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EXISTENCE OF POSITIVE SOLUTIONS OF FOURTH-ORDER SINGULAR SUPERLINEAR BOUNDARY VALUE PROBLEMS*

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This paper investigates fourth-order superlinear singular two-point boundary value problems and obtains some necessary and sufficient conditions for existence of C^2 or C^3 positive solutions on the closed interval.

Key Words : Positive Solution; Fourth-Order; Boundary Value Problem; Necessary and Sufficient Condition

1. INTRODUCTION

This paper discusses the fourth-order singular two-point boundary value problem

$$\left.\begin{aligned} &u^{(4)}(t) = f(t, u(t), u''(tt)), \qquad && 0 < t < 1, \\ &u(0) = u(1) = u''(0) = u''(1) = 0, && \end{aligned}\right\} \qquad \ldots (1)$$

where $f \in C((0), 1) \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$ and is *quasi-homogeneous* with respect to the last two variables, namely, there are constants $\lambda, \mu, \alpha, \beta; N_1, M_1, N_2, M_2, 0 < \lambda \le \mu < \infty,\ 0 \le \alpha \le \beta < 1;$ $1 < \lambda + \alpha \le \mu + \beta.\ 0 < N_1 \le 1 \le M_1;\ 0 < N_2 \le 1 \le M_2$ such that for all $0 < t < 1, u \ge 0$ and $v \le 0$,

(a_1) if $0 < c \le N_1$,

$$c^{\mu} f(t, u, v) \le f(t, cu, v) \le c^{\lambda} f(t, u, v) \qquad \ldots (2)$$

if $c \ge M_1$,

$$c^{\lambda} f(t, u, v) \le f(t, cu, v) \le c^{\mu} f(t, u, v) \qquad \ldots (3)$$

(a_2) if $0 < c \le N_2$,

$$c^{\beta} f(t, u, v) \le f(t, u, c\, v) \le c^{\alpha} f(t, u, v) \qquad \ldots (4)$$

if $c \ge M_2$,

$$c^{\alpha} f(t, u, v) \le f(t, u, c\, v) \le c^{\beta} f(t, u, v) \qquad \ldots (5)$$

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A typical function satisfying (a_1) and (a_2) is $f(t, u, v) = \sum_{i=1}^{n} p_i(t)\, u^{\alpha_i} (-v)^{\beta_i}$ where $p_i \in C((0, 1), R^+)$; $\alpha_i > 1$, $0 \le \beta_i < 1$, $i = 1, 2, ..., n$. Such boundary value problems have been termed as Lidstone problems. When f is monotone in u, Wong and Agarwal[11, 12] have given the conditions about the existence to the positive solutions and the multiple solutions. Recently, many attentions have been paid to the singular or nonsingualr fourth-order boundary value problems, and been paid to the singular or nonsingular fourth-order boundary value problems, and various forms of equation have been discussed (see [1, 2, 4-6, 8-10, 13]). When the function f involves the second derivative u''. O'Regan considered the singular case where $f(t, u, u'')$ is singular at $u = 0$ or $u'' = 0$, while in[9] singularity occurs at $t = 0$ or $t = 1$. Under the conditions that function f satisfies *quasi-homogeneous* conditions. Using a modified upper and lower solution method, Chen and Zhang[3] established necessary and sufficient conditions for existence of positive solutions to second-order sublinear boundary value problems on a half-line. When f does not involve u'', using a similar method, Wei[10] obtained necessary and sufficient conditions for existence of positive solutions to the fourth-order problem (1) in the sublinear case.

In this paper, under the conditions that f involves u'' and is superlinear on u, we define an appropriate Banach space and cone, then apply a fixed point theorem in cones to the superlinear problem (1) and obtain necessary and sufficient conditions for existence of a C^2 positive solution and a C^3 positive solution. By C^2 positive solution, we mean that a function $u \in C^2[0, 1] \cap C^4(0, 1)$ satisfying (1), and $u(t) > 0$, $u''(t) < 0$ for $t \in (0, 1)$. A C^3 solution can be similarly defined, while we need $u \in C^3[0, 1] \cap C^4(0, 1)$.

Our main results are the two following theorems.

***Theorem* 1** — *The boundary value problem (1) has a C^2 positive solution u, if and only if,*

$$\int_0^1 t(1-t) f(t, t(1-t), -1)\, dt < \infty. \qquad ... (6)$$

***Theorem* 2** — *The boundary value problem (1) has a C^3 positive solution u, if and only if,*

$$\int_0^1 f(t, t(1-t), -t(1-t))\, dt < \infty. \qquad ... (7)$$

2. Preliminary Results

To prove Theorems 1 and 2, we will finish some preliminary work. First, we state a fixed point theorem in a cone as follows :

Lemma 1[6] — Let E be a Banach space and P a cone in E. Suppose that Ω_1 and Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator satisfying

$$\| Tx \| \le \| x \| \text{ for } x \in P \cap \partial\Omega_1 \text{ and } \| Tx \| \ge \| x \| \text{ for } x \in P \cap \partial\Omega_2,$$

then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $E = \{ u \in C^2 [0, 1] : u(0) = u(1) = 0, u''(0) = u''(1) = 0 \}$. Define the norm $\| u \|$ for every $u \in E$ by

$$\| u \| = | u |_0 + | u'' |_0,$$

where $| \cdot |_0$ is the usual sup-norm for continuous functions over [0, 1]. It can be shown that E equipped with the norm $\| \cdot \|$ is a Banach space.

