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STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER INVOLVING A CERTAIN LINEAR OPERATOR

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Let the classes $S_o^*(b)$ and $K_o(b)$ consist of functions which are starlike and convex of complex order b as introduced by Nasr and Aouf[7, 8]. The main object of the present paper is to investigate the starlike and convex functions of complex order involving a certain linear operator defined by means of Hadamard product.

Key Words : linear Operator; Hadamard Product; Starlike and Convex Functions of Complex Order

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad \text{... (1.1)}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to the class A is said to be starlike of complex order $b\,(b \in C\setminus\{0\})$ if and only if $z^{-1} f(z) \neq 0\ (z \in U)$ and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \quad (z \in U). \qquad \text{... (1.2)}$$

We denote by $S_o^*(b)$ the subclass of A consisting of functions which are starlike of complex order b. Further, let $S_1^*(b)$ denote the class of functions $f(z) \in A$ satisfying

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < |b| \quad (b \in C\setminus\{0\}). \qquad \text{... (1.3)}$$

We note that $S_1^*(b)$ is a subclass of $S_o^*(b)$ (see[3]).

A function $f(z)$ belonging to the class A is said to be convex of complex order $b\,(b \in C\setminus\{0\})$ if and only if $f'(z) \neq 0\ (z \in U)$ and

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in U). \qquad \text{... (1.4)}$$

We denote by $K_o(b)$ the subclass of A consisting of functions which are convex of complex order b. Furthermore, let $K_1(b)$ denote the class of functions $f(z) \in A$ satisfying

$$\left| \frac{zf''(z)}{f'(z)} \right| < |b| \quad (b \in C \setminus \{0\}). \qquad \text{... (1.5)}$$

We note that

$$f(z) \in K_\circ(b) \Leftrightarrow zf'(z) \in S_\circ^*(b) \qquad \text{... (1.6)}$$

and

$$f(z) \in K_1(b) \Leftrightarrow zf'(z) \in S_1^*(b) \qquad \text{... (1.7)}$$

for $b \in C \setminus \{0\}$.

We also have $K_1(b) \subset K_\circ(b)$.

A function $f(z)$ belonging to the class A is said to be close-to-convex of complex order $b\,(b \in C \setminus\{0\})$ if and only if there exists a function $g(z) \in K_\circ(d)\,(d \in C\setminus\{0\})$ satisfying the condition

$$\text{Re}\left\{ 1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1 \right) \right\} > 0, \quad (z, \in U). \qquad \text{... (1.8)}$$

We denote by $C_\circ(b)$ the subclass of A consisting of functions which are close-to-convex of complex order b.

The classes $S_\circ^*(b)$ and $K_\circ(b)$ introduced and studied by Nasr and Aouf[7, 8] and the classes $S_1^*(b), K_1(b)$ and $C_\circ(b)$ introduced by Choi[3].

Remark 1 : Setting $b = 1 - \alpha\,(0 \le \alpha < 1)$, we observe that $S_\circ^*(1-\alpha) = S^*(\alpha), K_\circ(1-\alpha) = K(\alpha)$ and $C_\circ(1-\alpha) = C(\alpha)$, where $S^*(\alpha), K(\alpha)$ and $C(\alpha)$ denote the usual classes of starlike, convex and close-to-convex of real order α, respectively,

For the functions $f_j(z)\,(j = 1, 2)$ defined by

$$f_j(z) = \sum_{k=0}^{\infty} a_{k+1,j}\, z^{k+1} \qquad \text{... (1.9)}$$

let $f_1 * f_2(z)$ denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ defined by

$$f_1 * f_2(z) = \sum_{k=0}^{\infty} a_{k+1},\, a_{k+1,2}\, z^{k+1}. \qquad \text{... (1.10)}$$

