Indian J. pure appl. Math., 34(7) : 1089-1100, July 2003

$D\delta$-SUPERCONTINUOUS FUNCTIONS

J. K. KOHLI

Department of Mathematics, Hindu College, University of Delhi, Delhi 110 007

AND

DAVINDER SINGH

Department of Mathematics, University of Delhi, Delhi 110 007

(*Received* 16 *April* 2002; *accepted* 25 *February* 2003)

A new strong form of continuity called "$D\delta$ -supercontinuity" is introduced. Sufficient conditions on domain/range are given for a continuous function to be $D\delta$-supercontinuous. Basic properties of $D\delta$ -supercontinuous functions are studied. The class of $D\delta$ -supercontinuous functions constitutes a proper subclass of each of the classes of (1) strongly θ-continuous functions due to Noiri (*J. Korean Math. Soc.*, **16** (1980), 161-66), and (2) D-supercontinuous functions (*Indian J. Pure Appl. math.*, **32(2)** (2001), 227-235); and properly contains the class of Z-supercontinuous functions (*Indian J. Pure Appl. Math,*. **33(7)** 2002, 1097-1108. A new class of spaces called "δ-completely regular spaces" is introduced. The class of δ-completely regular spaces properly includes the class of completely $G\delta$- regular spaces (*Can. J. Math.*, **36(5)** (1984), 783-794) which in turn includes the class of completely regular spaces. Moreover, the behaviour of δ-completely regular spaces under $D\delta$-super continuous functins is investigated.

Key Words : Supercontinuous function, Z-supercontinuous function; D-supercontinuous function; Strongly θ-Continuous function, D_δ-Continuous function; δ-Completely regular space; D_δ-Completely regular space

1. INTRODUCTION

Several weak and strong variants of continuity occur in the literature. The strong variants of continuity with which we shall be dealing in this paper include [1, 5, 6 8, 10, 11, 12, 13, 14]. Certain of these strong forms of continuity coincide with continuity if the domain / range space is suitably augmented. In this paper we introduce a new strong form of continuity called "D_δ-supercontinuity", which coincides with continuity if domain or range is a D_δ-completely regular space[7]. The class of D_δ-supercontinuous functions properly contains the class of Z-supercontinuous functions[5] and hence contains all clopen maps[13], all perfectly continuous functions of Noiri[12] and all strongly continuous functions initiated by Levine[8] Moreover, the class of D_δ-supercontinuous functions constitutes a proper subclass of each of the classes of

(1) Strongly θ-continuous functions[11] (which in turn constitute a proper subclass of the class of supercontinuous functions initiated by Munshi and Bassan[10] and

(2) D-supercontinuous functions[6]

Basic properties of D_δ -supercontinuous functions are elaborated in Section 3. In Section 4, we consider the notions of D_δ -quotient topology and D_δ -quotient space and compare it with the standard quotient topology, D-quotient topology[6] and Z-quotient topology[5]. The notion of a δ-completely regular space is introduced in Section 5, wherein the behaviour δ-completely regular spaces under D_δ-supercontinuous functions is investigated. In Section 6, we consider D_δ-complete regularization of a space and conclude with alternative proofs of certain results of preceeding sections.

2. Preliminaries and Basic Definitions

Definitiont 2.1[9] — A subset H of a space X is said to be a regular G_δ-set if H is an intersection of a sequence of closed sets whose interiors contain H, i.e., if $H = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \overset{\circ}{F_n}$, where each F_n is a closed subset of X. The complement of a regular G_δ-set is called a regular F_σ-set.

Definition 2.2 — A function $f : X \to Y$ from a topological space X into a topological space Y is said to be D_δ-supercontinuous at a point $x \in X$ if for every open set U containing $f(x)$ there exists a regular F_σ-set V containing x such that $f(V) \subset U$. The function f is said to be D_δ-supercontinuous if it is D_δ-supercontinuous at each $x \in X$.

Definition 2.3[8] — A function $f : X \to Y$ is said to be strongly continuous if $f(\overline{A}) \subset f(A)$ for all $A \subset X$.

Definition 2.4[12] — A function $f : X \to Y$ is said to be perfectly continuous if for every open set $V \subset Y, f^{-1}(V)$ is clopen in X.

Definition 2.5[1] — A function $f : X \to Y$ is said to be completely continuous if for each open set $V \subset Y, f^{-1}(V)$ is regularly open.

