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NOTE ON "NEW GENERALIZATIONS OF HARDY'S INTEGRAL INEQUALITY"

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In this paper, we give some new generalizations of Hardy's integral inequalities.

Key Words: Hardy's integral inequality

1. INTRODUCTION

The classical Hardy inequality[1] states that for $f(x) \geq 0$ and $p > 1$,

$$\int_0^\infty \left[\frac{1}{x} \int_0^x f(t)\, dt \right]^p dx < q^p \int_0^\infty f^p(t)\, dt \qquad \ldots (1)$$

where $q = \left(\frac{p}{p-1} \right)$ is the best possible constant.

Yang Bicheng[2] gave generalization of (1).

$$\int_a^\infty \left[\frac{1}{x} \int_a^x f(t)\, dt \right]^p dx < q^p \int_a^\infty [1 - \theta_p(t)] f^p(t)\, dt \qquad \ldots (2)$$

where $\theta_p(t) = (1/p) \sum_{k=1}^{\infty} \binom{p}{k+1} (-1)^{k-1} (a/t)^{k/q} > 0$ for $t > a > 0$, and $\theta_p(a) = 1/q$.

Recently, James Adedayo Oguntuase[3] also gave generalization of (2).

$$\int_a^\infty \left(\frac{1}{x^{(1-\frac{1}{r})}} \int_a^x f(t)\, dt \right)^p dx < q^{(1-1/r)p} \left(1 - \frac{1}{r} \right)^{(1-1/r)p} \int_a^\infty [1 - \theta_p(t)] f^p(t)\, dt \qquad \ldots (3)$$

where

$$\theta_p(t) = \frac{1}{\left(1 - \frac{1}{r}\right)p} \sum_{k=1}^{\infty} \binom{(1-\frac{1}{r})p}{k+1} (-1)^{k-1} \left(\frac{a}{t} \right)^{\frac{k}{(1-1/r)q}} > 0 \text{ for } t > a > 0, \qquad \ldots (4)$$

and $$\theta_p(a) = \frac{1}{\left(1 - \frac{1}{r}\right)} q. \qquad \text{... (5)}$$

2. Remarks on Previous Work

In (Ref. 3), to prove the result, authors gave the following proposition.

Proposition — Let $a \geq 0, p > 1, 1/p + 1/q = 1 - 1/r, f \geq 0, r > 1$, and $0 < \int_a^\infty f^p(t)\,dt < \infty$. Then there exists a real number $x_0 \in (a, \infty)$ such that for any $x > x_0$, the following inequality is true:

$$\int_a^x f(t)\,dt < q^{1/q}\left(1 - \frac{1}{r}\right)^{1/q}\left(\frac{1}{x^{(1-1/r)q}} - \frac{1}{a^{(1-1/r)q}}\right)^{1/q}$$

$$\left(\int_a^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p}.$$

In the proof of the Proposition, author used the following result for

$1/p + 1/q = 1 - 1/r, r > 1.$

$$\int_0^x \frac{1}{t^{(1-1/r)pq}} f(t)\, t^{-\frac{1}{(1-1/r)pq}}\,dt \leq \left(\int_0^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p}$$

$$\left(\int_0^x t^{-1/(1-1/r)p}\,dt\right)^{1/q}$$

the above result is not true for $1/p + 1/q = 1 - 1/r$. For example, $f(x) = g(x) = 1, p = q = r = 3$, satisfy $1/p + 1/q = 1 - 1/r$,

$$\int_1^9 f(x)\,g(x)\,dx = 8, \left(\int_1^9 f^3(x)\,dx\right)^{1/3} = \left(\int_1^9 g^3(x)\,dx\right)^{1/3} = \left(\int_1^9 1^3\,dx\right)^{1/3} = 2$$

but have

$$\int_1^9 f(x)\,g(x)\,dx > \left(\int_1^9 f^3(x)\,dx\right)^{1/3}\left(\int_1^9 g^3(x)\,dx\right)^{1/3}$$

It follows that the result of (Ref. 3) is not true.

In what follows, we give our result for $1/p + 1/q = 1 - 1/r, r > 1$.

