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SEPARATION AXIOMS IN TERMS OF θ-CLOSURE AND δ-CLOSURE OPERATORS

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We introduce and characterize some bitopological separation axioms in terms of quasi-coincidence, quasi-neighborhood of Pu and Liu[16], θ-closure and δ-closure operators as initiated in[8] in fuzzy setting. Also, we introduce and study the concepts of $r\,\theta$-connectedness and $r\,\delta$-connectedness for fuzzy sets in fuzzy bitopological spaces in Sostak's sense as a weaker version of r-connectedness.

Key Words : FP Regular; FP Almost Regular; FP Uryshon; FP-T_2, $r\,\theta$-Connected; $r\,\delta$-connected; Fuzzy Bitopological Spaces

1. INTRODUCTION

Sostak[18] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology[1]. In[19, 20], Sostak gave some rules and showed how such an extension can be realized. Chattopadhyay *et al.*[2, 3] have redefined the same concept. In[5, 17], Ramadan and his colleagues gave a similar definition, namely smooth topological space for lattice $L = [0, 1]$. It has been developed in many directions[6, 12, 19, 20]. Höhle and Sostak[7] introduce the concept of an L-fuzzy topologies and establish their corresponding convergence theory for any lattice L. Kumar[13] introduced the notions of fuzzy δ-cluster point and fuzzy θ-cluster points in fuzzy bitopological spaces in the sense of Chang's fuzzy topology. Kim[8] introduced r-δ-cluster (r-θ-cluster) point and δ-closure (θ-closure) operators in fuzzy bitopological spaces in view of the definition of Sostak. It is a good extension of the notions of Kumar.

In this paper, we shall give various characterizations of FP regularity and FP almost regularity wih the help of quasi-neighborhood[16], θ-closure and δ-closure operators[8]. FP Urysohn and FP-T_2 axioms are defined and their relation with FP almost regularity is obtained. Also, we present and investigate the notions of $r\,\theta$-connectedness and $r\,\delta$-connectedness relative to a fuzzy bitopological space in view of the definition Sostak, and investigate the relationship with r-connectedness. We compare all these forms of connectedness and investigate their properties in almost regular, semi-regular and regular fuzzy bitopological spaces.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$, $I_0 = (0, 1]$ and I^X be a family of all fuzzy sets. For $\alpha \in I, \alpha(x) = \alpha \ \forall\ x \in X$. $P_t(X)$ is the family of all fuzzy points in X. For $\mu, \lambda \in I^X$, μ is called quasi-coincident with λ, denoted by $\mu\, q\, \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu\, \bar{q}\, \lambda$. The indices $i, j \in \{1, 2\}$ and $i \neq j$. Notation and notions not described in this paper are standard and usual.

Definition 1.1[18] — A mapping $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions :

(01) $\tau(\underline{0}) = \tau(\underline{1}) = 1$.

(02) $\tau(\lambda \wedge \mu) \geq \tau(\lambda) \wedge \tau(\mu)$ for all $\lambda, \mu \in I^X$.

(03) $\tau(\vee_{j \in J} \lambda_j) \geq \wedge_{j \in J} \tau(\lambda_j)$ for all $\lambda_j \in I^X$.

The pair (X, τ_1, τ_2) is called a fuzzy bitopological space (fbts, for short) where τ_1 and τ_2 are fuzzy topologies on X.

Definition 1.2[2] — A mapping $C : I^X \times I_\circ \to I^X$ is called a fuzzy closure operator on X if for each $\lambda, \mu \in I^X$ and $r, s \in I_\circ$, it satisfies the following conditions :

(1) $C(\underline{0}), r) = \underline{0}$.

(2) $\lambda \leq C(\lambda, r)$.

(3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$.

(4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$.

The pair (X, C) is clled a fuzzy closure space.

A fuzzy closue space (X, C) is topological if for $\lambda \in I^X$ and $r \in I_\circ$,

(5) $C(C(\lambda, r), r) = C(\lambda, r)$.

***Theorem* 1.3**[2] —*let (X, τ_1, τ_2) be an fbts. For each $r \in I_\circ, \lambda \in I^X$, we define an operator $C_{\tau_i} : I^X \times I_\circ \to I^X$ as follows :*

$$C_{\tau_i}(\lambda, r) = \wedge\left\{ \mu \in I^X \mid \lambda \leq \mu,\ \tau_i(1 - \mu) \geq r \right\}.$$

Then (X, C_{τ_i}) is a topological fuzzy closure space.

***Theorem* 1.4**[8] — *Let (X, τ_1, τ_2) be an fbts. For each $r \in I_\circ, \lambda \in I^X$, we define the operator $I_{\tau_i} : I^X \times I_\circ \to I^X$ as follows :*

$$I_{\tau_i}(\lambda, r) = \wedge\left\{ \mu \in I^X \mid \lambda > \mu,\ \tau_i(\mu) \geq r \right\}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_\circ$, the operator I_{τ_i} satisfies the following conditions :

(1) $I_{\tau_i}(\underline{1} - \lambda, r) = \underline{1} - C_{\tau_i}(\lambda, r)$.

(2) If $I_{\tau_i}(C_{\tau_j}(\lambda, r), r) = \lambda$, then $C_{\tau_i}(I_{\tau_j}(\underline{1} - \lambda, r), r) = \underline{1} - \lambda$.

(3) $I_{\tau_i}(\underline{1}, r) = \underline{1}$.

(4) $I_{\tau_i}(\lambda, r) \leq \lambda$.

(5) $I_{\tau_i}(\lambda, r) \wedge I_{\tau_i}(\mu, r) = I_{\tau_i}(\lambda \wedge \mu, r)$.

(6) $I_{\tau_i}(\lambda, r) \geq I_{\tau_i}(\lambda, s)$ if $r \leq s$.

(7) $I_{\tau_i}(I_{\tau_i}(\lambda, r) = I_{\tau_i}(\lambda, r)$.

Definition 1.5[4, 8] — Let (X, τ_1, τ_2) be an fbts, $\mu \in I^X, x_t \in P_t(X), r \in I_\circ$.

(1) μ is called a r-open Q_{τ_i}-neighborhood of x_t if $x_t \, q \, \mu$ with $\tau_i(\mu) \geq r$.

(2) μ is called a r-open $R_{\tau_j}^{\tau_i}$-neighborhood of x_t if $x_t \, q \, \mu$ with $\mu = I_{\tau_i}(C_{\tau_j}(\mu, r), r)$.

We denote

$$Q_{\tau_i}(x_t, r) = \left\{ \mu \in I^X \mid x_t \, q \, \mu, \, \tau_i(\mu) \geq r \right\},$$

$$R_{\tau_j}^{\tau_i}(x_t, r) = \left\{ \mu \in I^X \mid x_t \, q \, \mu = I_{\tau_i}(C_{\tau_j}(\mu, r), r) \right\}$$

Definition 1.6[8] — Let (X, τ_1, τ_2) be an fbts, $\lambda \in I^X, x_t \in P_t(X), r \in I_\circ$.

(1) x_t is called a $r - \tau_i$ cluster point of λ if for every $\mu \in Q_{\tau_i}(x_t, r)$, we have $\mu \, q \, \lambda$.

(2) x_t is called a $r - (\tau_i \, \tau_j)$ δ-cluster point of λ if for every $\mu \in R_{\tau_j}^{\tau_i}(x_t, r)$, we have $\mu \, q \, \lambda$.

(3) An (τ_i, τ_j) δ-closure operator is a mapping $D_{\tau_j}^{\tau_i} : I^X \times I_\circ \to I^X$ defined as follows :

$$D_{\tau_j}^{\tau_i}(\lambda, r) = \vee \{x_t \in P_t(X) \mid x_t \text{ is a } r - (\tau_i, \tau_j) \ \delta\text{-cluster point of } \lambda\}.$$

(4) x_t is called a $r - (\tau_i, \tau_j)$ θ-cluster point of λ if for every $\mu \in Q_{\tau_i}(x_t, r)$, we have $C_{\tau_j}(\mu, r) \, q \, \lambda$.