Let $G(t, s)$ be the Green function of the second-order boundary value problem

$$\begin{cases} -u''(t) = 0, \\ u(0) = u(1) = 0, \end{cases}$$

that is,
$$G(t, s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Let
$$h(t, s) = \int_0^1 G(t, \tau) G(\tau, s) \, d\tau.$$

Then $h(t, s)$ is the Green function of the homogeneous fourth-order boundary value problem corresponding to (1). It is easily seen that for $0 \le t, s \le 1$, $G(t, s) \le G(s, s)$, and for $1/4 \le t \le 3/4$, $0 \le s \le 1$, $G(t, s) \ge \frac{1}{4} G(s, s)$. By calculating, it can be shown that

$$h(t, s) \le t(1-t) s(1-s) \le s(1-s) \quad 0 \le t \le 1, 0 \le s \le 1 \qquad \dots (8)$$

$$h(t, s) \ge \frac{1}{12} s^2 (1-s)^2 \ge \frac{1}{4^3} s(1-s) \quad 1 | 4 \le t \le 3/4, 0 \le s \le 1. \qquad \dots (9)$$

Denote
$$P = \{ u \in E \,|\, u(t) \ge 0, u''(t) \le 0, 0 \le t \le 1$$

$$\min \left\{ u(t), \frac{1}{4} \le t \le \frac{3}{4} \right\} \ge \frac{1}{4} | u |_0; \ \min \left\{ -u''(t), \frac{1}{4} \le t \le \frac{3}{4} \right\} \ge \frac{1}{4} | u'' |_0.$$

It can be easily seen that P is a cone in E.

Next, we define an operator $T : P \to E$ by

$$(Tu)(t) = \int_0^1 h(t, s) f(s, u(s), u''(s)) \, ds, \quad u \in P. \qquad \dots (10)$$

Using the Green function, for every $u \in P$, we will have an estimate for $u(t)$ in terms of the magnitude of its second derivative, namely, for $t \in [0, 1]$,

$$u(t) = \int_0^1 G(t, s)(-u''(s)) \, ds$$

$$\le \left(\int_0^t s(1-t) \, ds + \int_t^1 t(1-s) \, ds \right) | u'' |_0$$

$$= \frac{1}{2} t(1-t) | u'' |_0. \qquad \dots (11)$$

For every $u \in P$, let c_1 and c_2 be two positive numbers such that

$$C_1 \geq \max\left\{ M_1, |u''|_0/2N_1 \right\} \text{ and } c_2 \geq \max\left\{ M_2, |u''|_0/N_2 \right\}.$$

If f is quasi-homogeneous, by using of the property (8) of $h\,(t,\ s)$, then from (a_1) and (a_2),

$$|Tu\,(t)| = \int_0^1 h(t, s) f(s, u\,(s), u''\,(s))\, ds$$

$$\leq \int_0^1 s\,(1-s) f\left(s, \frac{u\,(s)}{c_1\, s\,(1-s)}\, c_1\, s(1-s), (-1)\, c_2\, \frac{-u''\,(s)}{c_2} \right) ds$$

$$\leq \int_0^1 s\,(1-s)\, c_1^{\mu} \left(\frac{|u''|_0}{2c_1} \right)^{\lambda} c_2^{\beta} \left(\frac{|u''|_0}{c_2} \right)^{\alpha} f(s, s\,(1-s), -1)\, ds$$

$$= \left(\frac{1}{2} \right)^{\lambda} c_1^{\mu-\lambda}\, c_2^{\beta-\alpha}\, |u''|_0^{\lambda+\alpha} \int_0^1 s\,(1-s) f(s, s\,(1-s), -1)\, ds.$$

Hence, T is well defined on P provided that (6) holds.

Furthermore, under the condition that (6) holds, for $u \in P$, by using of (11) and Fubini's theorem, we have

$$\int_0^1 G\,(t, \tau) \int_0^1 G\,(\tau, s) f(s, u\,(s), u''\,(s))\, ds\, d\tau$$

$$= \int_0^1 \int_0^1 G\,(t, \tau)\, G\,(\tau, s) f(s\, u\,(s), u''\,(s))\, d\tau\, ds.$$

Indeed, (10) can be rewritten as

$$(Tu)\,(t) = \int_0^1 G\,(t, \tau) \int_0^1 G\,(\tau, s) f(s, u(s), u''\,(s))\, ds\, d\tau. \qquad \text{... (12)}$$

Then, if the integral equation $u = Tu$ has a positive solution, the boundary value problem (1) must has a positive solution.

3. Existence of C^2 Positive Solution

In this section, on the basis of some Lemmas, we prove Theorem 1.

Lemma 2 — If (6) holds, then $T\,(P) \subset P$.