Definition 2.6[13] — A function $f : X \to Y$ is said to be clopen if for each $x \in X$ and each open set $V \subset Y$ containing $f(x)$ there is a clopen set $U \subset X$ with $x \in U$ and $f(U) \subset V$.

Definition 2.7[11] — A function $f : X \to Y$ is said to be strongly θ-continuous if for each $x \in X$ and each open set V containing $f(x)$ there is an open set $U \subset X$ with $x \in U$ and $f(\overline{U}) \subset V$.

Definition 2.8[5] — A function $f : X \to Y$ is said to be Z-supercontinuous if for each $x \in X$ and each open set U containing $f(x)$ there exists a cozero set V containing x such that $f(V) \subset U$.

Definition 2.9[2] — A collection β of subsets of a space X is called an open complementary system if it consists of open sets such that for every $B \in \beta$ there exist $B_1, B_2, \ldots\ldots \in \beta$ with $B = \bigcup \{ X \setminus B_i : i \in N \}$.

Definition 2.10[2] — A subset F of a space X is called a strongly open F_σ-set if there exists a countable open complementary system $\beta(F)$ with $F \in \beta(F)$. The complement of a strongly open F_σ-set is called a strongly closed G_δ-set.

Definition 2.11[14] — A function $f: X \to Y$ is said to be D^*-supercontinuous if for each open set $U \subset Y$ containing $f(x)$ there exists a strongly open F_σ-set V containing x such that $f(V) \subset U$.

Definition 2.12[10] — A function $f: X \to Y$ is said to be supercontinuous if for each $x \in X$ and each open set $U \subset Y$ containing $f(x)$ there exists an open set N containing x such that $f((\overline{N})^o) \subset U$.

Definition 2.13[6] — A function $f: X \to Y$ is said to be D-supercontinuous if for each $x \in X$ and each open set U containing $f(x)$ there exists an open F_σ-set V containing x such that $f(V) \subset U$.

The following diagram well illustrates the relationships that exist among D_δ - supercontinuous functions and other strong variants of continuity defined above.

strongly continuous

⇓

perfectly continuous ⇒ **completely continuous**

⇓

clopen map

⇓ ⇓

Z – supercontinuous

⇙ ⇘

D* - supercontinuous D_δ - supercontinuous ⇒ strongly θ - continuous ⇒ supercontinu

⇘ ⇙ ⇙

D – supercontinuous ⇒ **continuous**

However, none of the above implications is reversible as will be shown in the sequel. Noiri[2] gave examples to show that a clopen map need not be perfectly continuous and that a perfectly continuous map need not be strongly continuous. Moreover, Noiri showed that a completely continuous map need not be perfectly continuous. In[6] it is shown that the notion of a D-supercontinuous function is independent of the notions of supercontinuous function, strongly θ-continuous function and completely continuous function. For examples of D-supercontinuous function but not D^*-supercontinuous, D^*-supercontinuous function but not Z-supercontinuous, Z-supercontinuous but not clopen and supercontinuous function but not Z-supercontinuous (see [14])

Example 2.14 — Consider the space A defined on page 504 of *E.* Hewitt[3] which is a D_δ-completely regular space (see Definition 3.2) but not completely regular. it turns out that the identity function defined on A is D_δ-supercontinuous but not Z-supercontinuous.

Example 2.15 — Let $X = N$ be the set of positive integers equipped with cofinite topology. Then X is a D-completely regular space[2] but not D_δ-completely regular. The identity function on X is a D^*-supercontinuous function but not D_δ-supercontinuous.

Example 2.16 — Let $X = Y$ be the mountain chain space due to Heldermann[2] which is a regular space but not a D_δ -completely regular space. The identity map from X onto Y is a strongly θ-continuous function which is not D_δ-supercontinuous.

3. Basic Properties of $D\delta$-Supercontinuous Functions

A set G in a topological space X is said to be d_δ-open if for each $x \in G$, there exists a regular F_σ-set H such that $x \in H \subseteq G$. The complement of a d_δ-open set will be referred to as a d_δ-closed set.

***Theorem* 3.1** — *For a function $f: X \to Y$, the following statements are equivalent.*

(a) f is D_δ-supercontinuous.

(b) Inverse image of every open subset of Y is a d_δ-open set in X.

(c) Inverse image of every closed subset of Y is a d_δ-closed set in X.

(d) For each point x of X and for each open set U containing $f(x)$, there is a d_δ-open set V containing x such that $f(V) \subset U$.

PROOF : Easy.

Definition 3.2[7] — A space X is called a D_δ-completely regular if X has a base consisting of regular F_σ-sets.

