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ON THE DYNAMICS OF THE EXTENSION OF A SINGULAR SHIFT

G. RAMHARTER

Institut für Analysis, 114/2, *Technische Universität Wien*, *A*-1040 *Wiedner Hauptstrasse* 8-10, *Vienna, Austria*
E-mail: gerhard.ramharter+e1142@tuwien.ac.at

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We consider the two-dimensional extension W of the transformation $Tx = [1/x] - 1/x$ which induces a singular measure on the unit square, and we give a geometrical interpretation of its dynamics via a diagonally invariant lattice function which we recognize as a function on the orbit space of W. We also prove, in the spirit of P. Levy's classical work on the Gauss shift, some metrical properties of the continued fraction generated by T. As an application, we give a metrical contribution to a lattice covering problem and to one-sided Diophantine approximation.

Key Words : Singular Number-Theoretic Transformation; Frequency of Digits; Hausdorff Dimension; Lattice Covering; Functional Equation; One-sided Diophantine Approximation

1. INTRODUCTION

Let J be the half-open unit interval (0, 1]. Each $x \in J$ has a unique non-terminating continued fraction expansion

$$x = \frac{1}{\underline{a_1}} \frac{1}{\underline{a_2}} \cdots \frac{1}{\underline{a_n}} \cdots = \langle a_1, a_2, ..., a_n, \ldots \rangle \qquad \text{... (1)}$$

with partial denominators $a_n \in N \backslash \{1\}$. This continued fraction is obtained by iterating the shift $T : x \mapsto [1/x] - 1/x$ (where $[\xi]$ means the least integer not smaller than ξ). Obviously, T has the effect of skipping the leading entry $a_1(x) = [1/x]$ in the expansion (1). We remark that all relevant topics of the elementary theory of this "reduced regular" continued fraction (in Perron's terminology; see [18]) can be found in[23]. Quadratic irrationalities are characterized by (ultimately) periodic expansions with a period $\neq (2)_\infty$, and rational numbers by non-terminating expansions with $\limsup_{n \to \infty} a_n = 2$. Occasionally it will be convenient to represent rationals by terminating expansions $\langle a_1, ..., a_n \rangle$ or in the form $\langle a_1, ..., a_n, \infty, \infty, \ldots \rangle$.

This type of expansion has been used, for example, in algebraic geometry to describe the support polygon of a two-dimensional lattice in a sector about the origin (see H. Cohn[5]; similar questions were considered in[14]) or to describe one-sided analogues of the Lagrange and Markov spectra (see [19] and [6]). Danzer, Murphy and Reay[7] took sections of support polygons for a clever construction of translational lattice covering prototiles; we remark that the expansion (1) could alternatively be employed for their purpose. For interesting work on the ergodic properties of the above shift and related maps we refer to [10] and [21].

We define a two-dimensional extension W of T as follows: given any point $(x, y) \in J^2$, let

$$(\xi, \eta) = W(x, y) = (Tx, S(x, y)) \qquad \text{... (2a)}$$

where $S(x, y) = 1/([1/x] - y)$. If $x = \langle a_1, a_2, a_3 \ldots \rangle$ and $y = \langle a_0, a_{-1}, \ldots \rangle$, then obviously

$Tx = \langle a_2, a_3, \dots \rangle$, $S(x, y) = \langle a_1, a_0, a_{-1} \dots \rangle$; in other words, W is a bijection from J^2 onto itself which simply transfers the leading entry of x from the first to the second variable. Clearly, the inverse mapping is given by

$$(x, y) = W^{-1}(\xi, \eta) = (S(\eta, \xi), T\eta) \qquad \dots (2b)$$

and has the reverse effect. So the symbolic dynamics of W and W^{-1} can be understood as a shift on the family of doubly infinite sequences $(a_n)_{n \in Z}$ with entries $a_n \in N \backslash \{1\}$, these sequences being in a one-to-one correspondence with the orbits under W.

We remark that this is an analogue to Nakada's natural extension of the Gauss shift $x \mapsto 1/x - [1/x]$. Nakada's seminal paper[17] has motivated extensive work on the subject; we mention[13, 3, 4] (see also [20] for some further references).

In section 2 we present a geometric algorithm in the lattice plane which can be viewed as an inhomogeneous analogue of the Szekeres algorithm (for a description of the latter see [11], [20] or Szekeres' original paper[22]). Our algorithm sets up a one-to-one corespondence between classes of diagonally equivalent lattices and doubly infinite integer sequences, which in turn can be identified with the orbits under W (Theorem 1). The procedure is intimately connected with an inhomogeneous problem of the first kind (for the terminology and an overview of related problems the reader is referred to the survey article by R. P. Bambah[1]). This problem is concerned with the values of the lattice function $\delta(L)$ introduced with definitions (3) below.