PROOF : Let $u \in P$. Obviously, $(Tu)\,(t) \geq 0$ and $-(Tu)''\,(t) \geq 0$. For $1/4 \leq t \leq 3/4$, we claim that

$$(Tu)(t) \geq \frac{1}{4} | Tu |_0.$$

It follows from (10) and (12) that

$$| Tu |_0 \leq \int_0^1 G(\tau, \tau) \int_0^1 G(\tau, s) f(s, u(s), u''(s))\, ds\, d\tau. \quad \ldots (13)$$

On the other hand, for $1/4 \leq t \leq 3/4$, the property of Green function $G(t, s)$ together with (13) gives

$$(Tu)(t) \geq \frac{1}{4} \int_0^1 G(\tau, \tau) \int_0^1 G(\tau, s) f(s, u(s), u''(s))\, ds\, d\tau \geq \frac{1}{4} | Tu |_0. \quad \ldots (14)$$

Next, we claim that

$$-(Tu)''(t) \geq (1/4) | (Tu)'' |_0 \text{ for } t \in [1/4, 3/4].$$

In fact, from

$$-(Tu)''(t) = \int_0^1 G(t, s) f(s, u(s), u''(s))\, ds,$$

it follows from the property of Green function $G(t, s)$ that

$$| (Tu)'' |_0 \leq \int_0^1 G(s, s) f(s, u(s), u''(s))\, ds$$

and, for $1/4 \leq t \leq 3/4$,

$$-(Tu)''(t) \geq \frac{1}{4} \int_0^1 G(s, s) f(s, u(s), u''(s))\, ds \geq \frac{1}{4} | (Tu)'' |_0. \quad \ldots (15)$$

We now conclude that $T : P \rightarrow P$ from (14) and (15) and complete the proof. □

Lemma 3 — If (6) holds, then T is a completely continuous operator on P.

PROOF : If $u_n \in P$ and $u_n \rightarrow u_0$ in E as $n \rightarrow \infty$, then we have that $u_0 \in P$ and that $\{ \| u_n \| \}$ is bounded. This means that there exists a constant L such that $\| u_n \| \leq L$, $n \geq 1$. Then $| u_n'' |_0 \leq L$. As a result, from (11), we have

$$u_n(t) \leq \frac{L}{2} t(1 - t). \quad \ldots (16)$$

Let c_1 and c_2 be two positive numbers such that $c_1 \geq \max \{ M_1, L/2N_1 \}$ and $c_2 \geq \max \{ M_2, L/N_2 \}$. It follows from (a_1), (a_2) and (8) that

$$|(Tu_n)(t)| \le \int_0^1 s(1-s) f(s, u_n(s), u_n''(s))\, ds$$

$$\le \int_0^1 s(1-s)\, c_1^{\mu} \left(\frac{|u_n''|_0}{2c_1} \right)^{\lambda} c_2^{\beta} \left(\frac{|u_n''|_0}{c_2} \right)^{\alpha} f(s, s(1-s), -1)\, ds$$

$$\le \left(\frac{1}{2} \right)^{\lambda} c_1^{\mu-\lambda} c_2^{\beta-\alpha} L^{\lambda+\alpha} \int_0^1 s(1-s) f(s, s(1-s), -1)\, ds < \infty.$$

Now, by (6), we conclude that T is continuous on P from Lebesgue's dominant convergence theorem.

To prove T is a compact operator, we will show that for every bounded sequence $\{u_n\}$ in P, the sequence $\{Tu_n\} \subset P$ has a convergent subsequence in E. Since $\{Tu_n\}$ is bounded in E, $\left\{ |(Tu_n)''|_0 \right\}$ is bounded and hence $\{Tu_n(t)\}$ is equicontinuous. By Ascoli-Alzela's lemma, it suffices to show that $\left\{ (Tu_n)''(t) \right\}$ is equicontinuous. By Ascoli-Alzela's lemma, it suffices to show that $\{(Tu_n)''(t)\}$ is equicontinuous. Let C_0 be a positive number such that $\|u_n\| \le C_0$, $n = 1, 2, \ldots$ It follows from (11) that $u(t) \le \frac{C_0}{2} t(1-t)$. Again, choose a $c \ge \max\left\{ M_1, M_2, C_0/(2N_1), C_0/N_2 \right\}$. Then

$$(Tu)'''(t) = \int_0^t sf(s, u(s), u''(s))\, ds - \int_t^1 (1-s) f(s, u(s), u''(s))\, ds$$

$$\le \int_0^t sf(s, u(s), u''(s))\, ds + \int_t^1 (1-s) f(s, u(s), u''(s))\, ds$$

$$\le C_1 \left(\int_0^t sf(s, s(1-s), -1)\, ds + \int_t^1 (1-s) f(s, s(1-s), -1)\, ds \right)$$

$$=: F(t),$$

where $C_1 = 2^{-\lambda} c^{\mu-\lambda+\beta-\alpha} C_0^{\lambda+\alpha}$. Since, in view of (6),

$$\int_0^1 F(t)\, dt = C_1 \int_0^1 \int_0^t sf(s, u(s))\, ds\, dt + C_1 \int_0^1 \int_t^1 (1-s) f(s, u(s))\, ds$$

$$= 2C_1 \int_0^1 s(1-s) f(s, s(1-s))\, ds < \infty,$$

we have the equicontinuity of the sequence $\left\{ (Tu_n)''(t) \right\}$ from the uniform continuity of the convergent integral of $F(t)$ with respect to the Lebesgue measure over $[0, 1]$.