Now, we define the function $\varphi(a, c, ; z)$ be defined by

$$\varphi(a\, c, ; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \; (z \in U), \qquad \text{... (1.11)}$$

for $c \neq 0, -1, -2, \ldots$, where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & (\text{if } k = 0), \\ x(x+1)\ldots(x+k-1), & (\text{if } k \in N = 1, 2, \ldots). \end{cases}$$

Corresponding to the function $\varphi(a, c,; z)$, Carlson and Shafer[2] defined a linear operator $L(a, c)$ on A by

$$L(a, c) f(z) = \varphi(a, c; z) * f(z) = \left(\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \right) * f(z) \quad \dots (1.12)$$

for $f(z) \in A$ and $c \neq 0, -1, -2, \dots$.

Remark 2 : If $c > a > 0, L(a, c) f(z)$ has integral representation

$$L(a, c) f(z) = \int_0^1 u^{-1} f(uz)\, d\mu(u),$$

where μ satisfies

$$d\mu(u) = \frac{u^{a-1}(1-u)^{c-a-1}}{\beta(a, c-a)} du$$

and $\int_0^1 d\mu(u) = 1$, where $\beta(a, c-a)$ is the familiar Beta function.

Remark 3 : For $f(z) \in A$

$$L(n+1, 1) fd(z) = \frac{z}{(1-z)^{n+1}} * f(z) = D^n f(z),$$

the symbol $D^n f(z)$ was introduced by Ruscheweyh[10] and is called the Ruscheweyh derivative of $f(z)$ of nth order.

$$L(c+1, c+2) f(z) = \frac{c+1}{z^c} \int_0^c f(t)\, dt = J_c f(z) \quad \dots (1.13)$$

where $c + 1 > 0$. The operator J_c was introduced by Bernardi[1]. In particular, the operator J_1 was studied earlier by Libera[4], and Livingston[5]

2. Main Results

In order to prove our main results, we shall require the following lemmas to be used in sequel.

Lemma 1 — (Miller and Mocanu[6]) Let $\phi(u, v)$ be complex valued function,

$\phi: D \to C, D \subset C \times C$ (C is the complex plane).

and let $u = u_1 + iu_2$, $v = v_1 + i v_2$. Suppose that the function $\phi(u, v)$ satisfies

(*i*) $\phi(u, v)$ is continuous in D;

(*ii*) $(1, 0) \in D$ and Re $\{\phi(1, 0)\} > 0$;

(*iii*) for all $iu_2, v_1) \in D$ such that $v_1 \leq -\frac{\left(1 + u_2^2\right)}{2}$, Re $\left\{ \phi(iu_2, v_1) \right\} \leq 0$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the unit disc U, such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0 \ (z \in U), \qquad \ldots (2.1)$$

then Re $\{p(z)\} > 0$ $(z \in U)$.

Lemma 2 — (Ruscheweyh and Sheil-Small[11]) — Let $\zeta(z)$ and $g(z)$ be analytic in U and satisfy $\zeta(0) = g(0) = 0$, $\zeta'(0) \neq 0$ and $g'(0) \neq 0$. Suppose that for each $\sigma(|\sigma| = 1)$ and $\rho(|\rho| = 1)$

$$\zeta(z) * \left(\frac{1 + \rho\sigma z}{1 - \sigma z}\right) g(z) \neq 0 \quad (z \in U \setminus \{0\}).$$

then for each function $F(z)$ analytic in the unit disc U and satisfying the inequality Re $\{F(z)\} > 0$ $(z \in U)$, we have

$$\operatorname{Re}\left(\frac{\zeta * G(z)}{\zeta * g(z)}\right) > 0 \quad (z \in U),$$

where $G(z) = F(z)\, g(z)$.