In section 3 we discuss some metrical aspects of the dynamical process on J^2 driven by W. In contrast to Nakada's extension of the Gauss shift which has an invariant measure wih destiny $(\log 2)^{-1}(1 + xy)^{-2}\,dxdy$, the present transformation W does not preserve a regular invariant measure on its domain. In fact, almost all orbits under W (in the sense of a natural probability measure on the orbit space) concenrate near the boundary of J^2. This follows from the singularity of the expansion (1) for which we give a proof in the spirit of P. levy's classical work[15] on the regular continued fraction transformation (Theorem 2). As an immediate consequence we find that one has $\delta(L) = \infty$ for almost all lattices (Corollary). In this context we propose to investigate a funcational equation which arises as fixed point of the mapping[17]

In the final section 4 we apply the results of the preceding sections to give a contribution to the metrical theory of an asymmetric analogue of the Lagrange spectrum. This set is made up of the one-sided approximation constants $\lambda(\vartheta)$ (see (18)) which can also be represented in terms of the expansion (1). We prove a sharp metrical result about the set Θ of badly approximable numbers (i.e. numbers with $\lambda(\vartheta) < \infty$): Θ is a null-set in the sense of the Lebesgue $(t^1-$ measure, but has infinite measure with respect to any measure function $t^{1-\varepsilon}$, $\varepsilon > 0$ (Theorem 3).

2. A Geometric Interpretation of the Action of W Via a Lattice Covering Problem

We introduce a lattice function which makes sense in arbitrary dimensions. Given a (non-degenerate) lattice L, with determinant $d(L)$, in the space $\mathcal{L}$ of d-dimensional lattices, we put

$$\delta(L) = \sup_P \{\mathrm{vol}(P) \,/\, d(L) : L \text{ is not a covering lattice for } P\} \qquad \dots (3a)$$

where P runs through the d-dimensional parallelepipeds with faces parallel to the coordinate axes. Letting C denote the unit cube $(0, 1)^d$ and $\mathcal{D}$ the group of non-singular diagonal $(d \times d)$-matrices, we may replace (3a) by the equivalent definition

$$\delta(L) = \sup_{D \in \mathcal{D}} \{|\det(D)|/d(L) : L \text{ is admissible for some translate of } DC\} \quad \text{.... (3b)}$$

Note the diagonal invariance $\delta(L) = \delta(DL)$ $(L \in \mathcal{L}, D \in \mathcal{D})$, which means that δ is a function on the space $\mathcal{L}/\mathcal{D}$ of classes of diagonally equivalent lattices. This should be compared with the inhomogeneous problem of second kind involving the covering constants

$$\rho(L) = \inf_P \{\text{vol}(P) / d(L) : L \text{ is a covering lattice for } P\}$$

$$= \inf_{D \in \mathcal{D}} \{|\det(D)|/d(L) : L \text{ is not admissible for any translate of } DC\}$$

which was studied by Bambah, Dumir and Hans-Gill[2]

For the following, let $d = 2$. It will become clear that $\delta(L)$ is actually a function on the orbit space of W. To avoid tedious casework, we restrict our attention to lattices having no point in common with the coordinate axes (apart from the origin o), but there is no difficulty to eliminate this restriction. We describe a process which provides, for a given two-dimensional lattice L, the family of all lattice point free rectangles $P = DC + z, z \in \mathbb{R}^2$, which are "*extremal*" in the sense that each of their edges contains a lattice point in its relative interior. Clearly any extremal rectangle is associated with a lattice cell made up of four boundary lattice points $\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{a}'', \boldsymbol{a}'''$, say, listed in counter-clock-wise order. Note the relation $a' - a = a'' - a'''$. After suitable translation, we may assume that $\boldsymbol{a}$ is the origin $\boldsymbol{o}$, and that $\boldsymbol{o}$ is contained in the left vertical edge. There are two different types of extremal rectangles. we will say that P is of *upper type* (resp. *lower type*) if the point $\boldsymbol{a}''$ on the right vertical edge is contained in the first (resp. fourth) quadrant.

Our first aim is to show how to construct the locality of upper type rectangules starting from any chosen upper type rectangle P_0, say, with associated lattice cell $\boldsymbol{o}, \boldsymbol{a}'$ $(= a'' - a'''), a'', a'''$. Put $b_0 = a''$ and $b_1 = a'''$. After suitable diagonal transformation, we may assume that the matrix (b_0, b_1) is a lattice basis of the form $B_0 = (b_0, b_1) = \begin{pmatrix} 1 & x_0 \\ y_0 & 1 \end{pmatrix}$. Let

$$x_0 = \langle a_1, a_2, a_3 \dots \rangle \text{ and } y_0 = \langle a_0, a_{-1}, a_{-2}, \dots \rangle$$