Therefore, T is a compact operator on P and the proof of Lemma 3 is complete. □

We are now begin to prove our Theorem 1.

PROOF OF THEOREM 1 : *Necessity*. Let $u \in C^2[0, 1] \cap C^4(0, 1)$ be a positive solution of (1), say, $u(t) \geq 0$ and $u''(t) \leq 0$ for $0 \leq t \leq 1$. It follows from $u(0) = u(1) = 0$ that $u'(0) > 0$ and $u'(1) < 0$. Consequently, there must be a positive number k such that $u(t) \geq kt(1-t)$. Notice that $u^{(4)}(t) \geq 0$ and $u''(0) = u''(1) = 0$, then $u''(t) \neq 0$ for $t \in (0, 1)$. Moreover, for $t \in (0, 1)$, $1/(-u''(t)) \geq 1/|u''|_0$. Let $c_1 \geq \max\{M_1, 1/(kN)\}$; $c_2 \geq \max\{M_2 |u''|_0, 1/N_2\}$. Then for $0 < t < 1$, $t(1-t)/(c_1 u(t)) < N_1$; $c_2/(-u''(t)) \geq M_2$. Hence,

$$f(t, t(1-t), -1) = f\left(t, c_1 \frac{t(1-t)}{c_1 u(t)} u(t), \frac{1}{c_2} \frac{c_2}{-u''(t)} u''(t)\right)$$

$$\leq c_1^{\mu} \left(\frac{t(1-t)}{c_1 u(t)}\right)^{\lambda} \left(\frac{1}{c_2}\right)^{\alpha} \left(\frac{c_2}{-u''(t)}\right)^{\beta} f(t, u(t), u''(t))$$

$$\leq c_1^{\mu-\lambda} \left(\frac{1}{k}\right)^{\lambda} c_2^{\beta-\alpha} (-u''(t))^{-\beta} f(t, u(t), u''(t)). \qquad \text{... (17)}$$

It follows from $-u''(t) > 0$ $(t \in (0, 1))$ and $u''(0) = u''(1) = 0$ that there exists $t_0 \in (0, 1)$ such that $u'''(t_0) = 0$. It is easy to see that $-u'''(t) > 0$ $(0 < t < t_0)$ and $-u'''(t) \leq 0$ $(t_0 < t < 1)$. Hence, integrating (17) t from to t_0,

$$\int_t^{t_0} [-u''(s)]^{\beta} f(s\, s(1-s), -1)\, ds \leq c_1^{\mu-\lambda} k^{-\lambda} c_2^{\beta-\alpha} [-u'''(t)].$$

Noting that $-u''(t)$ is nondecreasing on $[0, t_0]$, we have

$$[-u''(t)]^{\beta} \int_t^{t_0} f(s, s(1-s), -1)\, ds \leq c_1^{\mu-\lambda} k^{-\lambda} c_2^{\beta-\alpha} [-u'''(t)],$$

and
$$\int_t^{t_0} f(s, s(1-s), -1)\, ds \leq c_1^{\mu-\lambda} k^{-\lambda} c_2^{\beta-\alpha} \frac{-u'''(t)}{[-u''(t)]^{\beta}}. \qquad \text{... (18)}$$

In view of $\beta < 1$, integrating (18) from 0 to t_0, we get

$$\int_0^{t_0} \int_{\tau}^{t_0} \frac{f(s, s(1-s), -1)}{ds} d\tau \leq c_1^{u-\lambda} k^{-\lambda} c_2^{\beta-\alpha} \int_0^{t_0} \frac{d(-u''(t))}{[-u''(t)]^{\beta}}$$

$$= c_1^{\mu-\lambda} k^{-\lambda} c_2^{\beta-\alpha} (1-\beta)^{-1} (-u''(t_0))^{1-\beta} < \infty.$$

Then, $\int_0^{t_0} sf(s, s(1-s), -1)\,ds < \infty$. On the other hand, we can also prove $\int_{t_0}^1 (1-s) f(s, s(1-s), -1)\,ds < \infty$. The above two inequalities give $\int_0^1 s(1-s) f(s, s(1-s), -1)\,ds < \infty$.

Hence, we obtain (6) and complete the proof of the necessity.

Sufficiency : Let $\Omega_1 = \{ u \in E \mid \| u \| < r \}$, where

$$r \le \min \left\{ \left(2^{1-\lambda} \int_0^1 s(1-s) f(s, s(1-s), -1)\,ds \right)^{1/1-(\lambda+\alpha)}, 2N_1, N_2 \right\}.$$

Let $u \in \partial \Omega_1 \cap P$. Then $\| u \| = | u |_0 + | u'' |_0 = r$, and $| u |_0 \le r, | u'' |_0 \le r$. It follows from (11) that

$$u(t) \le \frac{1}{2} t(1-t) | u'' |_0 \le \frac{r}{2} t(1-t) \le N_1 t(1-t); -u''(t) \le N_2. \quad \ldots (19)$$