3. Main Results

Lemma 3.1 — Let $0 < b < \infty, p > 1, 1, p + 1/q = 1 - 1/r, f \geq 0, r > 1$, and $0 < \int_0^b f^p(t)\,dt < \infty$. Then for any $x \in (0, b)$, the following inequality is true :

$$\int_0^x f(t)\,dt \leq \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{1-1/p} \left(1 - \frac{1}{r}\right)^{1-1/p} x^{1 - \frac{1}{p} - \frac{1}{p+q}}$$

$$\times \left(\int_0^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p} \qquad \ldots (6)$$

Proof : Let $x \in (0, b)$. Then by Hölder's inequality, we have

$$\int_0^x f(t)\,dt = \int_0^x t^{\frac{1}{(1-1/r)pq}} f(t)\, t^{-\frac{1}{(1-1/r)pq}}\,dt$$

$$\leq \left(\int_0^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p} \left(\int_0^x \left(t^{-\frac{1}{(1-1/r)pq}}\right)^{p/(p-1)} dt\right)^{1-1/p}$$

$$= \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{1-1/p} \left(1 - \frac{1}{r}\right)^{1-1/p} \left(x^{1 - \frac{1}{(1-1/r)q(p-1)}}\right)^{1-1/p}$$

$$\times \left(\int_0^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p}$$

$$= \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{1-1/p} \left(1 - \frac{1}{r}\right)^{1-1/p} x^{1 - \frac{1}{p} - \frac{1}{p+q}}$$

$$\times \left(\int_0^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p}$$

This completes the proof.

Lemma 3.2 — Let $a \geq 0, p > 1, 1/p + 1/q = 1 - 1/r, f \geq 0, r > 1$ and $0 < \int_a^\infty f^p(t)\,dt < \infty$. Then there exists a real number $x_0 \in (a, \infty)$ such that for any $x > x_0$, the following inequality is true :

$$\int_a^x f(t)\,dt < \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{1-1/p} \left(1 - \frac{1}{r}\right)^{1-1/p}$$

$$\times\left(x^{1-\frac{1}{(1-1/r)q(p-1)}}-a^{1-\frac{1}{(1-1/r)q(p-1)}}\right)^{1-1/p}$$

$$\times\left(\int_a^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p} \qquad \ldots (7)$$

PROOF : By Hölder's inequality, we have

$$\int_a^x f(t)\,dt = \int_a^x \frac{1}{t^{(1-1/r)pq}} f(t)\, t^{-\frac{1}{(1-1/r)pq}}\,dt$$

$$\leq \left(\int_a^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p} \left(\int_a^x \left(t^{-\frac{1}{(1-1/r)pq}}\right)^{p/(p-1)} dt\right)^{1-1/p}$$

$$= \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{1-1/p}\left(1-\frac{1}{r}\right)^{1-1/p}$$

$$\times\left(x^{1-\frac{1}{(1-1/r)q(p-1)}}-a^{1-\frac{1}{(1-1/r)q(p-1)}}\right)^{1-1/p}$$

$$\times\left(\int_a^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\right)^{1/p}. \qquad \ldots (8)$$

We need to show that there exists a real number $x_0 \in (a, \infty)$, such that (8) does not assume equality for any $x > x_0$. Otherwise, there exists $x = x_n \in (a, \infty)$, where $n = 1, 2, 3, \ldots, x_n \uparrow \infty$, such that (8) becomes an equality. By the same argument there exists a real number $c > 0$, and N, such that for $n > N$.

$$\left(\frac{1}{t^{(1-1/r)pq}} f(t)\right)^p = c\left(t^{-\frac{1}{(1-1/r)pq}}\right)^{p/p-1)} \quad a.e. \text{ in } [a, x_n].$$

Hence $$\int_a^{x_n} f^p(t)\,dt = \int_a^{x_n} c\,\frac{t^{-\frac{1}{(1-1/r)q(p-1)}}}{t^{\frac{1}{(1-1/r)q}}}\,dt = \int_a^{x_n} ct^{\frac{-p}{(1-1/r)q(p-1)}}\,dt \to \infty \text{ as } n \to \infty.$$

This is a contradiction to the fact that $0 < \int_a^\infty f^p(t)\,dt < \infty$. Hence, (7) holds true and the proof is complete.