(5) An (τ_i, τ_j) θ-closure operator is a mapping $T_{\tau_j}^{\tau_i} : I^X \times I_\circ \to I^X$ defined as follows :

$$T_{\tau_j}^{\tau_i}(\lambda, r) = \vee \{x_t \in P_t(X) \mid x_t \text{ is a } r - (\tau_i, \tau_j) \ \theta\text{-cluster point of } \lambda\}.$$

Theorem 1.7[8] — *Let (X, τ_1, τ_2) be an fbts. For each $\lambda, \mu, \rho \in I^X$ and $r \in I_\circ$, we have the following properties :*

(1) $T_{\tau_j}^{\tau_i}(\lambda, r) = \wedge \left\{ \mu \in I^X \mid \lambda \leq I_{\tau_j}(\mu, r), \, \tau_i \underline{1} - \mu) \geq r \right\},$

(2) $D^{\tau_i}_{\tau_j}(\lambda, r) = \wedge\left\{ \mu \in I^X \mid \lambda \leq \mu\, \mu = C_{\tau_i}(I_{\tau_j}(\mu, r), r) \right\}$.

(3) $C_{\tau_i}(\lambda, r) = \vee \{x_t \in P_t(X) \mid x_t$ is a $r - \tau_i$ cluster point of $\lambda\}$.

(4) x_t is a $r - (\tau_i, \tau_j)$ θ-cluster point of λ iff $x_t \in T^{\tau_i}_{\tau_j}(\lambda, r)$.

(5) x_t is a $r - (\tau_i, \tau_j)$ δ-cluster point of λ iff $x_t \in D^{\tau_i}_{\tau_j}(\lambda, r)$.

(6) x_t is a $r - \tau_i$ cluster point of λ iff $x_t \in C_{\tau_i}(\lambda, r)$.

(7) $R^{\tau_i}_{\tau_j}(x_t, r) \subset Q_{\tau_i}(x_t, r)$.

(8) If $\rho = C_{\tau_i}(I_{\tau_j}(\rho, r), r)$, then $D^{\tau_i}_{\tau_j}(\rho, r) = \rho$.

(9) $\lambda \leq C_{\tau_i}(\lambda, r) \leq D^{\tau_i}_{\tau_j}(\lambda, r) \leq T^{\tau_i}_{\tau_j}(\lambda, r)$.

(10) If $\tau_j(\lambda) \geq r$, then $C_{\tau_i}(\lambda, r) = D^{\tau_i}_{\tau_j}(\lambda, r) = T^{\tau_i}_{\tau_j}(\lambda, r)$.

(11) $C_{\tau_i}(I_{\tau_j}(\lambda, r), r) = C_{\tau_i}(I_{\tau_j}(C_{\tau_i}(I_{\tau_j}(\lambda, r), r), r), r)$

(12) $I_{\tau_i}(C_{\tau_j}(\lambda, r), r) = I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(C_{\tau_j}(\lambda, r), r), r), r)$.

(13) $D^{\tau_i}_{\tau_j}(D^{\tau_i}_{\tau_j}(\lambda, r), r) = D^{\tau_i}_{\tau_j}(\lambda, r)$.

Definition 1.8[9, 10, 11] — Let (X, τ_1, τ_2) and (Y, η_1, η_2) be fbts's. Let $f: X \to Y$ be a mapping

(1) f is called FP δ-continuous iff for each $\mu \in R^{\eta_i}_{\eta_j}(f(x)_t, r)$, there exists $\lambda \in R^{\tau_i}_{\tau_j}(x_t, r)$, such that $f(\lambda) \leq \mu$.

(2) f is called *FP* weakly θ-continuous iff for each $\mu \in Q_{\eta_i}(f(t)_t, r)$, there exists $\lambda \in Q_{\tau_i}(x_t, r)$, such that $f(\lambda) \leq C_{\eta_j}(\mu, r)$.

(3) f is called *FP* θ-continuous iff for each $\mu \in Q_{\eta_i}(f(x)_t, r)$ there exists $\lambda \in Q_{\tau_i}(x_t, r)$, such that $f(C_{\tau_j}(\lambda, r)) \leq C_{\eta_j}(\mu, r)$.

(4) f is called *FP* weakly δ-continuous iff for each $\mu \in Q_{\eta_i}(f(x)_t, r)$, there exists $\lambda \in Q_{\tau_i}(x_t, r)$, such that $f(I_{\tau_i}(C_{\tau_i}(\lambda, r), r)) \leq C_{\eta_j}(\mu, r)$.

***Theorem* 1.9**[9, 10, 11] — *Let $f: (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)$ be a mapping*

(a) f is FP θ-continuous iff $T^{\tau_i}_{\tau_j}(f^{-1}(\mu), r) \leq f^{-1}(T^{\eta_i}_{\eta_j}(\mu, r))$ for each $\mu \in I^Y, r \in I_\circ$.

(b) f is FP δ-continuous iff $D^{\tau_i}_{\tau_j}(f^{-1}(\mu), r) \leq f^{-1}(D^{\eta_i}_{\eta_j}(\mu, r))$ *for each* $\mu \in I^Y$, $r \in I_\circ$.

(c) f is FP weakly δ-continuous iff $D^{\tau_i}_{\tau_j}(f^{-1}(\mu), r) \leq f^{-1}(T^{\eta_i}_{\eta_j}(\mu, r))$ *for each* $\mu \in I^Y$, $r \in I_\circ$.

(d) f is FP weakly θ-continuous iff $C_{\tau_i}(f^{-1}(\mu), r) \leq f^{-1}(T^{\eta_i}_{\eta_j}(\mu, r))$ *for each* $\mu \in I^Y$, $r \in I_\circ$.

2. Fuzzy Pairwise Regular Spaces

Definition 2.1 — Let (X, τ_1, τ_2) be an fbts. For each $\lambda \in I^X$, $r \in I_\circ$.

(1) λ is called $r-(\tau_i, \tau_j)$ fuzzy θ-closed (resp. fuzzy δ-closed) iff $\lambda = T^{\tau_i}_{\tau_j}(\lambda, r)$ (resp. $\lambda = D^{\tau_i}_{\tau_j}(\lambda, r)$). We define

$$\Delta^{\tau_i}_{\tau_j}(\lambda, r) = \wedge\left\{ \mu \in I^X \mid \lambda \leq \mu, \mu = D^{\tau_i}_{\tau_j}(\mu, r) \right\},$$

$$\Theta^{\tau_i}_{\tau_j}(\lambda, r) = \wedge\left\{ \mu \in I^X \mid \lambda \leq \mu, \mu = T^{\tau_i}_{\tau_j}(\mu, r) \right\}.$$

(2) The complement of a $r-(\tau_i, \tau_j)$ fuzzy θ-closed (resp. fuzzy δ-closed) sets are called $r-(\tau_i, \tau_j)$ fuzzy θ-open (resp. fuzzy δ-open).

(3) λ is called $r-(\tau_i, \tau_j)$-fuzzy regular open (resp. regular closed) if $\lambda = I_{\tau_i}(C_\tau(\lambda, r), r)$ (resp. $\lambda = C_{\tau_i}(I_{\tau_j}(\lambda, r), r)$).

In this paper, $r-(\tau_i, \tau_j)-f\theta c$, $r-(\tau_i, \tau_j)-f\theta o$, $r-(\tau_i, \tau_j)-f\delta c$, $r-(\tau_i, \tau_j)-f\delta o$, $r-(\tau_i, \tau_j)$-from and $r-(\tau_i, \tau_j)$-fro and $r-(\tau_i, \tau_j)$–frc are abbreviated to $r-(\tau_i, \tau_j)$ fuzzy θ-closed,$r-(\tau_i, \tau_j)$ fuzzy θ-open, $r-(\tau_i, \tau_j)$ fuzzy δ-closed, $r-(\tau_i, \tau_j)$ fuzzy δ-open, $r-(\tau_i, \tau_j)$ fuzzy regular open and $r-(\tau_i, \tau_j)$ fuzzy regular closed, respectively.

***Theorem* 2.2** — *Let* (X, τ_1, τ_2) *be an fbts. For each* $\lambda \in I^X$ *and* $r \in I_\circ$, *we have the following properties :*

(1) $\Delta^{\tau_i}_{\tau_j}(\lambda, r) = D^{\tau_i}_{\tau_j}(\lambda, r)$.

(2) $\Delta^{\tau_i}_{\tau_j}(\lambda, r)$ is $r-(\tau_i, \tau_j)$ fuzzy δ-closed.

(3) $\Theta^{\tau_i}_{\tau_j}(\lambda, r) = T^{\tau_i}_{\tau_j}(\Theta^{\tau_i}_{\tau_j}(\lambda, r), r)$, i.e., $\Theta^{\tau_i}_{\tau_j}(\lambda, r)$ is $r-(\tau_i, \tau_j)$ fuzzy θ-closed.

(4) $T^{\tau_i}_{\tau_j}(\lambda < r) \leq \Theta^{\tau_i}_{\tau_j}(\lambda, r)$.