In view of $(a_1), (a_2)$, (8) and (19), we have

$$Tu(t = \int_0^1 h(t, s) f(s, u(s), u''(s))\,ds$$

$$= \int_0^1 h(t, s) f\left(s, \frac{u(s)}{s(1-s)} s(1-s), (-1)(-u''(s)) \right) ds$$

$$\le \int_0^1 h(t, s) \left(\frac{u(s)}{s(1-s)} \right)^{\lambda} (-u''(s))^{\alpha} f(s, s(1-s), -1)\,ds$$

$$\le 2^{-\lambda} r^{\lambda+\alpha} \int_0^1 s(1-s) f(s, s(1-s), -1)\,ds.$$

and

$$|Tu|_0 \le 2^{-\lambda} r^{\lambda+\alpha} \int_0^1 s(1-s) f(s, s(1-s), -1)\,ds, \quad u \in \partial \Omega_1 \cap P. \quad \ldots (20)$$

On the other hand,

$$-(Tu)''(t) = \int_0^1 G(t, s) f(s, u(s), u''(s))\,ds$$

$$\le \int_0^1 s(1-s) \left(\frac{u(s)}{s(1-s)} \right)^{\lambda} (-u''(s))^{\alpha} f(s, s(1-s), -1)\,ds$$

$$\leq 2^{-\lambda} r^{\lambda+\alpha} \int_0^1 s(1-s) f(s, s(1-s), -1)\, ds.$$

and so $$|(Tu)''\|_0 \leq 2^{-\lambda} r^{\lambda+\alpha} \int_0^1 s(1-s) f(s, s(1-), -1)\, ds. \quad \text{... (21)}$$

Recalling that $\lambda + \alpha > 1$, from (24) and (25), we have

$$\| Tu \| = | Tu |_0 + |(Tu)''|_0 \leq 2^{1-\lambda} r^{\lambda+\alpha} \int_0^1 s(1-s) f(s, s(1-s), -1)\, ds$$

$$\leq r = \| u \|, \quad u \in \partial \Omega_1 \cap P.$$

Set $\Omega_2 = \{ u \in E \mid \| u \| < R \}$, where

$$R \geq \max \left\{ \left[2^{\alpha - 5\lambda - 6} 3^{-2(\lambda+\alpha)} 17 \int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds \right]^{1/(1-(\lambda+\alpha))} \right.$$

$$288 M_1, \frac{9}{2} M_2.$$

Let $u \in \partial \Omega_2 \cap P$, then $\| u \| = | u |_0 + | u'' |_0 = R, | u |_0 \leq R, | u'' |_0 \leq R$. From (11), we have

$$| u |_0 \leq \frac{1}{8} | u'' |_0, \quad | u'' |_0 \geq \frac{8}{9} R. \quad \text{... (22)}$$

Also, by the definition of the cone P, we have that for $1/4 \leq t \leq 3/4$,

$$u(t) = \int_0^1 G(t, s)(-u''(s))\, ds \geq \int_{1/4}^{3/4} G(t, s)(-u''(s))\, ds$$

$$\geq \frac{1}{4^2} | u'' |_0 \int_{1/4}^{3/4} G(s, s)\, ds \geq \frac{1}{2^8} | u'' |_0,$$

and hence, $$| u |_0 \geq \frac{1}{2^8} | u'' |_0. \quad \text{... (23)}$$

Since $u \in P$, from (23), for $1/4 \leq t \leq 3/4$, we have

$$\frac{u(t)}{t(1-t)} \geq 4u(t) \geq | u |_0 \geq \frac{1}{2^8} | u'' |_0, \quad \text{... (24)}$$

and so, from (22) and (24), for $1/4 \leq t \leq 3/4$,

$$-u''(t) \geq \frac{1}{4}|u''|_0 \geq \frac{1}{4}\frac{8}{9}R = \frac{2}{9}R \geq M_2. \qquad \ldots (25)$$

$$\frac{u(t)}{t(1-t)} \geq \frac{1}{2^8}\frac{8}{9}R = \frac{1}{288}R \geq M_1. \qquad \ldots (26)$$

For $1/4 \leq t \leq 3/4$, in view of the property (9) of $h(t, s)$, from (25) and (26), we have

$$\begin{aligned}
(Tu)(t) &= \int_0^1 h(t,s) f(s, u(s), u''(s))\, ds \\
&\geq \int_{1/4}^{3/4} h(t,s) f(s, u(s), u''(s))\, ds \\
&\geq \frac{1}{4^3}\int_{1/4}^{3/4} s(1-s) f(s, u(s), u''(s))\, ds \\
&\geq \frac{1}{4^3}\int_{1/4}^{3/4} s(1-s)\left(\frac{u(s)}{s(1\ s)}\right)^{\lambda} (-u''(s))^{\alpha} f(s, s(1-s), -1)\, ds \\
&\geq \frac{1}{2^6}\left(\frac{R}{2^5 3^2}\right)^{\lambda}\left(\frac{2R}{3^2}\right)^{\alpha}\int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds \\
&= 2^{-(6+5\lambda-\alpha)} 3^{-2(\lambda+\alpha)} R^{\lambda+\alpha}\int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds.
\end{aligned}$$

and hence,
$$|(Tu)|_0 \geq 2^{-(6+5\lambda-\alpha)} 3^{-2(\lambda+\alpha)} R^{\lambda+\alpha}\int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds. \qquad \ldots (27)$$