Since in a D_δ-completely regular space every open set is d_δ-open, in view of the above theorem it follows that every continuous function defined on a D_δ-completely regular space is D_δ-supercontinuous.

Definition 3.3 — Let X be a topological space and let $A \subset X$. A point $x \in X$ is said tobe a d_δ-**adherent point** of A if every regular F_σ-set containing x intersects A. Let $[A]_{d\,\delta}$ denote the set of all d_δ-adherent points of A. Clearly a set A is d_δ-closed if and only if $[A]_{d\,\delta} = A$.

***Theorem* 3.4** — *A function $f: X \to Y$ is D_δ-supercontinuous if and only if $f([A]_{d\,\delta}) \subset \overline{f(A)}$ for every set $A \subset X$.*

PROOF : Suppose f is D_δ-supercontinuous. Since $\overline{f(A)}$ is closed in Y, by Theorem 3.1 $f^{-1}(\overline{f(A)})$ is a d_δ-closed in X. Again, since $A \subset f^{-1}(\overline{f(A)})$, $[A]_{d\,\delta} \subset [f^{-1}(\overline{f(A)})]_{d\,\delta} = f^{-1}(\overline{f(A)})$ and so $f([A]_{d\,\delta}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$.

Conversely, suppose $f([A]_{d\,\delta}) \subset \overline{f(A)}$ for every set $A \subset X$. Let F be any closed set in Y. Then $f([f^{-1}(F)]_{d\,\delta}) \subset \overline{f(f^{-1}(F))} \subset \overline{F} = F$ and hence $[f^{-1}(F)]_{d\,\delta} \subset f^{-1}(F)$. Thus $[f^{-1}(F)]_{d\,\delta} = f^{-1}(F)$ which shows that f is D_δ-supercontinuous.

***Theorem* 3.5** — *A function f from a space X into a space Y is D_δ-supercontinuous if and only if $[f^{-1}(B)]_{d\,\delta} \subset f^{-1}(\overline{B})$ for every set $B \subset Y$.*

PROOF : Suppose f is D_δ-supercontinuous. By Theorem 3.1 $f^{-1}(\overline{B})$ is d_δ-closed in X for every $B \subset Y$ and so $f^{-1}(\overline{B}) = [f^{-1}(\overline{B})]_{d\,\delta}$.

Conversely, let F be any closed set in Y. Then $[f^{-1}(F)]_{d\delta} \subset f^{-1}(\overline{F}) = f^{-1}(F)$. Again, since $f^{-1}(F) \subset \overline{f^{-1}(F)} \subset [f^{-1}(F)]_{d\delta}, f^{-1}(F) = [f^{-1}(F)]_{d\delta}$ which in turn implies that f is D_δ-supercontinuous.

Definitin 3.6 — A filter base $\mp$ is said to d_δ-converge to a point x (written an $\underset{\mp \to x}{d_\delta}$) if every regular F_δ-set containing x contains a member of $\mp$.

***Theorem* 3.7** — *A function $f: X \to Y$ is D_δ-supercontinuous if and only if for each $x \in X$ and each filter base $\mp$ that d_δ-converges to $x, f(\mp) \to f(x)$.*

PROOF : Assume that f is D_δ-supercontinuous and let $\mp \to x$. Let W be an open set containing $f(x)$. Then $x \in f^{-1}(W)$ and $f^{-1}(W)$ is d_δ-open. Let H be a regular F_σ-set such that $x \in H \subset f^{-1}(W)$ and so $f(H) \subset W$. Since $\underset{\mp \to x}{d_\delta}$, there exists $U \in \mp$ such that $U \subset H$ and hence $f(U) \subset f(H) \subset W$. Thus $f(\mp) \to f(x)$.

Conversely, let W be an open subset of Y containing $f(x)$. Now the filter base $\aleph x$ consisting of all regular F_δ-sets containing x d_δ-converges to x and so by hypothesis $f(\aleph x) \to f(x)$. Hence there exists a member $f(N)$ of $f(\aleph x)$ such that $f(N) \subset W$. Since $N \in \aleph x, N$ is a regular F_σ-set containing x. Thus f is D_δ-supercontinuous.

***Theorem* 3.8** — *If $f: X \to Y$ is D_δ-supercontinuous and $f(X)$ is endowed with the subspace topology, then $f: X \to f(X)$ is D_δ-supercontinuous.*

PROOF : Since $f: X \to Y$ is D_δ-supercontinuous, for every open subset U of Y, $f^{-1}(U \cap f(X)) = f^{-1}(U) \cap f^{-1}(f(X)) = f^{-1}(U) \cap X = f^{-1}(U)$ is d_δ-open. Hence $f: X \to f(X)$ is D_δ-cupercontinuous.