Lemma 3 (Ruscheweyh and Sheil-Small[11]) — Let $\zeta(z)$ be convex and $g(z)$ be starlike in U. Then for each function $F(z)$ analytic in the unit disc U and satisfying the inequality *Re* $\{F(z)\} > 0$ $(z \in U)$, we have

$$\operatorname{Re}\left(\frac{\zeta * Fg(z)}{\zeta * g(z)}\right) > 0 \quad (z \in U).$$

Applying the above lemmas we derive

***Theorem* 1** — *Let $f(z) \in S_o^*(b)$ $(b \in C \setminus \{0\})$ and let*

$$\varphi(a, c, ; z) * \left(\frac{1 + \rho\sigma z}{1 - \sigma z}\right) f(z) \neq 0 \ (z \in U \setminus \{0\}). \qquad \ldots (2.2)$$

for each $\sigma(|\sigma| = 1)$ and $\rho(|\rho| = 1)$, and for $c \neq 0, -1, -2, \ldots$. Then we have

$$L(a, c) f(z) \in S_o^*(b).$$

PROOF : It is sufficient to show that

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z(L(a, c) f(z))'}{L(a, c) f(z)} - 1\right)\right\} > 0, \qquad \ldots (2.3)$$

for $z \in U$. Since

$$1 + \frac{1}{b}\left(\frac{z(L(a, c) f(z))'}{L(a, c) f(z)} - 1\right) = 1 + \frac{1}{b}\left(\frac{L(a, c)(zf'(z))}{L(a, c) f(z)} - 1\right)$$

$$= \frac{\varphi(a, c, ; z) * [(b-1) f(z) + zf'(z)]}{\varphi(a, c, z) * bf(z)} \qquad \ldots (2.4)$$

putting $\zeta(z) = \varphi(a, c, ; z)$, $g(z) = bf(z)$ and $F(z) = 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)$ in Lemma 2, we can see that

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{z(L(a,c)f(z))}{L(a,c)f(z)} - 1\right)\right\} > 0,$$

which completes the proof of Theorem 1.

Corollary 1 — Let the function $f(z)$ defined by (1.1) be in the class $S_\circ^*(b)$ and let

$$D^n\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)f(z) \neq 0 \quad (z \in U\backslash\{0\})$$

for each $\sigma(|\sigma| = 1)$ and $\rho(|\rho| = 1)$. Then $D^n f(z)$ belongs to the class $S_\circ^*(b)$.

Corollary 2 — Let the function $f(z)$ defined by (1.1) be in the class $S_\circ^*(b)$. and let

$$L(c+1, c+2)\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)f(z) \neq 0 \quad (z \in U\backslash\{0\})$$

for each $\sigma(|\sigma| = 1)$ and $\rho(|\rho| = 1)$. Then $J_c f(z)$ belongs to the class $S_\circ^*(b)$.

Corollary 3 — Let $\varphi(a, c, ; z)$ be convex and let $f(z) \in S_1^*(b)$ $(|b| < 1)$. Then $L(a, c)f(z) \in S_\circ^*(b)$.

PROOF : From the hypothesis, we obtain

$$f(z) \in S_1^*(b) \subset S^*(0) = S^* \quad (|b| < 1).$$

By applying Lemma 3 in view of Theorem 1, we have the desirous result immediately.

***Theorem* 2** — *Let $f(z) \in K_\circ(b)$ $(b \in C\backslash\{0\})$. and let*

$$L(a, c, ; z)\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)zf'(z) \neq 0 \quad (z \in U\backslash\{0\})$$

for each $\sigma(|\sigma| = 1)$ and $\rho(|\rho| = 1)$, and for $c \neq 0, -1, -2\ldots$. Then we have

$$L(a\ c)f(z) \in K_\circ(b).$$

PROOF : Applying (1.6) and Theorem 1, we observe that

$$f(z) \in K_\circ(b) \Leftrightarrow zf'(z) \in S_\circ^*(b) \Rightarrow L(a,c)\, zf'(z) \in S_0^*(b) \Rightarrow$$

$$z(L(a,c)f(z))' \in S_\circ^*(b) \Rightarrow L(a,c)f(z) \in K_\circ(b)$$

which evidently proves Theorem 2.