On the other hand, from (25) and (26),

$$\begin{aligned}
-(Tu)''(t) &= \int_0^1 G(t,s) f(s, u(s), u''(s))\, ds \\
&\geq \frac{1}{4}\int_{1/4}^{3/4} s(1-s)\left(\frac{u(s)}{s(1-s)}\right)^{\lambda} (-u''(s))^{\alpha} f(s, s(1-s), -1)\, ds \\
&\geq \frac{1}{2^2}\left(\frac{R}{2^5 3^2}\right)^{\lambda}\left(\frac{2R}{3^2}\right)^{\alpha}\int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds \\
&\geq 2^{-(2+5\lambda-\alpha)} 3^{-2(\lambda+\alpha)} R^{\lambda+\alpha}\int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds.
\end{aligned}$$

and hence, $$|(Tu)''|_0 \geq 2^{\alpha-5\lambda-2}\, 3^{-2(\lambda+\alpha)} R^{\lambda+\alpha} \int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds. \quad \ldots (28)$$

Now, from (27) and (28), we arrive at

$$\|Tu\| = |(Tu)|_0 + |(Tu)''|_0$$

$$\geq 17\,(2^{-(6+5\lambda-\alpha)})\, 3^{-2(\lambda+\alpha)} R^{\lambda+\alpha} \int_{1/4}^{3/4} s(1-s) f(s, s(1-s), -1)\, ds$$

$$\geq R = \|u\|.$$

Therefore, by Lemma 2, Lemma 3 and Lemma 1, the operator T has at least one fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is a positive $C^2\,[0, 1]$ solution to the boundary value problem (1).

The proof of Theorem 1 is complete. □

4. Existence of C^3 Positive Solution

In this section, we begin to prove Theorem 2. Let c be a positive number such that $c \geq M_2$, and $\frac{t(1-t)}{c} \leq N_2$. Noting that $t^{\beta}(1-t)^{\beta} \geq t(1-t)$ for $\beta < 1$, then for $t \in (0, 1)$, we have

$$f(t, t(1-t), -t(1-t)) = f\left(t, t(1-t), -c\frac{t(1-t)}{c}\right)$$

$$\geq c^{\alpha}\left(\frac{t(1-t)}{c}\right)^{\beta} f(t, t(1-t), -1)$$

$$= c^{\alpha-\beta} t^{\beta}(1-t)^{\beta} f(t, t(1-t), -1)$$

$$\geq c^{\alpha-\beta} t(1-t) f(t, t(1-t), -1).$$

Consequently, if $\int_0^1 f(t, t(1-t), -t(1-t))\, dt < \infty$, we have $\int_0^1 t(1-t) f(t, t(1-t), -1))\, dt < \infty$. This means (7) implies (6).

Now, we suppose (7) in Theorem 2 holds. Since (7) implies (6), Theorem 1 provides a C^2 positive solution $u \in P$. From (11), $u(t) \leq (1/2)\, t(1-t)\, |u''|_0$. In order to prove the Theorem 2, we prepare some basic work. For the u that we get above, let the function $g(t, z) = f(t, u(t), -z)$. Then $g \in C((0, 1) \times [0, +\infty), [0, +\infty))$. We firstly turn to consider the second-order two-point boundary value problem

$$\begin{cases} -z''(t) = g(t, z(t)), & 0 < t < 1, \\ z(0) = z(1) = 0. \end{cases} \quad \ldots (29)$$

It is easy to see that $g(t, z)$ is also quasi-homogeneous with respect to the second variable, namely, if $0 < c \leq N_2$,

$$c^{\beta} g(t, z) \le g(t, cz) \le c^{\alpha} g(t, z)$$

$$\text{if } c > M_2, \; c^{\alpha} g(t, z) \le g(t, cz) \le c^{\beta} g(t, z)$$

where M_2, N_2, α, β are those in (4) and (5). Choose a positive number c such that $c \ge \max \left\{ M_1, |u''|_0/2N_1 \right\}$. We have

$$g(t, z) = f(t, u(t) - z) = f\left(t, c \frac{u(t)}{ct(1-t)} t(1-t), -z \right)$$

$$\le c^{\mu} \left(\frac{u(t)}{ct(1-t)} \right)^{\lambda} f(t, t(1-t), -z) \le 2^{-\lambda} c^{\mu - \lambda} |u''|_0^{\lambda} f(t, t(1-t), -z). \quad \ldots (30)$$

Consequently, by (30), we have the following Lemma.

Lemma 4 — If (7) holds, $g(t, z)$ satisfies

$$\int_0^1 g(t, t(1-t))\, dt < \infty. \quad \ldots (31)$$

Next, we define upper solution and lower solution to (29). If $p \in [0, 1] \cap C^2(0, 1)$ satisfies

$$\begin{cases} -p''(t) \ge g(t, p(t)), & 0 < t < 1, \\ p(0) \ge 0; p(1) \ge 0, \end{cases} \quad \ldots (32)$$

$p(t)$ is called a upper solution of (29). similarly we define a lo ver solution of (29) to be a function $q(t)$ in $C[0, 1] \cap C^2(0, 1)$ that satisfies

$$\begin{cases} -q''(t) \le g(t, q(t)), & 0 < t < 1, \\ q(0) \le 0; q(1) \le 0. \end{cases} \quad \ldots (33)$$