PROOF : (1) From Theorem 1.7 (9.13), $\lambda \leq D^{\tau_i}_{\tau_j}(\lambda, r) = D^{\tau_i}_{\tau_j}(D^{\tau_i}_{\tau_j}(\lambda, r), r)$ implies $\Delta^{\tau_i}_{\tau_j}(\lambda, r) \leq D^{\tau_i}_{\tau_j}(\lambda, r)$.

Suppose $\Delta^{\tau_i}_{\tau_j}(\lambda, r) \not\geq D^{\tau_i}_{\tau_j}(\lambda, r)$. There exist $x \in X$ and $t \in I_\circ$ such that

$$\Delta^{\tau_i}_{\tau_j}(\lambda, r)(x) < t < D^{i}_{\tau_j}(\lambda, r)(x).$$

From the definition $\Delta^{\tau_i}_{\tau_j}(\lambda, r)$, there exists $\mu \in I^X$ and $\lambda \leq \mu = D^{\tau_i}_{\tau_j}(\mu, r)$ such that

$$\Delta^{\tau_i}_{\tau_j}(\lambda, r)(x) \leq \mu(x) < t < D^{\tau_i}_{\tau_j}(\lambda, r)(x).$$

On the other hand, since $\lambda \leq \mu$, we have

$$D^{\tau_i}_{\tau_j}(\lambda, r) \leq D^{\tau_i}_{\tau_j}(\mu, r) = \mu.$$

It is a contradiction. Hence $\Delta^{\tau_i}_{\tau_j}(\lambda, r) \geq D^{\tau_i}_{\tau_j}(\lambda, r)$.

(2) From Theorem 1.7 (13), it is trivial.

(3) Let $\lambda \leq \mu_j = T^{\tau_i}_{\tau_j}(\mu_j, r)$ for each $j \in \Gamma$. Then

$$\bigwedge_{j \in \Gamma} \mu_j \leq T^{\tau_i}_{\tau_j}\left(\bigwedge_{j \in \Gamma} \mu_j, r\right) \leq T^{\tau_i}_{\tau_j}(\mu_j, r) = \mu_j.$$

So, $\bigwedge_{j \in \Gamma} \mu_j = T^{\tau_i}_{\tau_j}(\wedge_{j \in \Gamma} \mu_j, r)$. Hence $\Theta^{\tau_i}_{\tau_j}(\lambda, r) = T^{\tau_i}_{\tau_j}(\Theta^{\tau_i}_{\tau_j}(\lambda, r), r)$.

Thus $\Theta^{\tau_i}_{\tau_j}(\lambda, r)$ is a $r-(\tau_i, \tau_j)$ fuzzy θ-closed set.

(4) SInce $\lambda \leq \Theta^{\tau_i}_{\tau_j}(\lambda, r)$, by (3), we have

$$T^{\tau_i}_{\tau_j}(\lambda, r) \leq T^{\tau_i}_{\tau_j}(\Theta^{\tau_i}_{\tau_j}(\lambda, r), r) = \Theta^{\tau_i}_{\tau_j}(\lambda, r).$$

In general, by Theorem 2.2 (1-2), an (τ_i, τ_j) δ-closure operator is $r-(\tau_i, \tau_j)$ δ-fuzzy closed for each $r \in I_o$, but an (τ_i, τ_j) θ-closure operator is not $r-(\tau_i, \tau_j)$ θ-fuzzy closed.

Example 2.3 — Let $X = \{x, y\}$ be a set. Let (X, τ_1, τ_2) be an fbts as follows :

$$\tau_1(\lambda) = \begin{cases} 1 & \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \lambda = \underline{0.7}, \\ \frac{1}{2}, & \lambda = \underline{0.4}, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \lambda \in \{\underline{0}, \underline{1}\} \\ \frac{1}{2}, & \lambda = \underline{0.5}, \\ 0, & \text{otherwise}. \end{cases}$$

From Theorem 1.7 (1), we obtain

$$T_{\tau_2}^{\tau_1}(\lambda, r) = \begin{cases} \underline{0}, & \lambda = \underline{0}, r \in I_o \\ \underline{0.6}, & \underline{0} \neq \lambda \leq 0.5, 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise,} \end{cases}$$

$$T_{\tau_1}^{\tau_2}(\lambda, r) = \begin{cases} \underline{0}, & \lambda = \underline{0}, r \in I_o \\ \underline{0.5}, & \underline{0} \neq \lambda \leq 0.4, 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise,} \end{cases}$$

Since

$$\underline{1} = T_{\tau_2}^{\tau_1}\left(T_{\tau_2}^{\tau_1}\left(\underline{0.5}, \frac{1}{2}\right), \frac{1}{2}\right) \neq T_{\tau_2}^{\tau_1}\left(\underline{0.5}, \frac{1}{2}\right) = \underline{0.6}$$

$$\underline{1} = T_{\tau_1}^{\tau_2}\left(T_{\tau_1}^{\tau_2}\left(\underline{0.4}, \frac{1}{2}\right), \frac{1}{2}\right) \neq T_{\tau_1}^{\tau_2}\left(\underline{0.4}, \frac{1}{2}\right) = \underline{0.5}$$

then $T_{\tau_2}^{\tau_1}\left(\underline{0.5}, \frac{1}{2}\right)$ is not $\frac{1}{2} - (\tau_1, \tau_2)$ θ-fuzzy closed and $T_{\tau_1}^{\tau_2}\left(\underline{0.4}, \frac{1}{2}\right)$ is not $\frac{1}{2} - (\tau_2, \tau_1)$ θ-fuzzy closed. Since

$$\Theta_{\tau_2}^{\tau_1}(\lambda, r) = \begin{cases} \underline{0}, \lambda = \underline{0}, r \in I_o \\ \underline{1}, \text{otherwise.} \end{cases} \quad \Theta_{\tau_2}^{\tau_1}(\lambda, r) = \begin{cases} \underline{0}, \lambda = \underline{0}, r \in I_o \\ \underline{1}, \text{otherwise.} \end{cases}$$

we have $\quad T_{\tau_j}^{\tau_i}(\lambda, r) \leq \Theta_{\tau_j}^{\tau_i}(\lambda, r).$

Definition 2.4[10] — An fbr's (X, τ_1, τ_2) is said to be fuzzy pairwise regular (briefly, FP regular) iff for each $x_t \in P_t(X)$ and each $\mu \in Q_{\tau_i}(x_t, r)$, there exists $\nu \in Q_{\tau_i}(x_t, r)$ such that $C_{\tau_j}(\nu, r) \leq \mu$.

Theorem 2.5 — *For an fbts $(X< \tau_1, \tau_2)$, the following statements are equivalent :*

(a) (X, τ_1, τ_2), is FP regular.

(b) For each $x_t \in P_t(X)$ and each $\lambda \in I^X$ with $\tau_i(\underline{1} - \lambda) \geq r$ $\forall$ $r \in I_o$ and $x_t \notin \lambda$ there exists a $\mu \in I^*$ with $\tau_j(\mu) \geq r$ $\forall$ $r \in I_o$ such that $x_i \in C_{\tau_i}(\mu, r)$ and $\lambda \leq \mu$.

(c) For each $x_t \in P_t(X)$ and each $\lambda \in I^X$ with $\tau_i(\underline{1} - \lambda) \geq r$ $\forall$ $r \in I_o$ and $x_t \notin \lambda$, there exists $\mu \in Q_{\tau_i}(x_t, r)$ and $\nu \in I^X$ with $\tau_j(\nu) \geq r$ $\forall$ $r \in I_o$ such that $\lambda \leq \nu$ and $\mu \bar{q} \nu$.

(d) For each $\xi \in I^X$ and $\lambda \in I^X$ with $\tau_i(\underline{1} - \lambda) \geq r$ $\forall$ $r \in I_o$ with $\xi \not\leq \lambda$, there exists $\mu \in I^X$ with $\tau_i(\mu) \geq r$ $\forall r C I_o$ and $\vee \in I^*$ with $\tau_j(\nu) \geq r$ $\forall r \in I_o$ such that with $\xi q \mu$, $\lambda \leq \nu$ and $\mu \bar{q} \nu$.

(e) For each $\xi \in I^X$ *and* $\mu \in I^X$ *with* $\tau_i(\mu) \geq r\ \forall\ r \in I_\circ$ *with* $\xi\, q\, \mu$, *there exists* $\nu \in I^X$ *with* $\tau_i(\nu) \geq r\ \forall\ r \in I_\circ$ *such that* $\xi\, q\, \mu \leq C_{\tau_j}(\nu, r) \leq \mu$.