Theorem 3.1 — *Let* $0 < a < b \leq \infty, p > 1, 1/p + 1/q = 1 - 1/r, f \geq 0, r > 1$ *and*

$0 < \int_a^b f^p(t)\,dt < \infty$. *Then*

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t)\, dt \right)^p dx < \left(\frac{p\,(p-1)}{(p+q)\,(p-1)-p} \right)^{p-1} \left(q\left(1-\frac{1}{r}\right)\right)^p$$

$$\times \left(1 - \left(\frac{a}{b}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}} \right)^{p-1}$$

$$\times \left(1 - \left(\frac{a}{b}\right)^{\frac{1}{(1-1/r)\,q}} \right) \int_a^b f^p(t)\, dt \qquad \text{... (9)}$$

PROOF : For $b=\infty$, by (7), we obtain

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t)\, dt \right)^p dx$$

$$< \left(\frac{pq\,(p-1)}{(p+q)\,(p-1)-p} \right)^{p-1} \left(1-\frac{1}{r} \right)^{p-1}$$

$$\int_a^b \frac{1}{x^p} \left(x^{1-\frac{1}{(1-1/r)\,q\,(p-1)}} - a^{1-\frac{1}{(1-1/r)\,q\,(p-1)}} \right)^{p-1}$$

$$\times \int_a^x t t^{\frac{1}{(1-1/r)\,q}} f^p(t)\, dt\, dx$$

$$= \left(\frac{pq(p-1)}{(p+q)\,(p-1)-p} \right)^{p-1} \left(1-\frac{1}{r} \right)^{p-1}$$

$$\int_a^b \left\{ \int_t^b x^{-1\frac{1}{(1-1/r)q}} \left(1 - \left(\frac{a}{x}\right)^{1-\frac{1}{(1-1/r)\,q(p-1)}} \right)^{p-1} dx \right\}$$

$$\times\ t t^{\frac{1}{(1-1/r)\,q}} f^p(t)\, dt$$

$$< \left(\frac{p\,(p-1)}{(p+q)\,(p-1)-p} \right)^{p-1} \left(q\left(1-\frac{1}{r}\right)\right)^p \left(1 - \left(\frac{a}{b}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}} \right)^{p-1}$$

$$\times \int_a^b \left(1 - \left(\frac{t}{b}\right)^{\frac{1}{(1-1/r)\,q}} \right) f^p(t)\, dt$$

$$< \left(\frac{p\,(p-1)}{(p+q)\,(p-1)-p} \right)^{p-1} \left(q\left(1-\frac{1}{r}\right)\right)^p \left(1 - \left(\frac{a}{b}\right)^{1-\frac{1}{(!-1/r)\,q\,(p-1)}} \right)^{p-1}$$

$$\times\left(1-\left(\frac{a}{b}\right)^{\frac{1}{(1-1/r)q}}\right)\times\int_a^b f^p(t)\,dt.$$

Similarly, by (6) the case for $b<\infty$ can be proved. This completes the proof of the theorem.

Theorem 3.2 — *Let* $a\geq 0, p>1, 1/p+1/q=1-1/r, f\geq 0, r>1,$ *and* $0<\int_a^\infty f^p(t)\,dt<\infty.$ *Then*

$$\int_a^\infty\left(\frac{1}{x^{1-\frac{1}{(1-1/r)q(p-1)+\frac{1}{p}}}}\int_a^x f(t)\,dt\right)^p dx<\left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^p\left(1-\frac{1}{r}\right)^p$$

$$\times\int_a^\infty[1-\theta_p(t)]f^p(t)\,dt \qquad \ldots (10)$$

where $$\theta_p(t)=1-\frac{1}{p}\left[1-\left(1-\left(\frac{a}{t}\right)^{1-\frac{1}{(1-1/r)(q(p-1)}}\right)^p\right]$$

$$\left(\frac{t}{a}\right)^{\frac{1}{(1-1/r)q}}\frac{1}{a^{(1-1/r)q(p-1)}}-\frac{1}{(1-1/r)p}.$$

Specially, as $a\geq 1$, we have $\theta_p(t)>0$.