be the respective (non-terminating) expansions. The lattice cell associated with P_0 is now $\{0, b_0 - b_1, b_0, b_1\}$. We create a new extremal rectangle P_1 - which we call the "*left neighbor*" of P_0- by moving the right vertical edge of P_0 to the left until $\boldsymbol{b}_1$ has become a vertex and then moving the upper horizontal edge upwards until it contains a new lattice point b_2. It is easy to see that one has $b_2 = -b_0 + a_1 b_1$, where $a_1 = [1/x_0]$. If $x_0 < \frac{1}{2}$ $(\Leftrightarrow a_1 \geq 3)$, then we can move the lower horizontal edge downwards until it contains the point $\boldsymbol{b}_1 - \boldsymbol{b}_2 (= \boldsymbol{b}_0 - (a_1 - 1)\boldsymbol{b}_1)$. If $x_0 > \frac{1}{2}$ $(\Leftrightarrow a_1 = 2)$, then the lower horizontal edge remains unmoved and we have $\boldsymbol{b}_1 - \boldsymbol{b}_2 = \boldsymbol{b}_0 - \boldsymbol{b}_1$. In both cases the cell $\{\boldsymbol{o}, \boldsymbol{b}_1 - \boldsymbol{b}_2, \boldsymbol{b}_1, \boldsymbol{b}_2\}$ defines the desired left neighbour rectangle P_1, and $(\boldsymbol{b}_1, \boldsymbol{b}_2)$ is a lattice basis obtained from $(\boldsymbol{b}_0, \boldsymbol{b}_1)$ by the unimodular transformation

$$B_0 U_1 = (b_0, b_1) U_1 = \begin{pmatrix} 1 & x_0 \\ y_0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} x_0 & x_0(-1/x_0 + a_1) \\ 1 & -y_0 + a_1 \end{pmatrix} = \begin{pmatrix} x_0 & x_0 x_1 \\ 1 & 1/y_1 \end{pmatrix} = (\boldsymbol{b}_1, \boldsymbol{b}_2) = B_1,$$

where $x_1 = \langle a_2, a_3, \dots \rangle = Tx_0, y_0 = \langle a_1, a_0, a_{-1}, \dots \rangle = S(x_0, y_0)$.

By iterating this procedure, we obtain an infinite sequence of left neighbors $P_0, P_1, \dots, P_n, \dots$ with associated cells $\{\boldsymbol{o}, \boldsymbol{b}_n - \boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{b}_{n+1}\}$ and bases $B_n = B_0 U_1 \dots U_n$,

where $U_n = \begin{pmatrix} 0 & -1 \\ 1 & a_n \end{pmatrix}$ and $B_n = (\boldsymbol{b}_n, \boldsymbol{b}_{n+1})$

$$= \begin{pmatrix} x_0 x_1 \dots x_{n-1} & x_0 x_1 \dots x_n \\ 1/(y_1 \dots y_{n-1}) & 1/(y_1 \dots y_n) \end{pmatrix} \quad (n \in N).$$

Extracting common factors, we arrive at a representation $B_n = D_n A_n$, with

$$D_n = \text{diag}\,(x_0 x_1 \dots x_{n-1}, 1/(y_1 \dots y_{n-1}),$$

$$A_n = \begin{pmatrix} 1 & x_n \\ y_n & 1 \end{pmatrix} = \begin{pmatrix} 1 & T^n x_0 \\ S^n(x_0, y_0) & 1 \end{pmatrix} \quad (n \in N),$$

which shows that the bases B_n are diagonally equivalent to "*reduced*" bases A_n with entries

$$(x_n, y_n) = (\langle a_{n+1}, a_{n+2}, \dots \rangle, \langle a_n a_{n-1}, \dots \rangle) = W^n(x_0, y_0) \quad (n \in N). \qquad \dots (4')$$

A perfectly analogous backward iteration of this procedure leads to an infinite sequence of extremal upper type "*right neighbors*" P_{-n}, with associated cells $\{\boldsymbol{o}, \boldsymbol{b}_{-n}, -\boldsymbol{b}_{-n+1}, \boldsymbol{b}_{-n}, \boldsymbol{b}_{-n+1}\}$ and lattice bases $B_{-n} = (\boldsymbol{b}_{-n}, \boldsymbol{b}_{-n+1})$ satisfying the recursive relations

$$B_{-n-1} = B_{-n}(U_{-n})^{-1} = \dots = B_0 (U_0)^{-1} \dots (U_{-n})^{-1}$$

$$= (D_0)^{-1} \dots (D_{-n})^{-1} A_{-n-1},$$

where $U_{-n} = \cdot \begin{pmatrix} 0 & -1 \\ 1 & a_{-n} \end{pmatrix}, A_{-n} = \begin{pmatrix} 1 & x_{-n} \\ y_{-n} & 1 \end{pmatrix},$

$$D_{-n} = diag\,(x_{-n-1} x_{-n} \dots x_{-1}, 1/(y_{-n} \dots y_{-1}),$$

$$(x_{-n}, y_{-n}) = (\langle a_{-n+1}, a_{-n+2}, \dots \rangle, \langle a_{-n}, a_{-n-1}, \dots \rangle) = W^{-n}(x_0, y_0) \, (n \in N). \quad \dots (4'')$$

Writing $\mathcal{A}(x, y) = \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}$, we have established the fundamental identity

$$A_n = \mathcal{A}(x_n, y_n) = \mathcal{A}(W^n(x_0, y_0)), \quad n \in \boldsymbol{Z}, \qquad \dots (4)$$

which reflects the announced bijection between $\boldsymbol{Z}$-indexed chains (A_n) of reduced bases of extremal upper type rectangles and W-orbits in J^2. We remark that the points $\boldsymbol{b}_n$ (associated with the sequence

$(a_n)_{n \in Z})$ are exactly the consecutive lattice points on the support polygon ("*sail*") of the lattice L in the first quadrant (a sail is the boundary of the convex hull of the lattice points in a sector or, specifically, a quadrant).