PROOF : (*a*) $\Rightarrow$ (*b*) Let $x_t \in P_t(X)$ and $\lambda \in I^X$ with $\tau_i(\underline{1} - \lambda) \geq r\ \forall r \in I_\circ$ such that $x_t \notin \lambda$. Then $x_t\, q\, (\underline{1} - \lambda)$. Put $\nu = \underline{1} - \lambda$. Then, $\nu \in Q_{\tau_i}(x_t, r)$. By FP regularity of X, there exists $\eta \in Q_{\tau_i}(x_t, r)$ such that $C_{\tau_j}(\eta, r) \leq \nu$. Put $\nu = \underline{1} - C_{\tau_j}(\eta, r)$. Then, $\tau_j(\mu) \geq r\ \ \forall r \in I_\circ$ (from the definition of C_{τ_j} and Definition 1.1 (03), such that $x_t\, q\, I_{\tau_i}(C_{\tau_j}(\eta, r), r)$ and hence

$$x_t \notin \underline{1} - I_{\tau_i}(\eta, r), r) = C_{\tau_i}(\underline{T} - C_{\tau_j}(\eta, r) = C_{\tau_i}(\mu, r).$$

Also, $$\lambda = \underline{1} - \nu \leq \underline{1} - C_{\tau_j}(\eta, r) = \mu.$$

(*b*) $\Rightarrow$ (*c*) Let $x_t \in P_t(X)$ and $\lambda \in I^X$ with $\tau_i(1 - \lambda) \geq r\ \ \forall r \in I_\circ$ and $x_t \notin \lambda$. By (b), there exists $\mu \in I^X$ with $\tau_j(\mu) \geq r\ \ \forall r \in I_\circ$ such that $x_t \notin C_{\tau_i}(\mu, r)$ and $\lambda \leq \mu$. Now, $x_t \notin C_{\tau_i}(\mu, r)$ implies that there exists $\nu \in Q_{\tau_i}(x_t, r)$ such that $\nu\, \bar{q}\, \mu$ (Theorem 1.7(3)). Thus (c) follows.

(*c*) $\Rightarrow$ (*d*) For each $\xi \in I^X$ and $\lambda \in I^X$ with $\tau_i(\underline{1} - \lambda) \geq r\ \ \forall r \in I_\circ$ with $\xi \not\leq \lambda$. From $\xi \not\leq \lambda$, there exists $x_t \in P_t(X)$ such that $x_t \in \xi$ and $x_t \notin \lambda$. By (c), there exists $\mu \in I^X$ with $\tau_i(\mu) \geq r\ \ \forall$ $r \in I_\circ$ and $\nu \in I^X$ with $\tau_j(\nu) \geq r\ \ \forall r \in I_\circ$ such that $x_t\, q\, \mu, \lambda \leq \nu$ and $\mu\, \bar{q}\, \nu$. Since $x_t \in \xi$ we have $\xi\, q\, \mu$.

(*d*) $\Rightarrow$ (*e*) For any $\xi \in I^X$ and any $\mu \in I^X$ with $\tau_i(\mu) \geq r\ \ \forall r \in I_\circ$, $\xi\, q\, \mu$ implies that $\xi \not\leq \underline{1} - \mu$, where $\tau_i(\underline{1} - (\underline{1} - \mu)) \geq r\ \ \forall r \in I_\circ$. By (d), there exists $\nu \in I^X$ with $\tau_i(\nu) \geq r\ \ \forall r \in I_\circ$ and $\eta \in I^X$ with $\tau_j(\eta) \geq r\ \ \forall r \in I_\circ$ such that $\xi\, q\, \nu, \underline{1} - \mu \leq \eta$ and $\nu\, \bar{q}\, \eta$. Then $C_{\tau_j}(\nu, r)\, \bar{q}\, \eta$. Thus $C_{\tau_j}(\nu, r)\, \bar{q}\, \eta$. Thus

$$\xi\, q\, \nu \leq C_{\tau_j}(\nu, r) \leq \underline{1} - \eta \leq \mu.$$

(*e*) $\Rightarrow$ (*a*) It is clear

3. Fuzzy Pairwise Almost Regular Spaces

Definition 3.1[11] — An fbrs (X, τ_1, τ_2) is said to be fuzzy pairwise almost regular (briefly, FP almost regular) iff for each $x_t \in P_t(X)$ and each $\mu \in R^{\tau_i}_{\tau_j}(x_t, r)$, there exists $\nu \in R^{\tau_i}_{\tau_j}(x_t, r)$ such that $C_{\tau_j}(\nu, r) \leq \mu$.

***Theorem* 3.2** — *For an fbts* (X, τ_1, τ_2), *the following statements are equivalent :*

(a) (X, τ_1, τ_2) *is FP almost regular.*

(a) (X, τ_1, τ_2) *is FP almost regular.*

(b) For each $x_t \in P_t(X)$ *and each* $\mu \in Q_{\tau_i}(x_t, r)$*, there exists* $\nu \in R_{\tau_j}^{\tau_i}(x_t, r)$ *such that* $C_{\tau_j}(\nu, r) \leq I_{\tau_i}(C_{\tau_j}(\mu, r), r)$.

(c) For each $x_t \in P_t(X)$ *and each* $\mu \in Q_{\tau_i}(x_t, r)$*, there exists* $\nu \in Q_{\tau_i}(x_t, r)$ *such that* $C_{\tau_j}(\nu, r) \leq I_{\tau_i}(C_{\tau_j}(\mu, r), r)$.

(d) For each $x_t \in P_t(X)$ *and* $r-(\tau_i, \tau_j)$*-frc set* ξ *with* $x_t \notin \xi$*, there exists* $\nu \in Q_{\tau_i}(x_t, r)$ *and* $\mu \in I^X$ *with* $\tau_j(\mu) \geq r$ $\forall$ $r \in I_\circ$ *such that* $\xi \leq \mu$ *and* $C_{\tau_i}(\mu, r)\, \bar{q}\, (C_{\tau_j}(\nu, r)$.

(e) For every $r-(\tau_i, \tau_j)$ *-frc set* ξ *and each fuzzy point* $x_t \notin \xi$*, there exists* $\nu \in Q_{\tau_i}(x_t, r)$ *and* $\mu \in I^X$ *with* $\tau_j(\mu) \geq r$ $\forall$ $r \in I_\circ$*,* $\xi \leq \mu$ *such that* $C_{\tau_j}(\nu, r)\, \bar{q}\, \mu$.

(f) For any $\xi \in I^X$ *and any* $r-(\tau_i, \tau_j)$*-fro set* μ *with* $\xi\, q\, \mu$*, there exists a* $r-(\tau_i, \tau_j)$*-fro set* ν *such that* $\xi\, q\, \nu \leq C_{\tau_j}(\nu, r) \leq \mu$.

(g) For any $\xi \in I^X$ *and any* $r-(\tau_i, \tau_j)$*-frc set* μ *with* $\xi \not\leq \mu$*, there exists a* $r-(\tau_i, \tau_j)$*-fro set* ν *and a* $r-(\tau_i, \tau_j)$*-fro set* ν *and a* $r-(\tau_j, \tau_i)$*-fro set* η $\forall$ $r \in I_\circ$ *such that* $\xi\, q\, \nu, \mu \leq \eta$ *and* $\nu\, \bar{q}\, \eta$.

PROOF : The proof of (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ *(a)* Let $x_t \in P_t(X)$ and $\mu \in R_{\tau_j}^{\tau_i}(x_t, r)$. Then, by (c), there exists $\nu \in Q_{\tau_i}(x_t, r)$ such that

$$C_{\tau_j}(\nu, r) \leq I_{\tau_i}(C_{\tau_j}(\mu, r), r) = \mu.$$

Since $\nu \in Q_{\tau_i}(x_t, r), I_{\tau_i}(C_{\tau_i}(\nu, r), r) \in R_{\tau_j}^{\tau_i}(x_t, r)$. Also, since $\eta = I_{\tau_i}(C_{\tau_j}(\nu, r), r) \leq C_{\tau_j}(\nu, r)$, we have $C_{\tau_j}(\eta, r) \leq C_{\tau_j}(\nu, r)$ and hence $x_t\, q\, \eta \leq C_{\tau_j}(\eta, r) \leq C_{\tau_j}(\nu, r) \leq \mu$, where $\eta \in R_{\tau_j}^{\tau_i}(x_t, r)$.

