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A GEOMETRIC PROOF OF MALKIN'S STABILITY THEOREM

V. SUNDARAPANDIAN

Assistant Professor, Department of Mathematics, Indian Institute of Technology, Kanpur 208 016, *Uttar Pradesh, India*
E-mail: vsundara@iitk.ac.in

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Malkin's stability theorem is an important result in the stability theory of nonlinear systems, and it has many important applications in the control systems literature. In this paper, we present a new, short proof of this famous result using centre manifold theory.

Key Words : Malkin's Stability Theorem; Stability Theory; Centre Manifold Theory; Nonlinear Systems

1. INTRODUCTION

In this paper, we give a new, geometric proof of Malkin's stability theorem[1]. Malkin's stability theorem is useful in the stability analysis of *critical cases* of nonlinear systems; it has many important applications[2] in the conrol systems literature. Our quick proof of Malkin's stability theorem uses center manifold theory for flows[3].

This paper is organized as follows. In Section 2, we give some basic stability definitions introduced by Malkin. In Section 3, we state Malkin's stability theorem for nonlinear systems and give a quick geometric proof.

2. DEFINITIONS

In this section, we give the basic stability definitions introduced by Malkin for the stability of equilibria of nonlinear systems.

Definition 2.1[1] — Consider a nonlinear system described by

$$\dot{x} = g_m(x) + g_{m+1}(x) + \ldots + g_N(x) + r(x) \qquad \ldots (1)$$

where x is defined in an open neighbourhood of the origin of $I\!R^n$, $g_i(x)$, $(i = m, m+1, ..., N)$ are homogeneous polynomials of degree i, and the degree of the term $r(x)$, which is considered as a disturbance, is greater than N. Then the equilibrium $x = 0$ of the system (1) is called **stable in the** Nth **approximation** if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that the general solution of (1) satisfies the inequality

$$\| x(t, x_0) \| < \varepsilon, \text{ for } t > 0$$

whenever $\| x_0 \| < \delta$. The term $r(x)$ is arbitrary as long as it satisfies an estimate

$$\| r(x) \| < a \| x \|^{N+1}, a > 0. \qquad \square$$

The equilibrium $x = 0$ of the system (1) is called **asymptotically stable in the Nth approximation** if it is stable in the Nth approximation and if $x(t, x_0)$ tends to zero for small initial conditions $x(0)$.

The equilibrium $x = 0$ of the system (1) is called **unstable** in the Nth **approximation** if there exists an $\varepsilon_0 > 0$ and in each neighbourhood of the origin $\mathbb{R}^n$, there exists initial values x_0 such that $\| x(t, x_0) \|$ reaches ε_0 in finite time for each choice of $r(x)$ which satisfies an estimate

$$\| r(x) \| < a \| x \|^{N+1}, a > 0.$$

The number ε_0 is allowed to depend on a. □

Next, we illustrate Malkin's stability definitions with a scalar nonlinear differential equation.

Example 2.2 — Consider a scalar differential equation defined by

$$\dot{x} = \alpha x^N + r(x) \quad \text{... (2)}$$

where the state x is defined on $\mathbb{R}$ near $x = 0$, $N \geq 2$, and the disturbance $r(x)$ is a function vanishing at $x = 0$ together with all derivatives of order less than or equal to N.

Then it is easy to check that the equilibrium $x = 0$ is asymptotically stable in the Nth approximation if N is odd and $\alpha < 0$. It is also easy to check that the equilibrium $x = 0$ is unstable in the Nth approximation if N is odd and $\alpha > 0$ or if N is even.

These claims can be easily verified by taking the candidate Lyapunov function $V(x) = x^2$ and applying the stability theorems of Lyapunov. □

3. Malkin's Stability Theorem

In this section, we present a geometric proof of Malkin's stability theorem for nonlinear systems.

***Theorem* 3.1**[1] — *Consider a nonlinear system described by*

$$\dot{x}_1 = Ax_2 + p(x_1, x_2),$$

$$\dot{x}_2 = Bx_2 + q(x_1, x_2) \quad \text{... (3)}$$

where $x_1 \in \mathbb{R}^c, x_2 \in \mathbb{R}^s, A, B$ are constant matrices, B is Hurwitz, p, q are analytic functions vanishing at $(x_1, x_2) = (0, 0)$ together with all their first order partial derivatives, and the Taylor series expansion of $q(x_1, 0)$ begins with terms of degree at least $N + 1$, where $N \geq 2$. If the equilibrium $x_1 = 0$ of the reduced system

$$\dot{x}_1 = p(x_1, 0) \quad \text{... (4)}$$

is stable, asymptotically stable, or unstable in the Nth approximation, then the equilibrium $(x_1, x_2) = (0, 0)$ of the full system (3) is stable, asymptotically stable, or unstable respectively.