We turn to the lower type rectangles. Clearly, this case can be settled by passing to the lattice L^* obtained by reflection of L at the x-axis, and employing the "*dual*" sequence associated with L^*. The appropriate notion of duality is as follows: Given a doubly infinite sequence $\alpha = (a_n)_{n \in Z}$ of integers $a_n \geq 2$, we define the dual sequence $\alpha^* = (a^*_m)_{m \in Z}$ by writing α in the form $(..., (2)_k, r+3, (2)_l, s+3, (2)_m, ...)$ (that is, collecting strings of consecutive 2's of lengths ..., $k, l, m, \ldots \geq 0$, with intermediate integers ..., $r + 3$, $s + 3$, ... ≥ 3 (..., r, s, ... ≥ 0), and putting $\alpha^* = (..., k+3, (2)_r, l+3, (2)_s, m+3, ...)$. Note that evidently $\alpha^{**} = \alpha$. It is well known (see e.g. [19]) that the sail of L in the fourth quadrant, and so the sail of L^* in the first quadrant, are associated with the dual sequence $(a^*_m)_{m \in Z}$. Hence the identity (4), with $(a_n)_{m \in Z}$ replaced by $(a^*_m)_{m \in Z}$, carries over to an analogous chain $(P^*_m)_{m \in Z}$ of extremal lower type rectangles in L. It should be noted that selfdual sequences (a_n) give rise to identical orbits.

Summarizing, we have obtained an exhaustive description (modulo translations and diagonal equivalence) of the system of extremal rectangles which can be fit into a given lattice L, and we have proved

Theorem 1 — *Identity (4) sets up a bijection between $\mathcal{L}/\mathcal{D}$ and the orbit space of W.*

As a consequence, the function (3) can be expressed in terms of the associated orbit $(x_n, y_n)_{n \in Z}$ and its dual as follows

$$\delta(L) = \max\left\{\sup_{n \in Z} \frac{\text{vol}(P_n)}{\det(B_n)}, \sup_{m \in Z} \frac{\text{vol}(P^*_m)}{\det(B^*_m)}\right\}$$

$$= \max\left\{\sup_{n \in Z} \frac{2 - y_n}{1 - x_n y_n}, \sup_{m \in Z} \frac{2 - y^*_m}{1 - x^*_m y^*_m}\right\}. \quad \ldots (5)$$

THE SINGULARITY OF THE SHIFT T AND SOME APPLICATIONS

It is clear that the Lebesgue measure on J^2 induces a natural probability measure μ on $\mathcal{L}/\mathcal{D}$ (and so on the orbit space of W) via the representation $L = D\begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix} Z^2, (x, y) \in J^2$. This measure is absolutely continuous with respect to the usual measure on this space. As mentioned, W does not preserve a regular invariant measure on its domain. Our aim here is to prove that (μ^-) almost all orbits concentrate on the boundary of J^2. We deduce this from

Theorem 2 — *(i) Let any integer $k \geq 2$ be given. Then for all numbers $\xi = \langle a_1, a_2, \ldots \rangle \in J$ except for those in a set of Lebesgue measure 0 the average frequency*

$$f(k) = \lim_{N \to \infty} \frac{1}{N} \sum_{\substack{n=1 \\ a_n = k}}^{\infty} 1$$

exists for each integer $k \geq 2$, and one has $f(2) = \infty, f(k) = 0$ $(k \geq 3)$.

(ii) For almost all numbers the expansion (1) contains arbitrarily long strings of consecutive 2's.

Corollary — For μ-almost all two-dimensional lattices L, one has $\delta(L) = \infty$.

PROOF OF THEOREM 2 : Given a number $\xi = \langle a_1, a_2, \ldots \rangle = \langle a_1, \ldots, a_{n-1}; x_n \rangle \in J$, we consider the remainders $x_n = x_n(\xi) = 1/\langle a_n, a_{n+1}, \ldots \rangle$ $(n \in N)$ and the (reduced) convergents $p_n(\xi)/q_n(\xi) = \langle \alpha_1, \ldots, a_{n-1} \rangle$ $(n \geq 2)$. Letting $q_0 = p_1 = 0, q_1 = 1$, we have the relations $p_{n+1} = a p_n - p_{n-1}, q_{n+1} = a_n q_n - q_{n-1}$, $\xi = (x_n p_n - p_{n-1})/(x_n q_n - q_{n-1})$ $(n \in N)$. Given any integers $a_1, \ldots, a_n$ (≥ 2) and any real number $x \geq 1$, we introduce the interval $I_n(x) = I(a_1, \ldots, a_n; x) = (\langle a_1, \ldots, a_n \rangle, \langle a_1, \ldots, a_n; x_n \rangle]$ whose length is

$$\left| I(a_1, \ldots, a_n; x) \right| = \frac{1}{q_{n+1}(xq_{x+1} - q_n)}. \qquad \ldots (6)$$