(c) $\Rightarrow$ *(d)* Let ξ be any $r-(\tau_i, \tau_j)$-frc set in X and $x_t \in P_t(X)$ with $x_t \notin \xi$. Then $x_t\, q\, (\underline{1} - \xi)$, where $\underline{1} - \xi$ is $r-(\tau_i, \tau_j)$-fro. By (c), there is $\eta \in Q_{\tau_i}(x_t, r)$ such that

$$C_{\tau_j}(\eta, r) \leq I_{\tau_i}(C_{\tau_i}(\underline{1} - \xi, r), r) = \underline{1} - \xi.$$

Now $x_t\, q\, (I_{\tau_i}(C_{\tau_j}(\eta, r), r))$ and hence by hypothesis, there exists $\nu \in Q_{\tau_i}(x_t, r)$ such that $C_{\tau_j}(\nu, r) \leq I_{\tau_i}(C_{\tau_j}(\eta, r), r)$. Then $\xi \leq \underline{1} - C_{\tau_j}(\eta, r)$. Put $\mu = \underline{1} - C_{\tau_j}(\eta, r)$, then by the definition of C_{τ_j} and definition 1.1 (03), $\tau_j(\mu) \geq r$ $\forall$ $r \in I_\circ$. Hence

$$C_{\tau_i}(\mu, r) = \underline{1} - I_{\tau_i}(C_{\tau_j}(\eta, r), r) \leq \underline{1} - C_{\tau_j}(\nu, r),$$

i.e., $$C_{\tau_j}(\nu, r)\, \bar{q}\, C_{\tau_i}(\mu, r).$$

$(d) \Rightarrow (e)$ — Obvious.

$(e) \Rightarrow (f)$ — Let ξ be any fuzzy set and μ any $r-(\tau_i, \tau_j)$-fro set with $\xi q \mu$. Then $\xi \not\leq \underline{1} - \mu$. Hence there exists $x_t \in P_t(X)$ such that $x_t \in \xi$ and $\xi_t \notin \underline{1} - \mu$, where $\underline{1} - \mu$ is $r-(\tau_i, \tau_j)$-frc. By (e), there exists $\nu \in Q_{\tau_i}(x_t, r)$ and $\eta \in I^X$ with $\tau_j(\eta) \geq r\ \forall\ r \in I_\circ$ such that $\underline{1} - \mu \leq \eta$ and $C_{\tau_j}(\nu, r)\, \bar{q}\, \eta$. From $\nu \in Q_{\tau_i}(x_t, r)$ we have $x_t q \nu \leq I_{\tau_i}(C_{\tau_j}(\nu, r), r)$. Put $\nu_1 = I_{\tau_i}(C_{\tau_i}(\nu, r), r)$, we have $\xi q \nu_1$. Then ν_1 is $r-(\tau_i, \tau_j)$-fro set such that

$$\xi q \nu_1 \leq C_{\tau_j}(\nu_1, r) \leq C_{\tau_j}(\nu, r) \leq \underline{1} - \eta \leq \mu$$

$(f) \Rightarrow (g)$ — For any $\xi \in I^X$ and any $r-(\tau_i, \tau_j)$-frc set μ with $\xi \not\leq \mu$, we have $\xi q (\underline{1} - \mu)$ and hence by (f), there exists a $r-(\tau_i, \tau_j)$-fro set ν uch that $\xi q \nu \leq C_{\tau_j}(\nu, r) \leq \underline{1} - \mu$. Then ν is $r-(\tau_i, \tau_j)$-fro set, $\underline{1} - C_{\tau_j}(\nu, r)$ is a $r-(\tau_i, \tau_j)$-fro set such that $\xi q \nu, \mu \leq \underline{1} - C_{\tau_j}(\nu, r)$ and $\nu \bar{q} (\underline{1} - C_{\tau_j}(\nu, r))$.

$(g) \Rightarrow (a)$ — Let $x_t \in P_t(X)$ and $\mu \in R^{\tau_i}_{\tau_j}(x_t, r)$. Then $x_t \notin \underline{1} - \mu$, where $\underline{1} - \mu$ is $r-(\tau_i, \tau_j)$-frc. By (f), there exist $r-(\tau_i, \tau_j)$-fro set ν and $r-(\tau_j, \tau_i)$-fro set η such that $x_t q \nu, \underline{1} - \mu \leq \eta$ and $\nu \bar{q} \eta$. Then $C_{\tau_j}(\nu, r)\, \bar{q}, \eta$ as well. Thus $x_t q \nu \leq C_{\tau_j}(\nu, r) \leq \underline{1} - \eta \leq \mu$, where ν is $r-(\tau_i, \tau_j)$-fro Thus (X, τ_1, τ_2) is FP almost regular.

***Theorem* 3.3**[11] — *An fbts (X, τ_1, τ_2) is FP almost regular iff for each $\lambda \in I^X$ and $r \in I_\circ$,*

$$T^{\tau_i}_{\tau_j}(\lambda, r) = D^{\tau_1}_{\tau_j}(\lambda, r).$$

***Theorem* 3.4** — *An fbts (X, τ_1, τ_2) is FP almost regular iff for any $r-(\tau_i, \tau_j)$ – frc set λ in X, $r \in I_o$, $T^{\tau_i}_{\tau_j}(\lambda, r) = \lambda$.*

PROOF : The necessary part follows from Theorem 3.3 and the fact that a $r-(\tau_i, \tau_j)$-frc set is $r-(\tau_i, \tau_j)-f\delta c$.

Conversely, let λ be any $r-(\tau_i, \tau_j)$-frc and let $x_t \in P_t(X)$ such that $x_t \notin \lambda$. Then $x_t \notin T^{\tau_i}_{\tau_j}(\lambda, r)$ so there exists $\mu \in Q_{\tau_i}(x_t, r)$ such that $C_{\tau_j}(\mu, r)\, \bar{q}\, \lambda$. Thus $\lambda \leq \underline{1} - C_{\tau_j}(\mu, r) = \nu$ (say). Moreover, $C_{\tau_j}(\mu, r)\, \bar{q}\, \nu$. Hence, by Theorem 3.2 $(e) \Rightarrow (a)$), (X, τ_1, τ_2) is FP almost regular.

Lemma 3.5 — If $\lambda, \mu \in I^X, r \in I_\circ$ such that $\lambda \bar{q} \mu$ where μ is $r-(\tau_j - \tau_i)-f\delta o$, then $D^{\tau_j}_{\tau_i}(\lambda, r)\, \bar{q}\, \mu$..

PROOF : Since $\lambda \leq \underline{1} - \mu = D_{\tau_i}^{\tau_j}(\underline{1} - \mu, r)$, by Theorem 1.7 (13),

$$D_{\tau_i}^{\tau_j}(\lambda, r) \leq D_{\tau_i}^{\tau_j}(D_{\tau_i}^{\tau_j}(\underline{1} - \mu, r), r) = D_{\tau_i}^{\tau_j}(\underline{1} - \mu, r) = \underline{1} - \mu.$$

Lemma 3.6 — A fuzzy set μ is $r-(\tau_i, \tau_j)-f\delta o$ in an fbts $X, r \in I_0$, iff for every $x_t \in P_t(X)$ with $x_t\, q\, \mu$, there exists a $r-(\tau_i, \tau_j)$-fro set ν such that $x_t\, q\, \nu \leq \mu$.

PROOF : Let μ be $r-(\tau_i, \tau_j)$-$f\delta o$ and $x_t\, q\, \mu$. Then $x_t \notin \underline{1} - \mu$. Then there exist a $r-(\tau_j\, \tau_i)$-fro set ν such that $x_t\, q\, \nu$ and $\nu \bar{q} (\underline{1} - \mu)$, since $\underline{1} - \mu$ is $r-(\tau_i, \tau_j)-f\delta c$. Now ν being $r-(\tau_i, \tau_j)-$fro is $r-(\tau_i, \tau_j)-f\delta o$ as well such that

$$x_t\, q\, \nu \leq \underline{1} - (\underline{1} - \mu) = \mu.$$

Conversely, suppose $\underline{1} - \mu \neq D_{\tau_j}^{\tau_i}(\underline{1} - \mu, r)$. Then there exist $x \in X$ and $t \in (0, 1)$ such that

$$(\underline{1} - \mu)(x) < t < D_{\tau_j}^{\tau_i}(\underline{1} - \mu, r)(x).$$

Since $x_t\, q\, \mu$, there exists a $r-(\tau_i, \tau_j)-f\delta o$ set ν such that $x_t\, q\, \nu \leq \mu$. It implies

$$\underline{1} - \mu \leq (\underline{1} - \nu) = C_{\tau_i}(\tau_{\tau_i}(\underline{1}\ \tau_j(1 - \mu, r), r).$$