Corollary 4 — Let the function $f(z)$ defined by (1.1) be in the class $K_\circ(b)$ and let

$$D^n\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)zf'(z) \neq 0 \quad (z \in U\backslash\{0\})$$

for each $\sigma(|\sigma|=1)$ and $\rho(|\rho|=1)$. Then $D^n f(z)$ belongs to the class $K_\circ(b)$.

Corollary 5 — Let the function $f(z)$ defined by (1.1) be in the class $K_\circ(b)$ and let

$$L(c+1, c+2)\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)zf'(z) \neq 0 \quad (z \in U\setminus\{0\})$$

for each $\sigma(|\sigma|=1)$ and $\rho(|\rho|=1)$.. Then $J_c f(z)$ belongs to the class $K_\circ(b)$.

Corollary 6 — Let $\varphi(a, c, ; z)$ be convex and let $f(z) \in K_1(b)$ $(|b|<1)$. Then $L(a, c)f(z) \in K_\circ(b)$.

***Theorem* 3** —*Let $f(z) \in C_\circ(b)$ $(b \in C\setminus\{0\})$ and $h(z) \in S_\circ^*(b)$ $(b \in C\setminus\{0\})$, and let*

$$\varphi(a, c, ; z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right)h(z) \neq 0 \quad (z \in U\setminus\{0\}).$$

for each $\sigma(|\sigma|=1)$ and $\rho(|\rho|=1)$, and for $c \neq 0, -1, -2, \ldots$. Then we have

$$L(a, c)f(z) \in C_\circ(b).$$

PROOF : By Theorem 1, if $h(z) \in S_\circ^*(b)$, then $L(a, c)h(z) \in S_\circ^*(b)$. It is sufficient to show that

$$\text{Re}\left\{1+\frac{1}{b}\left(\frac{(zL(a, c)f(z))'}{L(a, c)h(z)}-1\right)\right\} > 0,$$

for $z \in U$. Since

$$1+\frac{1}{b}\left(\frac{z(L(a, c)f(z))'}{L(a, c)h(z)}-1\right) = 1+\frac{1}{b}\left(\frac{zL(a, c)(zf'(z))}{L(a, c)h(z)}-1\right)$$

$$= \frac{\varphi(a, c, ; z) * [(b-1)h(z)+zf'(z)]}{\varphi(a, c, ; z) * bh(z)}$$

putting $\zeta(z) = \varphi(a, c, ; z)$, $g(z) = bh(z)$ and $F(z) = 1+\frac{1}{b}\left(\frac{zf'(z)}{h(z)}-1\right)$ in Lemma 2, we can see that

$$\text{Re}\left\{1+\frac{1}{b}\left(\frac{z(L(a, c)f(z))'}{L(a, c)h(z)}-1\right)\right\} > 0,$$

which completes the proof of Theorem 3.

Corollary 7 — Let the function $f(z)$ defined by (1.1) be in the class $C_\circ(b)$ and $h(z) \in S_\circ^*(b)$ and let

$$D^n\left(\frac{1+\rho\sigma z}{1-\sigma z}\right)h(z) \neq 0 \quad (z \in U\setminus\{0\})$$

for each $\sigma(|\sigma|=1)$ and $\rho(|\rho|=1)$. Then $J_c f(z)$ belongs to the class $C_\circ(b)$.