Definition 3.9 — A function $f: X \to Y$ is said to be d_δ-open (d_δ-closed) if $f(A)$ is open (closed) in Y for every regular F_σ-set (regular G_δ-set) A in X.

***Theorem* 3.10** — *If $f: X \to Y$ is D_δ-supercontinuous and $g: Y \to Z$ is continuous, then gof is D_δ-supercontinuous. In particular, the composition of D_δ-supercontinuous functions is D_δ-supercontinuous. Further, if gof is D_δ-supercontinuous and f is a d_δ-open surjection, then g is continuous.*

PROOF : Let G be an open subset of Z. Then $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$ is d_δ-open in X and hence gof is D_δ-supercontinuous.

For the proof of the last part of theorem, observe that since f is a d_δ-open surjection, $f[f^{-1}(g^{-1}(G))] = g^{-1}(G)$ is open in Y. This proves that g is continuous.

Remark 3.11 : In general D_δ-supercontinuity of gof need not imply even the continuity of f. For example let X be the real line with cofinite topology, $Y = \{0, 1\}$ be the two point Sierpinski space [15] and let $f: X \to Y$ be defined by

$$f(x) = \begin{cases} 0, \text{ if x is irrational} \\ 1, \text{ if x is rational} \end{cases}$$

Let $Z = \{0, 1\}$ be endowed with the indiscrete topology and let $g: Y \to Z$ be the identity map. Then gof and g are D_δ-supercontinuous. However, f is not continuous.

Corollary 3.12 — Let $f: X \to Y$ be D_δ-supercontinuous. If Z is a space containing Y as subspace, then the function $h: X \to Z$ defined by $h(x) = f(x)$ for each $x \in X$ is D_δ-supercontinuous.

PROOF : Since h is the composition of D_δ-supercontinuous function $f: X \to Y$ and the inclusion mapping $i: Y \to Z$, by above Theorem, it follows that h is D_δ-supercontinuous.

***Theorem* 3.13** — *Let $f: X \to Y$ be a d-open, D_δ-supercontinuous surjection and let $g: Y \to Z$ be any function. Then gof is D_δ-supercontinuous if and only if g is continuous.*

PROOF : Sufficiency is obvious. To prove necessity, let gof be D_δ-supercontinuous and let G be an open subset of Z. Then $(gof)^{-1}(G) = f^{-1}(g^{-1}(G))$ is d_δ-open in X Since f is a d_δ-open surjection, $f[f^{-1}(g^{-1}(G))] = g^{-1}(G)$ is open. Hence g is continuous.

***Theorem* 3.14** — *Let $\{f_\alpha : X \to X_\alpha \mid \alpha \in \Lambda\}$ be a family of functions and let $f: X \to \Pi_{\alpha \in \Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$. Then f is D_δ-supercontinuous if an only if each $f_\alpha : X \to X_\alpha$ is D_δ-supercontinuous.*

PROOF : Let $f: X \to \Pi_{\alpha \in \Lambda} X_\alpha$ be D_δ-supercontinuous. Then $f_\alpha = p_\alpha$ of, where p_α denotes the projection of X onto the αth-coordinate space X_α. Hence by Theorem 3.10, each f_α is D_δ-supercontinuous.

Conversely, suppose that each $f_\alpha : X \to X_\alpha$ is D_δ-supercontinuous. To show that the function f is D_δ-supercontinuous, in view of Theorem 3.1 it is sufficient to show that $f^{-1}(U)$ is d_δ-open for each open set U in the product space $\Pi_{\alpha \in \Lambda} X_\alpha$. Since the finite intersections and arbitrary unions of d_δ-open sets are d_δ-open, it suffices to prove that $f^{-1}(S)$ is d_δ-open for every subbasic open set S in the product space $\Pi_{\alpha \in \Lambda} X_\alpha$. Let $U_\beta \times \Pi_{\alpha \in \Lambda} X_\alpha$ be a subbasic open set in ΠX_α. Then $f^{-1}(U_\beta \times \Pi_{\alpha \in \Lambda} X_\alpha) = f^{-1}(p_\beta^{-1}(U_\beta)) = f_\beta^{-1} U_\beta)$ is d_δ-open. Hence f is D_δ-supercontinuous.