For consistency, we put $I_0(x) = (0, 1/x]$. Clearly, for any $a \in N$ and any system of integers $a_1, \ldots, a_{n-1}$ $(a_j \geq 2)$, the interval $I_{n-1}(a) = I(a_1, \ldots, a_{n-1}; a)$ is the disjoint union of the intervals $I(a_1, \ldots, a_{n-1}, a_n; 1)$, $a_n = a+1, a+2, \ldots$. Accordingly

$$| I_{n-1}(a) | = \frac{1}{q_n(aq_n - q_{n-1})} = \sum_{a_n = a+1}^{\infty} \frac{1}{q_{n+1}(q_{n+1} - q_n)}. \qquad \ldots (7a)$$

Taking $a = 1$, we obtain the identity

$$(| I_0(1) | =)\ 1 = \sum_{a_1, \ldots, a_n = 2}^{\infty} \frac{1}{q_{n+1}(q_{n+1} - q_n)}. \qquad \ldots (7a)$$

We introduce two distribution functions $1 - \Psi_n$ and $1 - \Phi_n$ over $[1, \infty)$ which turn out to be interrelated by the Stieltjes integral (13). It will be proved that they are both (uniformly) convergent to the common limit 1 over any interval $[\sigma, \infty)$, $\sigma > 1$.

For real $\xi = \langle a_1, a_2, \ldots \rangle \in (0, 1)$ we consider the rational variable

$$y = y(\xi) = 1/\langle a_n, a_{n-1}, \ldots, a_1 \rangle = \frac{q_{n+1}(\xi)}{q_n(\xi)} \in (1, \infty).$$

(Here the reader is reminded of the well-known symmetry $q_n(a_1, \ldots, a_{n-1}) = q_n(a_{n-1}, \ldots, a_1)$ of continuants). Clearly, the representation of a rational number > 1 by a terminating expansion with entries from $N \setminus \{1\}$ is unique; in particular, it has well-defined length $n = n(y) \in N$. Hence, given any rational number $y = 1\langle a_n, ..., a_1 \rangle > 1$, the probability $\psi_m(y)$ to have $y_m(\xi) = y$ for a number ξ chosen at random from I is non-zero only if $m = n(y)$, and in this case $\psi_m(y)$ is given by the measure of the set $\left\{ \xi = \langle b_1, b_2, \dots \rangle \mid 1/\langle b_n, ..., b_1 \rangle = y \right\}$. Obviously, this set is the interval $I(a_1, ..., a_n; 1)$. Hence, by (6),

$$\psi_m = 0\ (m \neq n(y)); \quad \psi_n(y) = \frac{1}{q_{n+1}(q_{n+1} - q_n)}. \qquad \dots (8)$$

For each $n \in N$ we introduce a non-increasing step-function Ψ_n over $[1, \infty)$ by putting

$$\Psi(1) = 1, \quad \Psi_n(z) = 1 - \sum_{\substack{a_n, ..., a_1 = 2 \\ y = 1/\langle a_n, ..., a_1 \rangle \leq z}}^{\infty} \psi_n(y) \quad (z \in (1, \infty)).$$

Clearly, $1 - \Psi_n$ is a distribution function. Note that $\lim\limits_{z \to 1} \Psi(z) = 1$ and $\lim\limits_{z \to \infty} \Psi(z) = 0$. We may write Ψ_n as a Stieltjes integral. Given $z = 1/\langle b_n, b_{n-1}, ..., b_1; c_1, c_2, \dots \rangle > 1$ (rational z here being written in the form $1/\langle b_n, ..., b_1; \infty \dots \rangle$), we have the representation

$$\Psi_n(z) = \int_{\zeta = \infty}^{z} d\,\Psi_n(\zeta) = 1 - \sum_{\substack{a_n, ..., a_1 = 2 \\ (a_n, ..., a_1) \leq (b_n, ..., b_1)}}^{\infty} \frac{1}{q_{n+1}(q_{n+1} - q_n)} \qquad \dots (9)$$

where $q_{n+1} = q(a_1, ..., a_n), q_n = (a_1, ..., a_{n-1})$. The order relation in the above sum is to be understood in the lexicographical sense, that is, $(a_n, ..., a_1) < (b_n, ..., b_1)$ iff $\{a_n < b_n\}$ or $\{a_n = b_n, a_{n-1} < b_{n-1}\}$ or $\{a_n = b_n, a_{n-1} = b_{n-1}, a_{n-2} < b_{n-2}\}$, etc. Now, let

$$y_n = \frac{1}{\langle a, a_{n-1}, ..., a_1 \rangle} = \frac{q_{n+1}}{q_n} = a - \frac{1}{y}, \text{ with } y = y_{n-1} = \frac{1}{\langle a_{n-1}, ..., a_1 \rangle} = \frac{q_n}{q_{n-1}}.$$