PROOF : Due to the term Ax_2 in the dynamics for $\dot{x}_1$, we cannot apply the centre manifold theorem[3] directly. Therefore, we first make a change of coordinates

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} I & -AB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then, under the new coordinates (here, we denote them by x_1 and x_2), we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} I & -AB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix} \begin{bmatrix} I & AB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} I & -AB^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} p(x_1 + AB^{-1}x_2, x_2) \\ q(x_1 + AB^{-1}x_2, x_2) \end{bmatrix}$$

i.e.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} p(x_1 + AB^{-1}x_2, x_2) - AB^{-1} q(x_1 + AB^{-1}x_2, x_2) \\ q(x_1 + AB^{-1}x_2, x_2) \end{bmatrix}$$

That is, we have

$$\dot{x}_1 = p(x_1 + AB^{-1}x_2, x_2) - AB^{-1} q(x_1 + AB^{-1}x_2, x_2),$$

$$\dot{x}_2 = Bx_2 + q(x_1 + AB^{-1}x_2, x_2). \qquad \ldots (5)$$

By the centre manifold theorem[3], the system (5) has a center manifold at the origin, the graph of a C^1 map, $x_2 = \pi(x_1)$ with π satisfying

$$\pi(0) = 0 \text{ and } D\pi(0) = 0.$$

By hypotheses, the expansion of $q(x_1, 0)$ begins with terms of degree $N + k$, where $k \geq 1$.

Our goal is to prove $\pi(x_1) = O(\|x_1\|^{N+k})$.

For this purpose, we consider the analytic function

$$F(x_1, x_2) = Bx_2 + q(x_1 + AB^{-1}x_2, x_2).$$

Then, $F(0, 0) = 0$ and

$$\frac{\partial F}{\partial x_2}(0, 0) = B.$$

Since B is Hurwitz, it is nonsingular. Therefore, it follows from the implicit function theorem [4] that there exist a $\delta > 0$ and a unique analytic function G such that, for $\|x_1\| < b$, we have

$$F(x_1, G(x_1)) = BG(x_1) + q(x_1 + AB^{-1}G(x_1), G(x_1)) = 0. \qquad \ldots (6)$$

Since the expansion of $q(x_1, 0)$ begins with terms of degree $N + k$, $q(0) = 0$, $Dq(0, 0) = 0$, and $G(x_1)$ is analytic, the expansion of $G(x_1)$ must also begin from terms of degree $N + k$.

Thus, we find that

$$(MG)(x_1) \overset{\Delta}{=} Dg(x_1)\,[p(x_1 + AB^{-1}G(x_1), G(x_1))]$$

$$= O(\|x_1\|^{N+k-1})\,[O(\|x_1\|^2) + O(\|x_1\|^{N+k})] = O(\|x_1\|^{N+k+1}).$$

Then, from centre manifold theory, it follows that

$$\pi(x_1) = G(x_1) + O(\|x_1\|^{N+k+1}) = O(\|x_1\|^{N+k}).$$

The flow on the centre manifold is governed by the equation

$$\dot{x}_1 = p(x_1 + AB^{-1}\pi(x_1), \pi(x_1)) - AB^{-1}q(x_1 + AB^{-1}\pi(x_1), \pi(x_1)),$$

i.e.

$$\dot{x}_1 = p(x_1, 0) + O(\|x_1\|^{N+k}). \qquad \text{... (7)}$$

If the equilibrium $x_1 = 0$ of the dynamics

$$\dot{x}_1 = p(x_1, 0) \qquad \text{... (8)}$$

is stable, asymptotically stable, or unstable in the Nth approximation, then it is also stable, asymptotically stable, or unstable for the system (7) respectively.

Hence, by the reduction principle of the centre manifold theory, we conclude that if the equilibrium $x_1 = 0$ of the system (8) is stable, asymptotically stable, or unstable in the Nth approximation, then the equilibrium $(x_1, x_2) = (0, 0)$ is stable, asymptotically stable, or unstable for the full system (3) respectively. This completes the proof. □

REFERENCES

1. I. G. Malkin, *Theory of Stability of Motion*, United States Atomic Energy Commission, Washington, 1959.
2. P. Chen, H. Qin, D. Cheng and Y. Hong, *IEEE Trans. Aut. Control*, **45** (2000), 2331-35.
3. A. Vanderbauwhede and S. A. Van Gils, *J. of Functional Analysis*, **72** (1987), 209-24.
4. W. Rudin, *Principles of Mathematical Analysis*, Mc-Graw Hill, Singapore, 1976.