***Theorem* 3.15** — *Let $f: X \to Y$ be a function and $g: X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. Then g is D_δ-supercontinuous if and only if f is D_δ-supercontinuous and X is a D_δ-completely regular space.*

PROOF : To prove necessity, suppose that g is D_δ-supercontinuous. By Theorem 3.10 $f = p_y \, o \, g$ is D_δ-supercontinuous, where p_y is the projection from $X \times Y$ onto Y. Let U be any open set in X and let $x \in U$. Then $U \times Y$ is an open set containing $g(x)$. Since g is D_δ-supercontinuous, there exists a regular F_σ-set W containing x such that $g(W) \subset U \times Y$. Thus $x \in W \subset U$, which shows that U is d_δ-open and so X is a D_δ-completely regular space.

To prove sufficiency, let $x \in X$ and let W be an open set continuing $g(x)$. There exist open sets $U \subset X$ and $V \subset Y$ such that $(x, f(x)) \in U \times V \subset W$. Since X is D_δ-completely regular, there exists a regular F_σ-set G_1 in X such that $x \in G_1 \subset U$. Since f is D_δ-supercontinuous, there exists a regular F_σ-set G_2 in X containing x such that $f(G_2) \subset V$. Let $G = G_1 \cap G_2$, then G is a regular F_σ-set containing x and $g(G) \subset U \times V \subset W$, which implies that g is D_δ-supercontinuous.

Definition 3.16[7] — A function $f: X \to Y$ is said to be D_δ-continuous if for each $x \in X$ and each regular F_σ-set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.

Lemma 3.17 — For a function $f: X \to Y$, the following statements are equivalent.

(*a*) f is D_δ-continuous.

(*b*) $f(\overline{A}) \subseteq [f(A)]_{d\delta}$ for all $A \subseteq X$.

(*c*) $\overline{f^{-1}(B)} \subseteq f^{-1}([B]_{d\delta})$ for all $B \subseteq Y$.

(*d*) Inverse image of every d_δ-closed set is closed.

(*e*) Inverse image of every d_δ-open set is open.

PROOF : (*a*) $\Rightarrow$ (*b*) : Let $y \in f(\overline{A})$. Choose $x \in \overline{A}$ such that $f(x) = y$. Let V be a regular F_σ-set containing y. Since f is D_δ-continuous, $f^{-1}(V)$ is an open set containing x. This

$f^{-1}(V) \cap A \neq \phi$ which in turn implies that $V \cap f(A) \neq \phi$ and consequently $y \in [f(A)]_{d\delta}$. Hence $f(\overline{A}) \subset [f(A)]_{d\delta}$.

(*b*) $\Rightarrow$ (*c*) Let B any subset of Y. Then $f(\overline{f^{-1}(B)}) \subseteq [f(f^{-1}(B))]_{d\delta}$ and consequently $\overline{f^{-1}(B)} \subseteq f^{-1}([B]_{d\delta})$.

(*c*) $\Rightarrow$ (*d*) Since a set A is d_δ-closed if and only if $A = [A]_{d\delta}$, therefore the implication (*c*) $\Rightarrow$ (*d*) is obvious.

(*d*) $\Rightarrow$ (*e*) Obvious.

(*e*) $\Rightarrow$ (*a*) This is immediate since every regular F_σ-set is d_δ-open and since a function is D_δ-continuous if and only if the inverse image of every regular F_σ-set is open.

***Theorem* 3.18** — *Let X, Y and Z be topological spaces and let the function $f: X \to Y$ be D_δ-continuous and $g: Y \to Z$ be D_δ-supercontinuous. Then $gof : X \to Z$ is continuous.*

PROOF : It is immediate in view of above Lemma and Theorem 3.1.

However, if $f: X \to Y$ is D_δ-continuous and $gof: X \to Z$ continuous, $g: Y \to Z$ may not be D_δ-supercontinuous.

Example 3.19 — Let R be the real line endowed with the countable topology.

Let $X = \{a, b\}$, $\tau = \{\phi, X, \{a\}\}$.

Let $f: R \to X$ be defined by:

$$f(x) = \begin{cases} a, & \text{if } x \text{ is irrational.} \\ b, & \text{if } x \text{ is rational.} \end{cases}$$

Let $Y = \{c, d\}$, $\mathcal{J} = \{\phi, Y, \{d\}\}$. Let $g : X \to Y$ be defined $g(a) = d$, $g(b) = c$. Then $f : R \to X$ is D_δ-continuous and $gof : R \to Y$ is continuous but $g : X \to Y$ is not D_δ-supercontinuous.