Using the relation $q_{n+1} = aq_n - q_{n-1}$, we derive directly from (8)

$$\psi_n(y_n) = \frac{1}{(aq_n - q_{n-1})((a-1)(q_n - q_{n-1}))} = \frac{y(y-1)}{(ay-1)((a-1)(y-1))}\psi_{n-1}(y).$$

From the definitions of Ψ_n and Ψ_{n-1} we infer the relation

$$\Psi_n\left(7a - \frac{1}{z}\right) - \Psi_n(a) = \int_{\infty}^{z} \frac{y(y-1)}{(ay-1)((a-1)(y-1))} d\,\Psi_{n-1}(y)$$

$$(a \in N \setminus \{1\},\ z \geq 1). \qquad \dots (10)$$

The second function to be defined is

$$\Phi_n(x) = \sum_{a_1, \dots a_{n-1}=2}^{\infty} |I(a_1, \dots, a_{n-1}; a)| = \sum_{a_1, \dots a_{n-1}=2}^{\infty} \frac{1}{q_n(xq_n - q_{n-1})}. \quad \dots (11)$$

Obviously, Φ_n is strictly decreasing and infinitely differentiable over $[1, \infty)$, with $\lim_{x \to \infty} \Phi_n(x) = 0$ and $\Phi_n(1) = 1$ (cf. (7b)), hence $1 - \Phi_n$ is a distribution function. The value $\Phi_n(x)$ may be interpreted as the probability to have $x_n(\xi) \geq x$ for a real number $\xi = \langle a_1, a_2, \dots \rangle$ chosen at random from J. Hence the probability for $a_{n-1}(\xi)$ to take a prescribed value $a \in N \setminus \{1\}$ is

$$f(a) = \Phi_n(a-1) - \Phi_n(a) \quad \dots (12)$$

We can write Φ_n as a Stieljes integral in terms of Ψ_{n-1} : inserting (8), with n replaced by $n-1$, into (11) yields

$$\Phi_n(x) = \sum_{a_1, \dots, a_{n-1}=2}^{\infty} \Psi_{n-1}(y_{n-1}) \frac{y_{n-1} - 1}{xy_{n-1} - 1} = \int_{\infty}^{1} \frac{y-1}{xy-1} \, d\,\Psi_{n-1}(y). \quad \dots (13)$$

It follows immediately from this representation that all derivatives of Φ_n of even order are positive on $[1, \infty)$. In particular, Φ_n is convex. By (13) and (10) we have, for $a = 2, 3, \dots,$

$$\Phi_n(a-1) - \Phi_n(a) = \int_{\infty}^{1} \left(\frac{y-1}{(a-1)y-1} - \frac{y-1}{ay-1} \right) d\,\Psi_{n-1}(y) = \Psi_n(a-1) - \Psi_n(a).$$

Since $\Psi_n(1) = \Phi_n(1)$, this implies

$$\Psi_n(a) = \Phi_n(a) \quad (a \in N).$$

We are now going to prove that the functions Φ_{n+1} and Ψ_{n+p} coincide at an infinity of points accumulating at 1. Inserting $z = z_p = 1 + 1/p = 1/\langle (2)_p, \infty, \infty, \dots \rangle$ $(p \in N)$ into (9) gives

$$1 - \Psi_{n+p}(z_p) = \sum_{\substack{a_{n+p}, \dots, a_1 = 2 \\ (a_{n+p}, \dots, a_{n+1}; a_n, \dots, a_1) \\ \leq (2, \dots, 2; \infty, \dots, \infty)}}^{\infty} \frac{1}{q_{n+p+1}(q_{n+p+1} - q_{n+p})}$$

$$= \sum_{a_n, \dots, a_1 = 2}^{\infty} \frac{1}{q_{n+p+1}(q_{n+p+1} - q_{n+p})}$$

where $q_{n+p+1} = q((2)_p; a_n, \dots, a_1)$, $q_{n+p} = q((2)_{p-1}; a_n, \dots, a_1)$. A straigtforward induction shows that $q_{n+p+1} = (p+1)q_{n+1} - pq_n$ and $q_{n+p+1} - q_{n+p} = q_{n+1} - q_n$. Using (7b) and (11), we obtain after some simple manipulations

$$1-\Psi_{n+p}(z_p)=\sum_{a_n,\dots,a_1=2}^{\infty}\left(\frac{1}{q_{n+1}(q_{n+1}-q_n)}-\frac{1}{q_{n+1}\left(\frac{p+1}{p}q_{n+1}-q_n\right)}\right)=1-\Psi_{n+1}(z_p),$$

hence

$$\Psi_{n+p}(z_p)=\Phi_{n+1}(z_p)\quad(n,p\in N). \qquad \dots (14)$$

From (11) and the obvious identity

$$\sum_{a;a_{n-1},\dots,a_1=2}^{\infty}\left(\frac{1}{q_n\left(\left(a-\frac{1}{x}\right)q_n-q_{n-1}\right)}-\frac{1}{q_n(aq_n-q_{n-1})}\right)=\sum_{a_n,\dots,a_1=2}^{\infty}\frac{1}{q_{n+1}(xq_{n+1}-q_n)}$$