***Theorem* 4** — *Let the function f(z) defined by (1.1) be in the class $S_\circ^*(b)$ $(b \in C\setminus\{0\})$ and let*

$$0 < B \leq \frac{1}{2} \qquad \text{... (2.5)}$$

Then we have

$$\text{Re}\left\{\frac{f(z)}{z}\right\}^{\frac{B}{b}} > \frac{1}{2B+1} \quad (z \in U) \qquad \text{... (2.6)}$$

PROOF : If we put

$$\beta = \frac{1}{2B+1}$$

and
$$\left\{\frac{f(z)}{z}\right\}^{\frac{B}{b}} > (1-\beta)\, p(z) + \beta. \qquad \text{... (2.7)}$$

where B satisfies (2.5) then $p(z)$ is regular in the unit disc U and $p(z) = 1 + p_1 z + p_2 z^2 + ...$ from (2.7) after taking the logarithmic differentiation we have that

$$\frac{B}{b}\left\{\frac{zf'(z)}{f(z)} - 1\right\} = \frac{(1-\beta)\, zp'(z)}{(1-\beta)\, p(z) + \beta}. \qquad \text{... (2.8)}$$

and from that we have

$$1 + \frac{1}{b}\left\{\frac{zf'(z)}{f(z)} - 1\right\} = 1 = \frac{(1-\beta)\, zp'(z)}{B\,[(1-\beta)\, p(z) + \beta]} \qquad \text{... (2.9)}$$

Since $zf(z) \in S_o^*(b)$ then from (2.9) we get

$$\text{Re}\left\{1 + \frac{(1-\beta)\, p'(z)}{B\,[(1-\beta)\, p(z) + \beta]}\right\} > 0 \quad (z \in U). \qquad \text{... (2.10)}$$

Let the function $\phi(u, v)$ defined by

$$\phi(u, v) = 1 + \frac{(1-\beta)\, v}{B\,[(1-\beta)\, u + \beta]}$$

(it is noted $u = p(z)$ and $v = z\,p'(z)$). Then $\phi(u, v)$ is continuous in $D = \left(C - \frac{-\beta}{1-\beta}\right) \times C$. Also $(1, 0) \in D$ and $\text{Re}\{\phi(1, 0)\} = 1 > 0$. Furthermore, for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{(1+u_2^2)}{2}$ we have

$$\text{Re}\{\phi(iu_2, v_1)\} = 1 + \text{Re}\left\{\frac{(1-\beta)\, v_1}{B\,[(1-\beta)\, iu_2 = \beta]}\right\} = 1 + \frac{\beta(1-\beta)\, v_1}{B\,[(1-\beta)^2 u_2^2 + \beta^2]}$$

$$\leq 1 - \frac{\beta(1-\beta)(1+u_2^2)}{2B\,[(1-\beta)^2 u_2^2 + \beta^2]} = \frac{(1-\beta)\,[2B(1-\beta) - \beta]\, u_2^2}{2B\,[(1-\beta)^2 u_2^2 + \beta^2]} \leq 0$$

because $0 \leq \beta < 1$ and $B < \frac{1}{2}$. Therefore, the function $\phi(u, v)$ satisfies the conditions in Lemma 1 This proves that Re $\{p(z)\} > 0$, for $z \in U$, that is, from (2.7),

$$\text{Re}\left\{\frac{f(z)}{z}\right\}^{\frac{B}{b}} > \beta \quad (z \in U)$$

which equivalent to the statement of Theorem 4.

Remark 4 : Taking $b = (1 - \alpha) \cos \lambda \, e^{-i\lambda}$ and $B = \gamma(1 - \alpha) \cos \lambda$ in Theorem 4, we the result obtained by Obradovic and Owa[9].

Corollary 9 — Let the function $f(z)$ defined by (1.1) be in the class $K_{\circ}(b)$ $(b \in C \setminus \{0\})$, and let

$$0 < B \leq \frac{1}{2}.$$

Then we have

$$\text{Re}\left\{f'(z)\right\}^{\frac{B}{b}} > \frac{1}{2B+1} \quad (z \in U) \tag{2.11}$$

PROOF : Note that $f(z) \in K_{\circ}(b)$ if and only if $zf'(z) \in S_{\circ}^{*}(b)$. Hence, replacing $f(z)$ by $zf'(z)$ in Theorem 4, we have the assertion (2.11)

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