Let
$$w(t) = \int_0^1 G(t, s)\, g(s, s(1-s))\, ds$$

$$= (1-t) \int_0^t s g(s, s(1-s))\, ds + t \int_t^1 (1-s)\, g(s, s(1-s))\, ds.$$

If (31) holds, we have $w \in C^1[0, 1]$, and

$$a_1 t(1-t) \le w(t) \le a_2 t(1-t). \quad \ldots (34)$$

where
$$a_1 = \int_0^1 s(1-s)\, g(s, s(1-s))\, ds; \; a_2 = \int_0^1 g(s, s(1-s))\, ds.$$

Let $p(t) = h_1 w(t)$, where $(\max \left\{ M_2, (h_1 a_2)/N_2 \right\})^{\beta - \alpha} (h_1 a_2)^{\alpha} < h_1$. Let $q(t) = h_2 w(t)$, where $(\max \{ 1/N_2, M_2/(h_2 a_1) \})^{\alpha - \beta} (h_2 a_1)^{\alpha} > h_2$. It should be noted here that

$$h_1 a_1 t(1-t) \le p(t) \le h_1 a_2 t(1-t),$$

$$h_2 a_1 t(1-t) \le q(t) \le h_2 a_2 t(1-t).$$

By choosing h_1 and h_2 properly under above conditions, we may have $p(t) \ge q(t)$.

Lemma 5 — Suppose (7) holds, then $p(t)$ is a upper solution of (29) and $q(t)$ is a lower solution of (29).

PROOF : From Lemma 4, we can obtain $p(t)$ and $q(t)$ as above. Choose a positive number c_1 such that $c_1 \ge \max\{M_2, (h_1 a_2)/N_2\}$ and $c_1^{\beta-\alpha}(h_1 a_2)^{\alpha} \le h_1$. From the value of h_1, we know that the c_1 is available.

$$g(t, p(t) = g\left(t, c_1 \frac{p(t)}{c_1 t(1-t)} t(1-t)\right)$$

$$\le c_1^{\beta}\left(\frac{p(t)}{c_1 t(1-t)}\right)^{\alpha} g(t, t(1-t))$$

$$\le c_1^{\beta-\alpha}(h_1 a_2)^{\alpha} g(t, t(1-t))$$

$$\le h_1 g(t, t(1-t)).$$

On the other hand,

$$-p''(t) = -h_1 w''(t) = h_1 g(t, t(1-t)).$$

Hence, $-p''(t) \ge g(t, p(t))$, that means $p(t)$ is a upper solution to (29). Choose a positive number c_2 such that $c_2 \ge \max\left\{1/N_2, M_2/(h_2 a_1)\right\}$ and $c_2^{\alpha-\beta}(h_2 a_1)^{\alpha} \ge h_2$.

$$g(t, q(t)) = g\left(t, \frac{1}{c_2}\frac{c_2 q(t)}{t(1-t)} t(1-t)\right)$$

$$\ge \left(\frac{1}{c_2}\right)^{\beta}\left(\frac{c_2 q(t)}{t(1-t)}\right)^{\alpha} g(t, t(1-t))$$

$$\ge c_2^{\alpha-\beta}(h_2 a_1)^{\alpha} g(t, t(1-t))$$

$$\ge h_2 g(t, t(1-t)).$$

On the other hand,

$$-q''(t) = -h_2 w''(t) = h_2 g(t, t(1-t)).$$

Hence, $-q''(t) \le g(t, q(t))$, that means $q(t)$ is a lower solution to (29). □

Lemma 5 — If (7) holds, then (29) has a $C^1[0, 1]$ solution $z(t)$ satisfying $q(t) < z(t) \le p(t)$.

PROOF : Let

$$g^{*}(t, z(t)) = \begin{cases} g(t, p(t)), & z(t) > p(t), \\ g(t, z(t)), & q(t) \le z(t) \le p(t), \\ g(t, q(t)), & z(t) < q(t), \end{cases} \quad \text{... (35)}$$

and consider the two-point boundary value problem

$$\begin{cases} -z''(t) = g^{*}(t, z(t)), & 0 < t < 1, \\ z(0) = z(1) = 0, \end{cases} \quad \text{... (36)}$$

Denote $X = C[0, 1]$, and define an operator $A : X \to X$ by

$$(Az)(t) = \int_0^1 G(t, s)\, g^{*}(s, z(s))\, ds. \quad \text{... (37)}$$

It is easy to see that a fixed point of A in X is a positive solution of (36). By using of the definition of g^{*} and the properties of $p(t)$ and $q(t)$, in view of Lemma 4, (7) implies that A is a completely continuous operator on X. The proof is similar to that in Lemma 3. Moreover, $A(X)$ is a bounded set. Hence, Schauder's fixed point theorem implies that A has a fixed point $z \in X$. To finish the proof it remains to show $q(t) \le z(t) \le p(t)$.