4. D_δ-QUOTIENT TOPOLOGY AND D_δ-QUOTIENT SPACES

Definition 4.1 — Let $f: X \to Y$ be a function from a topological space X onto a set Y. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is d_δ-open in X is called the D_δ-quotient topology and the map f is called the D_δ-quotient map.

Definition 4/2[5] — Let $f: X \to Y$ be a function from a topological sapce X onto a set Y. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is Z-open in X is called the Z-quotient topology.

We may recall that a set A in a space X is Z-open[5] (respectively, d-open[6] if it is expressible as a union of cozero sets (respectively, open F_σ-sets).

Definition 4.3[6] — Let $f: X \to Y$ be a function from a topological space X onto a set Y. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is d-open in X is called the D-quotient topology.

It is clear from the definitions that in general Z-quotient topology on Y is coarser than D_δ-quotient topology which is coarser than D-quotient topology and the same is coarser than the quotient topology on Y. All four coincide if X is a completely regular space. If X is D_δ-completely regular space, then the notions of D_δ-quotient topology, D-quotient topology and quotient topology coincide.

In case X is a D-regular space, D-quotient topology and quotient topology are identical.

Example 4.4 — Let (X, τ_c) denote the set of positive integers endowed with cofinite topology. Let $Y = X$ and let f denote the identity map on X. Then D_δ-quotient topology on Y is indiscrete while D-quotient topology is τ_c.

Example 4.5 — Let $(x, \tau) = A = k \cup \{a_+, a_-\}$ be the space as defined by Hewitt [3, p. 503-504]. Then X is a D_δ-completely regular space which is not completely regular see[7]. Let $Y = X$ and let f denote the identity map on X. Then Z-quotient topology on Y is strictly coarser than τ, since there exist regular G_δ-set in X which are not zero sets, while D_δ-quotient topology coincides with τ.

Theorem 4.6 — *Let f be a function from a topological space (X, τ_1) onto a topological space (Y, τ_2), where τ_2 is the D_δ-quotient topology on Y. Then f is D_δ-supercontinuous. Moreover, τ_2 is the finest topology on Y which makes $f: X \to Y$ D_δ-supercontinuous.*

PROOF : The D_δ-supercontinuity of f follows from the definition of D_δ-quotient topology. Let τ_3 be a topology on Y such that $f: (X, \tau_1) \to (Y, \tau_3)$ is D_δ-supercontinuous. Let G be a τ_3 open set in Y. By D_δ-supercontinuity of f, $f^{-1}(G)$ is d_δ-open in X. Now bv the definition of the D_δ-quotient topology, G is τ_2-open and hence $\tau_3 \subset \tau_2$.

Theorem 4.7 — *Let $f: X \to Y$ be a D_δ-quotient map. Then a function $g : Y \to Z$ is continuous if and only if gof is D_δ -supercontinuous.*

PROOF : If U is an open set in Z and gof is D_δ-supercontinuous, then $(gof)^{-1}(U) = f^{-1}(g^{-1}(U))$, which is d_δ-open in X. Since f is a D_δ-quotient map, $g^{-1}(U)$ is open in Y and so g is continuous. Again in view of definition of D_δ-quotient topology, converse is immediate.

5. D_δ-SUPERCONTINUOUS FUNCTIONS AND δ-COMPLETELY REGULAR SPACES

Definition 5.1[4] — A space X is said to be a G_δ-regular if for every closed G_δ-set F and a point $x \notin F$, there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$.

It turns out that every finite topological space is a G_δ-regular space which need not be a regular space.

Definition 5.2[7] — A space X is said to be δ-completely regular if for every regular G_δ-set A and a point $x \notin A$, there is a continuous function f from X into the closed unit interval [0, 1] such that $f(x) = 0$ and $f(A) = 1$.

If in Definition 5.2, 'regular G_δ-set' is replaced by 'closed G_δ-set', then we arrive at the definition of a completely G_δ-regular space [4].

completely regular

⇙ ⇘

completely G$_\delta$ - regular **regular**

⇓ ⇘ ⇓

δ - completely regular **G$_\delta$ - regular**

The notions of 'δ-completely regular space' and 'G_δ-regular space' are independent of each other.

The following diagram will illustrates the interrelations that exist among the standard separation axioms and the separation axioms defined above

None of the above implications is reversible as is shown by the following examples.