From Theorem 1.7 (2), $D_{\tau_j}^{\tau_i}(\underline{1} - \mu, r)\ (x) \leq (\underline{1} - \nu) < t$. It is a contradictions. Thus,

$$\underline{1} - \mu = D_{\tau_j}^{\tau_i}(\underline{1} - \mu, r), \text{ i.e., } \mu \text{ is } r-(\tau_i, \tau_j)-f\delta o.$$

***Theorem* 3.7** — *For an fbts* (X, τ_1, τ_2), *the following statements are equivalent :*

(a) (X, τ_1, τ_2) *is FP almost regular,*

(b) For each $r-(\tau_i, \tau_j)-f\delta o$ *set* ν *and each* $x_t \in P_t(X)$ *with* $x_t\, q\, \nu$, *there exists a* $r-(\tau_j, \tau_i)-f\delta o$ *set* μ *such that*

$$x_t\, q\, \mu < D_{\tau_i}^{\tau_j}(\mu, r) \leq \nu.$$

(c) For every $r-(\tau_i, \tau_j)-f\delta c$ *set* λ *and each* $x_t \in P_t(X)$ *with* $x_t \notin \lambda$, *there exists* $\mu, \nu \in I^X$ *where* μ *is* $r-(\tau_i, \tau_j) f\delta o$ *and* ν *is* $r-(\tau_i, \tau_j)-f\delta o$ *in* X *such that* $x_t\, q\, \mu, \lambda \leq \nu$ *and*

$$D_{\tau_i}^{\tau_j}(\mu, r)\, \bar{q}\, D_{\tau_j}^{\tau_i}(\nu, r).$$

PROOF : $(a) \Rightarrow (b)$: Let ν be $r-(\tau_i, \tau_j)-f\delta o$ such that $x_t\, q\, \nu$. By Lemma 3.6, there exists a $r-(\tau_i, \tau_j)-fro$ set η such that $x_t\, q\, \eta \leq \nu$. By (a), there exists $r-(\tau_i, \tau_j)$-fro set μ (which is also $(r-(\tau_i, \tau_j)-f\delta o)$ such that

$$x_t\, q\, \mu \le C_{\tau_j}(\mu, r) = D^{\tau_j}_{\tau_i}(\mu, r) \le \eta \le \nu,$$

since $\tau_i(\mu) \ge r\ \forall\, r \in I_\circ$ from Theorem 1.7 (10).

$(b) \Rightarrow (c)$ — Let λ be $r-(\tau_i, \tau_j)-f\delta c$ set in X such that $y\ x_t \notin \lambda$ i.e., $x_t\, q\, (\underline{1} - \lambda)$. By (b), there exists a $r-(\tau_i, \tau_j)-f\delta o$ set η such that $x_t\, q\, \eta \le D^{\tau_j}_{\tau_i}(\eta, r) \le \underline{1} - \lambda$. By applying (b) again, we get a $r-(\tau_i, \tau_j)-f\delta o$ set μ such that $x_q\, q\, \mu \le D^{\tau_j}_{\tau_i}(\mu, r) \le \eta$. Put $\nu\ \underline{1} - D^{\tau_j}_{\tau_i}(\eta, r)$. By Theorem 1.7 (13), ν is $r-(\tau_j, \tau_i)-f\delta o$ such that $\lambda \le \nu$ and $\nu\, \bar{q}\, \eta$. By Lemma 3.5, we get $D^{\tau_i}_{\tau_j}(\nu, r)\, \bar{q}\, \eta$ and hence $D^{\tau_i}_{\tau_j}(\nu, r)\, \bar{q}\, D^{\tau_j}_{\tau_i}(\mu, r)$.

$(c) \Rightarrow (a)$ —Let $\lambda \in I^X$. By Theorem 1.7 (9), it s true that $D^{\tau_i}_{\tau_j}(\lambda, r) \le T^{\tau_i}_{\tau_j}(\lambda, r)$. Now, let $x_t \in P_t(X)$ such that $x_t \notin D^{\tau_i}_{\tau_j}(\lambda, r)$. Since $D^{\tau_i}_{\tau_j}(\lambda, r)$ is $r-(\tau_i, \tau_j)-f\delta c$, by (c), there exists a $r-(\tau_i, \tau_j)-f\delta o$ set μ and a $r-(\tau_i, \tau_j)-f\delta o$ set ν such that $x_t\, q\, \mu, D^{\tau_i}_{\tau_j}(\lambda, r) \le \nu$ and $D^{\tau_i}_{\tau_j}(\nu, r)\, \bar{q}\, D^{\tau_j}_{\tau_i}(\mu, r)$. Since $\lambda \le D^{\tau_i}_{\tau_j}(\nu, r)$ and $C_{\tau_j}(\mu, r) = D^{\tau_j}_{\tau_i}(\mu, r)$ it follows that $\mu \in Q_{\tau_i}(x_t, r)$, such that $\lambda \le \underline{1} - C_{\tau_j}(\mu, r) = I_{\tau_j}(\underline{1} - \mu, r)$. By Theorem 1.7 (1), $T^{\tau_i}_{\tau_j}(\lambda, r)(x) \le (\underline{1} - \mu)(x) < t$, consequently $x_t \notin T^{\tau_i}_{\tau_j}(\lambda, r)$. Thus, for $\lambda \in I^X$, we have $D^{\tau_i}_{\tau_j}(\lambda, r) = T^{\tau_i}_{\tau_j}(\lambda, r)$. Thus by Theorem 3.3 (X, τ_1, τ_2) is FP almost regular.

Remark 3.8 : Kim *et al.*[11] proved that *FP* regular is *FP* almost regular, but the converse is not true.

Definition 3.9 — An fbts (X, τ_1, τ_2) is FP-T_2 iff for any $x_t \ne y_s \in P_t(X)$:

Case I — When $x \ne y$, there exists $\mu_1, \mu_2 \in I^X$ with $\tau_i(\mu_1) \ge r\ \forall\, r \in I_\circ$, $\mu_2 \in Q_{\tau_i}(x_t, r)$ and $\nu_1, \nu_2 \in I^X$ with $\nu_1 \in Q_{\tau_j}(y_s, r)$, $\tau_j(\nu_2) \ge r\ \forall\, r \in I_\circ$ such that $x_t \in \mu_1, \mu_1\, \bar{q}\, \nu_1$ and $y_s \in \nu_2, \mu_2\, \bar{q}\, \nu_2$.

Case II — When $x = y$ and $t < s$ (say), there exists $\mu, \nu \in I^X$ with $\tau_i(\mu) \ge r\ \forall\, r \in I_\circ$ and $\nu \in Q_{\tau_j}(y_s, r)$ such that $x_t \in \mu$ and $\mu\, \bar{q}\, \nu$.

Definition 3.10 — An fbts (X, τ_1, τ_2) is FP Urysohn iff for any $x_t \ne y_s \in P_t(X)$:

Case I — When $x \ne y$, there exists $\mu_1, \mu_2 \in I^X$ with $\tau_i(\mu_1) \ge r\ \forall\, r \in I_\circ$, $\mu_2 \in Q_{\tau_i}(x_t, r)$ and $\nu_1, \nu_2 \in I^X$ with $\nu_1 \in Q_{\tau_j}(y_s, r)$, $\tau_j(\nu_2) \ge r\ \forall\, r \in I_\circ$ such that $x_t \in \mu_1, C_{\tau_j}(\mu_1, r)\, \bar{q}\, C_{\tau_i}(\nu_1, r)$ and $y_s \in \nu_2, C_{\tau_j}(\mu_2, r)\, \bar{q}\, C_{\tau_i}(\nu_2, r)$.

Case I — When $x \neq y$, there exists $\mu_1, \mu_2 \in I^X$ with $\tau_i(\mu_1) \geq r \ \forall \ r \in I_\circ$, $\mu_2 \in Q_{\tau_i}(x_t, r)$ and $v_1, v_2 \in I^X$ with $v_1 \in Q_{\tau_j}(y_s, r)$, $\tau_j(v_2) \geq r \ \forall \ r \in I_\circ$ such that $x_t \in \mu_1$, $C_{\tau_j}(\mu_1, r) \bar{q} C_{\tau_i}(v_1, r)$ and $y_s \in v_2$, $C_{\tau_j}(\mu_2, r) \bar{q} C_{\tau_i}(v_2, r)$.

Case II — When $x = y$ and $t < s$ (say), there exists $\mu, v \in I^X$ with $\tau_i(\mu) \geq r \ \forall \ r \in I_\circ$ and $v \in Q_{\tau_j}(y_s, r)$ such that $x_t \in \mu$ and $C_{\tau_j}(\mu, r) \bar{q} C_{\tau_i}(v, r)$.