$(x\geq 1)$ we infer the recurrence

$$\Phi_{n+1}(x)=\sum_{a=2}^{\infty}\left(\Phi_n\left(a-\frac{1}{x}\right)-\Phi_n(a)\right). \qquad \dots (15)$$

We claim that, for any fixed $x>1$, one has

$$\lim_{n\to\infty}\Psi_n(x)=0. \qquad \dots (16)$$

The following identity is easily proved by induction. If $k,r,l,s,m,\dots\in N\cup\{0\}$ and $x=\langle k=3;(2)_r,l=3,(2)_s,m+3,\dots\rangle$, then $1-x=\langle(2)_{k+1};r+3,(2)_l,s+3,(2)_m,\dots\rangle$. This duality relation shows that a 2-string of length $p-3$ occurs (in the average) as frequently as a single entry p. Accordingly, there is an integer function $m(n,p)\,(\geq n)$ such that, for each integer $p\geq 3$, one has $\Psi_n(1/\langle(2)_p\rangle)=\Psi_n(z_p)\leq\Psi_{m(n,p)}(p)\,(n\in N)$. Since z_p is arbitrarily close to 1 if p is sufficiently large, the asserted relation (16) follows. Combining (16) with (14), we conclude that also $\lim_{j\to\infty}\Phi_j(x)=0\,(x>1)$. The monotonicity of Φ_n now implies that, for each $a\geq 3$, the frequency $f(a)=\Phi_n(a-1)-\Phi_n(a)$ (cf. (12)) tends to 0 as $n\to\infty$. This proves the assertion (*i*) of Theorem 2.

The value $1-\Psi_{n+p}(z_p)$ can be interpreted as the probability that a_n is followed by a string of at least p consecutive 2's. Since $\lim_{n\to\infty}(1-\Psi_{n+p}(z_p))=\lim_{n\to\infty}(1-\Phi_n(z_p))=1$ for any fixed $p>1$, assertion (*ii*) follows. The proof of Theorem 2 is complete.

PROOF OF THE COROLLARY : Formula (5) shows that $\delta(L)=\infty$ iff the doubly infinite sequence (a_n) associated with L contains arbitrarily long strings of 2's. By the preceding result this is true for μ-almost all lattices. This proves Corollary 1.

Remark : Let $\mathcal{M}$ denote the class of real convex functions over the interval $[1,\infty)$ which are in C^∞ on $(1,\infty)$. The recurrence (15) suggests to consider the mapping $\mathcal{S}$ on $\mathcal{M}$ defined by

$$\mathcal{S}\Phi(x)=\sum_{a=2}^{\infty}\left(\Phi\left(a-\frac{1}{x}\right)-\Phi(a)\right) \qquad \ldots (17)$$

which has the remarkable property that, for any $\Phi\in\mathcal{M}$ the image $\mathcal{S}\Phi$ is again in $\mathcal{M}$ The essence of our proof was the observation that the iteration $\Phi_{n+1}=\mathcal{S}\Phi_n$, starting with $\Phi_0(x)=1/x$, converges to the discontinuous limit $\hat{\Phi}(1)=1$, $\hat{\Phi}(x)=0\ (x>1)$. In particular, some routine analysis shows that $\Phi_2(x)=1-\gamma-H(2-1/x)\in\mathcal{M}$ where γ is the Euler constant and $H(x)=\frac{d}{dx}\log\Gamma(x)$ is the digamma function. It would be interesting to decide whether all iterates Φ_n can be described explicitly in terms of standard functions (such as polygamma functions) from $\mathcal{M}$ It seems certain that $\hat{\Phi}$ is the only solution of the functional equation $\Phi=\mathcal{S}\Phi$ in $\mathcal{M}$ but we have reasons to suspect that the iteration is not convergent whenever one starts with a function different from $1/x$. All this should be compared with the behaviour of the mapping $\mathfrak{Z}:\mathcal{M}\to\mathcal{M}$, $\mathfrak{Z}\Phi(x)=\sum_{a=1}^{\infty}(\Phi(a)-\Phi(a+1/x))$ studied by Levy[15]. It is known that $\mathfrak{Z}$ has a non-trivial fixed element in $\mathcal{M}$ given by $\log(1+1/x)/\log 2$, which arises as the limit of the iteration $\Phi_{n=1}=\mathfrak{Z}\Phi_n$ if one starts with $1/x$. It is also not known whether this process is globally convergent.