Example 5.3 — Let I be the closed interval [0, 1] endowed with usual topology u, and let H be the set of all rational numbers in the open interval (1/2, 1). Consider the space $(I, \mathcal{J})$ where $\mathcal{J}$ is the topology on I generated by $u \cup \{H\}$ taken as subbase. The space $(I, \mathcal{J})$ is a non-regular functionally Hausdorff space which is not G_δ-regular. Since the point 3/4 is a member of the open F_σ-set H but there exists no open set containing 3/4 whose closure is contained in H. However, $(I, \mathcal{J})$ is a δ-completely regular space.

Example 5.4 — Let X be the space given by Hewitt [3, p 503-04]. The space X is a regular space which is not a δ-completely regular space.

The following theorem shows that for the existence of an open D_δ-supercontinuous bijection the domain as well as the range must be a regular space.

***Theorem* 5.5** — *Let $f : X \to Y$ be a D_δ-supercontinuous, open bijection. Then X and Y both are regular spaces.*

PROOF : Let A be any closed subset of Y and let $y \notin A$. Then $f^{-1}(A) \cap f^{-1}(y) = \phi$. Since f is D_δ-supercontinuous by Theorem 3.1 $f^{-1}(A)$ is d_δ-closed and so $f^{-1}(A) = \bigcap_{\alpha \in \Lambda} F_\alpha$, where each F_α is a regular G_δ-set. Since f is one-one $f^{-1}(y)$ is a singleton and so there exist $\alpha_0 \in \Lambda$ such that $f^{-1}(y) \notin F_{\alpha 0}$. Let $F_{\alpha 0} = \bigcap_{i=1}^{\infty} H_i = \bigcap_{i=1}^{\infty} H_i^o$, where each H_i is a closed set. So there exists an n such that $f^{-1}(y) \notin H_n$. Then H_n^o and $X - H_n$ are disjoint open sets containing $f^{-1}(A)$ and $f^{-1}(y)$, respectively. Since f is an open one-one map, $f(H_n^o)$ and $f(X - H_n)$ are disjoint open sets containing A and y, respectively. Hence Y is a regular space Since regularity is a topological property and since f is a homeomorphism, X is also a regular space.

Definition 5.6 — A function $f: X \to Y$ is said to be a D_δ-homeomorphism if f is a bijection such that both f and f^{-1} are D_δ-supercontinuous.

Lemma 5.7 — A space X is δ-completely regular if and only if for every d_δ-closed set A and a point $x \notin A$, there is a continuous function f from X into the closed interval [0, 1] such that $f(x) = 0$ and $f(A) = 1$.

PROOF : Since every regular G_δ-set is d_δ-closed sufficiency is obvious. To prove necessity, let A be a d_δ-closed set in X not containing the point x. Then there exist a regular F_σ-set G containing x such that $G \cap A = \phi$. Now $X \setminus G$ is a regular G_δ-set in X. So by δ-complete regularity of X there exists a continuous function f from X into [0, 1] such that $f(x) = 0$ and $f(X \setminus G) = 1$. Since $A \subset X \setminus G, f(A) = 1$.

The following result shows that for the existence of a D_δ-homeomorphism both the domain and range must be completely regular spaces.

***Theorem* 5.8** — *If $f: X \to Y$ is a D_δ-homeomorphism, then both X and Y are regular spaces. Further, if X is a δ-completely regular space, then both X and Y are completely regular spaces.*

PROOF : Since every D_δ-homeomorphism is a homeomorphism and so an open map, first assertion is immediate in view of Theorem 5.5

Let $f: X \to Y$ be a D_δ-homeomorphism of a δ-completely regular space X onto a space Y. Let A be a closed set in Y and let $y \notin A$. Then $f^{-1}(y) = \{x\}$ is a singleton and x is a not in the d_δ-closed set $f^{-1}(A)$. By Lemma 5.7 there exists a continuous function $g: X \to [0, 1]$ such that $g(x) = 0$ and $g(f^{-1}(A)) = 1$. Let $h = gof^{-1}$. Since f is a D_δ-homeomorphism, h is well defined. Again, since f^{-1} is D_δ-supercontinuous, it follows that h is continuous. Moreover, $h(y) = 0$ and $h(A) = 1$ and thus Y is completely regular. Since f is a homeomorphism and since complete regularity is a topological invariant, X is also completely regular.

Next two results give sufficient conditions for a D_δ-supercontinuous function and a D-supercontinuous function to be Z-supercontinuous, respectively.