***Theorem* 3.11** —*Every FP almost regular FP-T_2 space is FP Urysohn.*

PROOF : Let $x_t \neq y_s \in P_t(X)$ in (X, τ_1, τ_2).

Case I — $x \neq y$. Since (X, τ_1, τ_2). is FP-T_2, there exists $\mu \in I^X$ with $\tau_i(\mu) \geq r \ \forall r \in I_\circ$ and $v \in I^X$ such that $x_t \in \mu$, $y_s q v$ and $\mu \bar{q} v$. This implies $C_{\tau_j}(\mu, r) \bar{q} v$ i.e., $v \leq \underline{1} - C_{\tau_j}(\mu, r)$. Since $y_s q v, \underline{1} - C_{\tau_j}(\mu, r) \in R^{\tau_i}_{\tau_j}(y_s, r)$. By FP almost regularity of (X, τ_1, τ_2), there exists $\lambda \in R^{\tau_j}_{\tau_i}(y_s, r)$ such that $C_{\tau_i}(\lambda, r) \leq \underline{1} - C_{\tau_j}(\mu, r)$ i.e., $C_{\tau_j}(\mu, r) \bar{q} C_{\tau_i}(\lambda, r)$. Similarly there exists $\mu_1 \in I^X$ with $\tau_i(\mu_1) \geq r$ $\forall r \in I_\circ$ and $v_1 \in I^X$ with $\tau_j(v_1) \geq r \ \forall r \in I_\circ$ such that $x_t q \mu_1, y_s \in v_1$ and $C_{\tau_j}(\mu_1, r) \bar{q} (C_{\tau_i}(v_1, r)$.

Case II — $x = y$, and $t < s$. Since (X, τ_1, τ_2) is FP-T_2, there exists $\mu \in I^X$ with $\tau_i(\mu) \geq r$ $\forall r \in I_\circ$ $v \in I^X$ with $\tau_j(v) \geq r \ \forall r \in I_\circ$ such that $x_t \in \mu, y_s q v$ and $\mu \bar{q} v$. Then as in Case (*i*) we have $C_{\tau_j}(\mu, r) \bar{q} C_{\tau_i}(\lambda, r)$, where $\lambda \in R^{\tau_j}_{\tau_i}(y_s, r)$. Hence (X, τ_1, τ_2) is FP Urysohn.

4. FUZZY PAIRWISE $r\,\theta$-CONNECTED AND $r\,\delta$-CONNECTED SETS

Definition 4.1 — Let (X, τ_1, τ_2) be an fbts. A pair (λ, μ) of non-null fuzzy sets is said to be fuzzy pairwise r-separation (FP r-separation, for short) relative to X iff $\lambda \bar{q} \mu, \lambda \bar{q} C_{\tau_1}(\mu, r)$ and $C_{\tau_2}(\lambda, r) \bar{q} \mu \ \ \forall r \in I_\circ$.

A fuzzy set γ in an fbts (X, τ_1, τ_2) is said to be fuzzy pairwise r-connected (FP r-connected, for short) iff there do not exist two fuzzy sets λ and μ in X such that (λ, μ) is an *FP* r-separation relative to X and $\gamma = \lambda \vee \mu$.

Definition 4.2 — Let (X, τ_1, τ_2) be an fbts. A pair (λ, μ) of non-null fuzzy sets is said to be fuzzy pairwise $r\,\theta$-separation (FP $r\,\theta$-separation, for short) relative to X iff $\lambda \bar{q} \mu, \lambda \bar{q} \Theta^{\tau_1}_{\tau_2}(\mu, r)$ and $\Theta^{\tau_2}_{\tau_1}(\lambda, r) \bar{q} \mu \ \ \forall \ r \in I_\circ$.

A fuzzy set γ in an fbts (X, τ_1, τ_2) is said to be fuzzy pairwise $r\,\theta$-connected (FP $r\,\theta$-connected, for short) iff there do not exist two fuzzy sets λ and μ in X such that (λ, μ) is an FP $r\,\theta$-separation relative to X and $\gamma = \lambda \vee \mu$.

Definition 4.3 — Let (X, τ_1, τ_2) be an fbts. A pair (λ, μ) of non-hull fuzzy sets is said to be fuzy pairwise $r\,\delta$-separation (FP $r\,\delta$-sepaation, for short) relative to X iff $\lambda \bar{q}\, \mu$, $\lambda \bar{q}\, \Delta^{\tau_1}_{\tau_2}(\mu, r)$ and $\Delta^{\tau_2}_{\tau_1}(\lambda, r)\, \bar{q}\, \mu\ \forall\ r \in I_\circ$.

A fuzzy set γ in an fbts (X, τ_1, τ_2) is said to be fuzzy pairwise $r\,\delta$-connected (FP $r\,\delta$-connected, for short) iff there do not exist two fuzzy sets λ and μ in X such that (λ, μ) is an FP $r\,\delta$-separation relative to X and $\gamma = \lambda \vee \mu$.

Remark 4.4 : From Theorem 1.7 (9) and Theorem 22 (1, 4), it is clear that every FP r-connected set is FP $r\,\delta$-connected and every Fp $r\,\delta$-connected set is FP $r\,\theta$-connected. The converses need not be true as the following examples show.

Example 4.5 — Let $X = I$ and (X, τ_1, τ_2) an fbts as $\tau_1, \tau_2 : I^X \to I$ defined by

$$\tau_1(\lambda) = \begin{cases} 1, & \lambda \in (\underline{0}, \underline{1}), \\ \frac{1}{2}, & \lambda = \lambda_1, \\ \frac{1}{3}, & \lambda = \lambda_3, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \lambda \in (\underline{0}, \underline{1}), \\ \frac{1}{2}, & \lambda = \lambda_2, \\ \frac{1}{3}, & \lambda = \lambda_4, \\ 0, & \text{otherwise}, \end{cases}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are fuzzy sets defined as follows

$$\lambda_1(0) = \frac{1}{4},\ \lambda_2(0) = \frac{1}{7},\ \lambda_3(0) = \frac{8}{9},\ \lambda_4(0) = \frac{1}{5},$$

$$\lambda_k(x) = \frac{1}{2}\ \forall\ x \in I_\circ,\ k = 1, 2, 3, 4.$$

Then (X, τ_1, τ_2) is an fbts. Consider a fuzzy set defined as follows

$$\gamma(x) = \begin{cases} \frac{6}{7}, & x = 0, \\ \frac{1}{2}, & \text{otherwise}. \end{cases}$$

Then, $\gamma = \mu \vee \nu$, where

$$\mu(x) = \begin{cases} \frac{6}{7}, & x = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad \nu(x) = \begin{cases} \frac{1}{10}, & x = 0, \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

Obviously, $\mu \bar{q}\, \nu$. Again

$$C_{\tau_2}\left(\mu, \frac{1}{3}\right) = \begin{cases} \frac{6}{7}, & x = 0, \\ \frac{1}{2}, & \text{otherwise}, \end{cases} \qquad C_{\tau_1}\left(\nu, \frac{1}{3}\right) = \begin{cases} \frac{1}{9}, & x = 0, \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

Obviously, $\mu \bar{q} C_{\tau_1}\left(\nu, \frac{1}{3}\right)$ and $C_{\tau_2}\left(\mu, \frac{1}{3}\right) \bar{q}\, \nu$. Then γ is not FP $\frac{1}{3}$-connected. For any representation $\gamma = \mu \vee \nu i$, where μ and ν are non-empty, either $\mu(0) = \frac{6}{7}$ or $\nu(0) = \frac{6}{7}$, by Theorem 1.7 (2) and 2.2 (1), then if $\mu(0) = \frac{6}{7}$ (resp. if $\nu(0) = \frac{6}{7}$, then $\Delta^{\tau_1}_{\tau_2}\left(\mu, \frac{1}{3}\right) = \underline{1}$ (resp. $\Delta^{\tau_1}_{\tau_2}\left(\nu, \frac{1}{3}\right) = 1$ so that γ is not representable as $\mu \vee \nu$, where (μ, ν) is an FP $\frac{1}{3} - \delta$-separation. Hence γ is FP $\frac{1}{3} - \delta$-connected.