For $z(t) \le p(t)$, we have $z(t) \le h_1 a_2 t(1-t)$. Choose a positive number c_1 such that $c_1 \ge \max\{M_2, (h_1 a_2)/N_2\}$ and $c_1^{\beta-\alpha}(h_1 a_2)^{\alpha} \le h_1$. Then, when $z(t) \le p(t)$, we have

$$g^{*}(t, z(t)) = g(t, z(t)) = g\left(t, c_1 \frac{z(t)}{c_1 t(1-t)} t(1-t)\right)$$

$$\le c_1^{\beta}\left(\frac{z(t)}{c_1 t(1-t)}\right)^{\alpha} g(t, t(1-t))$$

$$\le c_1^{\beta-\alpha}(h_1 a_2)^{\alpha} g(t, t(1-t))$$

$$\le h_1 g(t, t(1-t)).$$

Then, by the definition of the upper solution, for either case of $z(t) \le p(t)$ and $z(t) \ge p(t)$, we have

$$p''(t) - z''(t) = g^{*}(t, z(t)) - h_1 g(t, t(1-t)) \le 0.$$

The Maximum Principle[8] implies $z(t) \le p(t)$. Similarly, we can prove that $z(t) \ge q(t)$. Now we complete the proof. □

Obviously, $x(t) = -u''(t)$ satisfies (29), that is

$$\begin{cases} -x''(t) = g(t, x(t)), & 0 < t < 1, \\ x(0) = x(1) = 0, \end{cases}$$

Next, we will prove that $x \in C^1[0, 1]$.

Lemma 6 — Let $x(t)$ be defined as above, and (7) holds, then $x \in C^1\,[0, 1]$.

PROOF : From (7), Lemma 4 and Lemma 5 provide a solution $z \in C^1\,[0, 1]$ of (29). If $x \in C^1\,[0, 1]$, we finish the proof. So, we suppose that at least one of $x'(0)$ and $x'(1)$ does not exist. First, we suppose $x'(0) = +\infty$. Then we may suppose that there exists a point t_1 such that $x(t) \geq z(t)$ for $0 \leq t \leq t_1 \leq 1$, and $x(t_1) = z(t_1)$. It follows from $x'(0) = +\infty$ that for any positive number M, there must exist a neighbourhood $(0, \delta)$ such that $x(t) \geq Mz(t)$ $(t \in (0, \delta))$, where $\delta < t_1$. Here, we suppose $M > M_2 \geq 1$. Let

$$y(t) = x'(t)\, z(t) - x(t)\, z'(t), \quad 0 < t < \delta.$$

Then
$$\limsup_{t \to +0} \; y(t) \geq 0. \qquad \ldots (38)$$

Noting that $x(t)/z(t) \geq M > M_2 \geq 1$ $(t \in (0, \delta))$ and $|x(t)/z(t)|^{\beta} \leq x(t)/z(t)$, then for $t \in (0, \delta)$, we have

$$g(t, x) = g\left(t, \frac{x}{z} z\right) = g\left(t, \frac{x}{z} z\right) \leq \left(\frac{x}{z}\right)^{\beta} g(t, z) \leq \frac{x}{z}\, g(t, z).$$

This means that for $t \in (0, \delta), x''(t)/x(t) \geq z''(t)/z(t)$. In view of $y'(t) = x''(t)\, z(t) - x(t)\, z''(t)$, then $y'(t) \geq 0$. From (38), $y(t) \geq 0$, $(t \in (0, \delta))$. This implies $[x(t)/z(t)]' \geq 0$ $(t \in (0, \delta))$. Then $x(t) \leq \dfrac{x(\delta)}{z(\delta)} z(t)$ for $t \in (0, \delta)$. Consequently, we have $x'(0) < +\infty$. This contradicts that we have supposed. Now we have $x \in C^1\,[0, 1)$. For the cases of $x'(1) = -\infty$, we can similarly prove as above.

□

PROOF OF THEOREM 2 : We prove the sufficiency first.

It suffices to show that $-u'' = x \in C^1\,[0, 1]$. From Lemma 6, we obtain that $x \in C^1\,[0, 1]$. That means $u \in C^3\,[0, 1]$.

To prove the necessity, let there be a C^3 positive solution u of (1). The same reasoning at the beginning of the proof of Theorem 1 asserts that there exist $m_1 > 0$ and $m_2 > 0$ such that $u(t) \geq m_1\, t(1-t)$ and $-u''(t) \geq m_2\, t(1-t)$. Let c_1 and c_2 are two positive numbers such that $c_1 \geq \max\{M_1, 1/m_1 N_1\}$ and $c_2 \geq \max\{M_2, 1/m_2 N_2\}$. Then we have

$$f(t, t(1-t), -t(1-t)) \leq c_1^{\mu}\left(\frac{t(1-t)}{c_1\, u(t)}\right)^{\lambda} c_2^{\beta}\left(\frac{t\,1-t)}{-c_2\, u''(t)}\right)^{\alpha} f(t, u(t), u''(t))$$

$$\leq c_1^{\mu-\lambda}\, c_2^{\beta-\alpha}\left(\frac{1}{m_1}\right)^{\lambda}\left(\frac{1}{m_2}\right)^{\alpha} f(t, u(t), u''(t))$$

Hence,

$$\int_0^1 f(t, t(1-t), -t(1-t))\, dt \le c_1^{\mu+\lambda} c_2^{\beta-\alpha} \left(\frac{1}{m_1}\right)^{\lambda} \left(\frac{1}{m_2}\right)^{\alpha} [u^{(3)}(1) - u^{(3)}(0)] < +\infty$$

Thus, (7) holds and the proof is complete. □

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