***Theorem* 5.9** — *If $f: X \to Y$ is a D_δ-supercontinuous function and X is a δ-completely regular space, then f is Z-supercontinuous.*

PROOF : Let $x \in X$ and V be an open set containing $f(x)$. Since f is D_δ-supercontinuous, there exists a regular F_σ-set U containing x such that $f(U) \subset V$. Since X is a δ-completely regular space, there exists a continuous functions $h: X \to [0, 1]$ such tat $h(x) = 1$ and $h(X - U) = 0$, then $h^{-1}(0, 1]$ is a co zero set containing x and contained in U and so it maps into V. This shows that f is Z-supercontinuous.

***Theorem* 5.10** — *If $f: X \to Y$ is D-supercontinuous function and X is completely G_δ-regular space, then f is Z-supercontinuous.*

PROOF : Proof is similar to Theorem 5.9 and hence omitted.

Next we quote the following results pertaining to δ-completely regular spaces which are immediate consequences of the results in Sections 4 and 5 of [4] and immediately follow on substituting P = regular G_δ-set in the respective results therein.

***Theorem* 5.11** — *A D_δ-completely regular space is completely regular if and only if it is δ-completely regular.*

***Theorem* 5.12** — *Let X be a δ-completely regular space. Let K and F be disjoint subsets of X such that K is compact and F is a regular G_δ-set. Then there exists a continuous function $f: X \to [0, 1]$ such that $f(K) = 0$ and $f(F) = 1$.*

***Theorem* 5.13** — *Let X be a δ-completely regular space. Let K be a compact G_δ-subset of X which is expressible as a countable intersection of regular F_σ-sets. Then K is a zero set in X.*

6. D_δ-COMPLETE REGULARIZATION

In this section we show that if the domain of a D_δ-supercontinuous function f is retopologized in an appropriate way, then f is simply a continuous function.

Let (X, τ) be a topological space, and let β denote the collection of all regular F_σ-subsets of (X, τ). Since the intersection of two regular F_σ-sets is regular F_σ-set, the collection β is a base for a topology τ^* on X called the **D_δ-complete regularization** of τ. Clearly $\tau^* \subset \tau$. The space (X, τ) is D_δ-completely regular if and only if $\tau^* = \tau$.

Throughout the section, the symbol τ^* will have the same meaning as in the above paragraph.

***Theorem* 6.1** — *The function $f: (X, \tau) \to (Y, \mathcal{J})$ is D_δ-supercontinuous if and only if $f: (X, \tau^*) \to (Y, \mathcal{J})$ is continuous.*

***Theorem* 6.2** — *Let (X, τ) be a topological space. Then the following are equivalent.*

(a) The space (X, τ) is D_δ-completely regular.

(b) Every continuous function from (X, τ) into a space $(Y, \mathcal{J})$ is D_δ-supercontinuous.

PROOF : $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (a)$: Take $(Y, \mathcal{J}) = (X, \tau)$. Then the identity function 1_x on X is continuous, and hence D_δ-supercontinuous. Thus by Theorem 6.1, $1_x : (X, \tau^*) \to (X, \tau)$ is continuous. Since $U \in \tau$ implies $1_x^{-1}(U) = U \in \tau^*$, $\tau \subset \tau^*$. Therefore $\tau^* = \tau$, and so (X, τ) is a D_δ-completely regular space.

Many of the results studied in Section 3 follow now from Theorem 6.1 and the corresponding standard properties of continuous functions.

***Theorem* 6.3** — *Let* $f : (X, \tau) \to (Y, \mathcal{J})$ *be a function. Then*

(a) f *is* D_δ*-continuous if and only if* $f : (X, \tau) \to (Y, \mathcal{J}^*)$ *is continuous.*

(b) f *is* d_δ*-open if and only if* $f : (X, \tau^*) \to (Y, \mathcal{J})$ *is open.*

PROOF : Obvious

In the light of Theorem 6.1 and 6.3 Theorem 3.10 can be restated as follows. If $f : (X, \tau^*) \to (Y, \mathcal{J})$ is a continuous open surjection and $g : (Y, \mathcal{J}) \to (Z, \nu)$ is a function, then g is a continuous if and only if gof is continuous and Theorem 3.18 is simply the result that the composition gof of the continuous functions $f : (X, \tau) \to (Y, \mathcal{J}^*)$ and $g : (Y, \mathcal{J}^*) \to (Z, \nu)$ is continuous.

Moreover, D_δ-quotient topology on Y determined by $f : (X, \tau) \to Y$ in Section 4 coincides with the usual quotient topology on Y determined by $f : (X, \tau^*) \to Y$.

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