namely $(x_3, 0, \pm z_3)$, equidistant and forming triangles with the line joining the center of the shell and the second primary provided these points lie within the spherical shell. After perturbations, these triangular points lie near or far from the center of the shell according as ε' is negative or positive. For $K = (1+\varepsilon')(1-\mu)$, there are infinite number of equilibrium points in the x-y plane, lying on a circle of radius $\left(1 - \frac{1}{3}\varepsilon'\right)$ and having center as the second primary, provided the points lie inside the spherical shell. Thus, after perturbation the radius of the circle increases or decreases according as ε' is negative or positive. The circular and triangular points are always unstable. The equilibrium point

(*i*) $(x_{11} + p\,\varepsilon', 0, 0)$ is stable if $p_1 > 0, q_1 > 0$ and $D_1 > 0$.

(*ii*) $(x_{21} + q\,\varepsilon', 0, 0)$ is stable if

$$16\,\mu\,(K-1)^3 < (K - 1 - \varepsilon'\,\alpha_1)\,(\mu + \sqrt{\mu\,(-4 + 4K + \mu)})^3.$$

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