4. SOME METRICAL RESULTS ON ONE-SIDED DIOPHANTINE APPROXIMATION

We denote by J the set of irrational numbers in the unit interval. Given any $\vartheta\in J'$, we consider the one-sided approximation constant $\lambda(\vartheta)=\limsup\left\{\frac{1}{q(p-q\vartheta)}\mid q\in N, p\in Z, p-q\vartheta>0\right\}$.

It is known (see [19]) that this function can be expressed in terms of the expansion (1). If $\vartheta=1-\langle a_1,a_2,\ldots,a_n,\ldots\rangle$ then

$$\lambda(\vartheta)=\limsup\left\{a_n-\langle a_{n-1},a_{n-2},\ldots,a_1\rangle-\langle a_{n=1},a_{n+2},\ldots\rangle\mid n\in N\right\}. \qquad \ldots (18)$$

The spectrum Λ of values $\lambda(\vartheta)\ (\vartheta\in J')$ is a one-sided analogue to the Lagrange spectrum (see e.g.[6]) which is made up of the numbers $\bar{\lambda}(\vartheta)=\max\{\lambda(\vartheta),\lambda(1-\vartheta)\}$. it is also known (and follows easily from (18)) that $\min\Lambda=1$, and that Λ contains an entire halfray $[\rho,\infty)$, with an explicit $\rho>1$ (see Lindgren[16]). Lindgren also found a sequence of gaps with endpoints in Λ. It is an open question whether Λ contains some open interval below ρ. It seems likely that Λ is nowhere dense at least in a neighborhood of the minimum 1, perhaps with Hausdorff dimension < 1. The author[19] has proved that Λ is a perfect set. In the light of this it appears hopeless to determine the complete list of gaps in the lower spectrum.

In this section we apply the results of the previous section to show (part (i) of Th. 3) that $\lambda(\vartheta) = \infty$ holds almost everywhere, or equivalently, the set Θ of "badly approximable" numbers (that is, of numbers with $\lambda(\vartheta) < \infty$) is a nullset in the sense of the Lebesgue measure (which is based on the measure function t^1). Therefore it is natural to consider the fractional (Hausdorff) dimension which works with the scale of measure functions t^σ, $(0 <)\ \sigma < 1$ and assigns to each (Lebesgue) nullset a value in the range $0 \leq hdim \leq 1$. It turns out that, in this sense, assertion (*i*) cannot be improved; in fact, the t^σ-measure of Θ is infinite for all $\sigma < 1$ (part (*ii*) of Th. 3).

***Theorem* 3** — *(i) The set $\Theta = \{\vartheta \in J' \mid \lambda(\vartheta) < \infty\}$ of badly approximable numbers has Lebesgue measure 0.*

(ii) The Hausdorff dimension hdim (Θ) of the exceptional set Θ is equal to 1.

PROOF OF THEOREM 3 : It follows from the results of the previous section that the sequence $a_n(\vartheta)$ is unbounded almost everywhere (Note that this is not in collision with assertion (*i*) of Theorem 2 stating that the average frequency of the occurence of any $a \geq 3$ is zero a.e.). It is now clear from the representation (18) that $\lambda(\vartheta) < \infty$ holds a.e. This proves statement (i).

We proceed with the proof of (*ii*). Given $\vartheta \in J'$, let $[c_1\ \vartheta, \ldots, c_n(\vartheta), \ldots] = \frac{1}{c_1 +}\ \frac{1}{c_2 +} \cdots \frac{1}{c_n +} \cdots$ denote the ordinary (regular) continued fraction expansion of ϑ. It is well known (see e.g.[11] p. 402) that one has

$$\bar{\lambda}(\vartheta) = \limsup \{c_n + [c_{n-1}, c_{n-2}, \ldots, c_1] + [c_{n+1}, c_{n+2}, \ldots] \mid n \in N\}. \qquad \ldots (19)$$

We introduce the sets

$$\Theta_r = \{\vartheta\ J' \mid \lambda(\vartheta) \leq r + 2\},$$

$$\bar{\Theta}_r = \{\vartheta\ J' \mid \bar{\lambda}(\vartheta) \leq r + 2\},$$

$$\vec{\Theta}_r = \left\{\vartheta\ J' \mid \limsup_{n\to\infty} c_n(\vartheta) \leq r\right\}.$$

From $\bar{\lambda}(\vartheta) = \max\ \{\lambda(\vartheta), \lambda(1-\vartheta)\} \geq \lambda(\vartheta)$ and formula (19) we infer the inclusions.

$$\vec{\Theta}_r \subseteq \bar{\Theta}_r \subseteq \Theta_r.$$

An old result by Good[9] states that hdim $(\vec{\Theta}_r) \to 1$ as $r \to \infty$. Since $\Theta = \bigcup_{r=1}^{\infty} \Theta_r$, the equality hdim $(\Theta) = 1$ follows from the monotonicity of the dimension. The proof of Theoem 3 is complete.

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