Example 4.6 — Define an fts (X, τ_1, τ_2) as ain Example 3.3. From Theorem 1.7 (2) and Theorem 2.2 (1), we obtain

$$\Delta^{\tau_1}_{\tau_2}(\lambda, r) = \begin{cases} \underline{0}, & \lambda = \underline{0},\ r \in I_o \\ \underline{0.6}, & \underline{0} \neq \lambda \leq \underline{0.6}, 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

$$\Delta^{\tau_2}_{\tau_1}(\lambda, r) = \begin{cases} \underline{0}, & \lambda = \underline{0},\ r \in I_o \\ \underline{0.5}, & \underline{0} \neq \lambda \leq \underline{0.5}, 0 < r \leq \frac{1}{2}, \\ \underline{1}, & \text{otherwise}, \end{cases}$$

For $\underline{0.4} = \underline{0.3} \vee \underline{0.4}$ we have $\underline{0.3}\, \bar{q}\, \underline{0.4}$,

$$\underline{0.6} = \Delta^{\tau_1}_{\tau_2}\left(\underline{.3}, \frac{1}{3}\right) \bar{q}\, \underline{0.4},\quad \underline{0.3}\, \bar{q}\, \Delta^{\tau_2}_{\tau_1}\left(0.4, \frac{1}{3}\right) = \underline{0.5}.$$

Hence $(\underline{0.3}, \underline{0.4})$ is an FP $\frac{1}{3} - \delta$-separation and $\underline{0.4}$ is not FP $\frac{1}{3} - \delta$-connected.

For any representation $\underline{0.4} = \mu \vee \nu$, where μ and ν are non-empty, by Example 3.3, $\Theta^{\tau_i}_{\tau_j}(\lambda, r) = \underline{1}$ for $\lambda \in \{\mu, \nu\}$. Thus, $\underline{0.4}$ is FP $\frac{1}{3} - \theta$-connected.

Definition 4.7[9] — An fbts (X, τ_1, τ_2) is called FP semi-regular iff for each $\mu \in Q_{\tau_i}(x_t, r)$, there exists $\rho \in Q_{\tau_i}(x_t, r)$ with $I_{\tau_i}(C_{\tau_j}(\rho, r), r) \leq \mu$.

***Theorem* 4.8**[9] — *An fbts (X, τ_1, τ_2) is FP semi-regular iff*

$$D^{\tau_i}_{\tau_j}(\lambda, r) = C_{\tau_i}(\lambda, r),$$

for each $\lambda \in I^X$ and $r \in I_o$.

Ccorollary 4.9 — In a *FP* semi-regular space, the concept of *FP* r-connectedness and that of *FP r δ-connectedness are equivalent.*

Corollary 4.10 — In a *FP* almost regular space, *FP* $r\,\theta$-connectedness are equivalent because, by Theorem 3.3, $\Theta^{\tau_i}_{\tau_j}(\lambda, r) = \Delta^{\tau_i}_{\tau_j}(\lambda, r)$, for each $\lambda \in I^X$ and $r \in I_\circ$.

***Theorem* 4.11**[11] — *An fbts* (X, τ_1, τ_2) *is FP regular iff it is FP almost regular and Fp semi-regular.*

Corollary 4.12 — In a *FP* regular space, by Theorem 3.3 and Theorem 4.8, the notions of *FP* r-connectedness, *FP* $r\,\delta$-connectedness and *FP* $r\,\theta$-connectedness become identical.

***Theorem* 4.13** — *Let* $f: (X, \tau_1, \tau_2) \to (Y, \tau_1^*, \tau_2^*)$ *be a mapping and* $\lambda \in I^X$, $r \in I_\circ$.

(a) If λ *is FP r-connected in X and f is FP weakly* θ*-continuous, then* $f(\lambda)$ *is FP* $r\,\theta$*-connected in Y.*

(2) If λ *is FP* $r\,\delta$*-connected in X and f is FP weakly* δ*-continuous, then* $f(\lambda)$ *is Fp* $r\,\delta$*-connected in Y.*

(c) If λ *is FP* $r\,\delta$*-connected in X and f is Fp* δ*-continuous, then* $f(\lambda)$ *is FP* $r\,\theta$*-connected in Y.*

(d) If λ *is FP* $r\,\theta$*-connected in X and f is FP* θ*-continuous, then* $f(\lambda)$ *is FP* $r\,\theta$*-connected in Y.*

PROOF : (a) If possible, let (μ, ν) be an *FP* $r\,\theta$-separation relative to Y such that $f(\lambda) = \mu \vee \nu$. Suppose $\rho = \lambda \wedge f^{-1}(\mu)$ and $\eta = \lambda \wedge f^{-1}(\nu)$. Then, obviously $\lambda = \rho \vee \eta$. To arrive to a contradiction it suffices to show that (ρ, η) is an *FP* r-separation relative to X. Now, since $f(\lambda) \vee \mu \neq \underline{0}$ (otherwise $\mu = \underline{0}$), there exists $y \in Y$ such that $f(\lambda)(y) > 0$. Then for some $x \in f^{-1}(y)$, $\lambda(x) > 0$. Also, $f^{-1}(\mu)(x) = \mu(f(x)) > 0$. Thus, $\rho = \lambda \vee f^{-1}(\mu) \neq \underline{0}$. Similarly $\eta = \lambda \vee f^{-1}(\nu) \neq \underline{0}$. Now, $\rho \leq f^{-1}(\mu)$, by Theorem 1.9 (d) and Theorem 2.2 (4),

$$C_{\tau_2}(\rho, r) \leq C_{\tau_2}(f^{-1}(\mu), r) \leq f^{-1}(T^{\tau_2^*}_{\tau_1^*}(\mu, r)) \leq f^{-1}(\Theta^{\tau_2^*}_{\tau_1^*}(\mu, r)).$$

Again, since $\Theta^{\tau_2^*}_{\tau_1^*}(\mu, r)\,\overline{q}\,\nu$, we have $f^{-1}\left(\Theta^{\tau_2^*}_{\tau_1^*}(\mu, r)\right)\overline{q}\,f^{-1}(\nu)$ and hence $C_{\tau_2}(\rho, r)\,\overline{q}\,f^{-1}(\nu)$. But $\eta \leq f^{-1}(\nu)$. So that $\eta\,\overline{q}\,C_{\tau_2}(\rho, r)$. hence, (ρ, η) is an *FP* r-separation relative to X.

The proofs of (b)-(d) follow similarly as in (*a*) by using Theorem 1.9.

REFERENCES

1. C. L. Chang, *J. math. anal. appl.*, **24** (1968), 182-90.
2. K. C. Chatopadhyay and S. K. Samanta, *Fuzzy sets and systems*, **54** (1993), 207-12.
3. K. C. Chattopadhyay, R. N. Hazra, S. K. Samanta, *Fuzzy Sets and Systems*, **49** (1992), 237-42.
4. M. Demirci, *Fuzzy Sets and Systems*, **92** (1997), 123-28.
5. M. K. El Gayyar, E. E. Kerre and A. A. Ramadan, *Fuzzy sets and systems*, **92** (1994), 193-202.
6. U. Höhle and A. P. Sostak, *Fuzzy sets and systems*, **73** (1995), 131-49.
7. U. Höhle and A. P. Sostak, *Axiomatic Foundations of Fixed-Basis fuzzy topology*, *The Handbooks of Fuzzy sets series*, Volume 3, Kluwer Academic Publishers, Dordrecht (Chapter 3) (1999).

8. Y. C. Kim, A. A. Ramadan and S. E. Abbas, *J. Math. Sci.*, Special Vol. III (2000), 379-96.
10. Y. C. Kim, A. A. Ramadana dn S. E. Abbas, *J. Math. Sci.*, **3(4)** (2001), 543-62.
11. Y. C. Kim, A. A. Ramadan and S. E. Abbas, *J. Fuzzy Math.*, **10(2)** (2002), 475-92.
12. T. K. Kubiak and A. P. Sostak, *Fuzzy sets and Systems*, **87** (1997), 235-42.
14. S. S. Kumar, *Fuzzy Sets and Systems*, **103** (1999), 535-40.
15. S. S. Kumar, *J. Fuzzy math.*, **7(3)** (1999), 689-99.
16. P. M. Pu and Y. M. Liu, Fuzzy topology, *I. J. Math. anal. appl.* **76** (1980), 571-99.
17. A. A. Ramadan, *Fuzzy sets and systems*, **48** (1992), 371-75.
18. A. P. Sostak, *Suppl. Rend. Circ. Matem. Palerms ser*, II **11** (1985), 89-103.
19. A. . Sostak, *Zb. Rodova Univ. Nis, ser Math.*, **4** (1990), 7-14.
20. A. P. Sostak, *J. of Math. Sciences*, **78(6)** (1996), 662-701.