PROOF : Applying inequality (7), we have

$$\int_a^\infty\left(\frac{1}{x^{1-\frac{1}{(1-1/r)q(p-1)}+\frac{1}{p}}}\int_a^x f(t)\,dt\right)^p dx$$

$$<\left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1}\left(1-\frac{1}{r}\right)^{p-1}$$

$$\times\int_a^\infty\frac{1}{x^{p+1-\frac{p}{(1-1/r)q(p-1)}}}\left(x^{1-\frac{1}{(1-1/r)q(p-1)}}-a^{1-\frac{1}{(1-1/r)q(p-1)}}\right)^{p-1}$$

$$\times\int_a^x\frac{1}{t^{(1-1/r)q}}f^p(t)\,dt\,dx$$

$$=\left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^p\left(1-\frac{1}{r}\right)^p\int_a^\infty\int_t^\infty\left(1-\left(\frac{a}{x}\right)^{1-\frac{1}{(1-1/r)q(p-1)}}\right)^{p-1}$$

$$\times d\left(1-\left(\frac{a}{x}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}}\right)\left(\frac{t}{a}\right)^{\frac{1}{(1-1/r)\,q}}$$

$$\times a^{\frac{1}{(1-1/r)\,q\,(p-1)}-\frac{1}{(1-1/r)\,p}}\, f^{p}(t)\,dt$$

$$=\left(\frac{pq\,(p-1)}{(p+q)\,(p-1)-p}\right)^{p}\left(1-\frac{1}{r}\right)^{p}\int_{a}^{\infty}\frac{1}{p}\left[1-\left(1-\left(\frac{a}{t}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}}\right)^{p}\right]$$

$$\times\left(\frac{t}{a}\right)^{\frac{1}{(1-1/r)\,q}} a^{\frac{1}{(1-1/r)\,2\,(p-1)}-\frac{1}{(1-1/r)\,p}}\, f^{p}(t)\,dt$$

$$=\left(\frac{pq\,(p-1)}{(p+q)\,(p-1)-p}\right)^{p}\left(1-\frac{1}{r}\right)^{p}\int_{a}^{\infty}[1-\theta_p(t)]f^{p}(t)\,dt$$

where
$$\theta_p(t)=1-\frac{1}{p}\left[1-\left(1-\left(\frac{a}{t}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}}\right)^{p}\right]$$

$$\times\left(\frac{t}{a}\right)^{\frac{1}{(1-1/r)\,q}} a^{\frac{1}{(1-1/r)\,q\,(p-1)}-\frac{1}{(1-1/r)\,p}}.$$

Hence, inequality (10) is valid.

If $t > a \geq 1$, then by Bernoulli's inequality we have

$$1-p\left(\frac{a}{t}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}} < \left[1-\left(\frac{a}{t}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}}\right]^{p}.$$

From this we obtain

$$\theta_p(t) > 1-\frac{1}{p}\left[1-\left(1-p\left(\frac{a}{t}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}}\right)\right]\left(\frac{t}{a}\right)^{\frac{1}{(1-1/r)\,q}}$$

$$\times a^{\frac{1}{(1-1/r)\,q\,(p-1)}-\frac{1}{(1-1/r)\,p}}$$

$$=1-\left(\frac{a}{t}\right)^{1-\frac{1}{(1-1/r)\,q\,(p-1)}}\left(\frac{t}{a}\right)^{\frac{1}{(1-1/r)\,q}} a^{\frac{1}{(1-1/r)\,q\,(p-1)}-\frac{1}{(1-1/r)\,p}}$$

$$=1-t^{\frac{1}{(1-1/r)\,q\,(p-1)}-\frac{1}{(1-1/r)\,p}}>0.$$

This completes the proof of the theorem.

***Theorem* 3.3** — *Let* $0 < b \leq \infty, s \geq p > 1, 1/p + 1/q = 1 - 1/r, f \geq 0, r > 1,$ *and*

$0 < \int_0^b t^{-s+p} f^p(t)\, dt < \infty$. *Then*

(i) For $b \in (0, \infty)$, *we have*

$$\int_0^b x^{-s} \left(\int_0^x f(t)\, dt \right)^p dx$$

$$\leq \left(\frac{p(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \frac{\left(q\left(1-\frac{1}{r}\right)\right)^p}{(s-p)\left(1-\frac{1}{r}\right)q+1} \int_0^b \left[1 - \left(\frac{t}{b}\right)^{s-p+\frac{1}{(1-1/r)q}} \right]$$

$$\times t^{-s+p} f^p(t)\, dt \qquad \text{... (11)}$$

In particular, when $s = p$ we obtain

$$\int_0^b \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx$$

$$\leq \left(\frac{p(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \left(q\left(1-\frac{1}{r}\right)\right)^p \int_0^b \left[1 - \left(\frac{t}{b}\right)^{\frac{1}{(1-1/r)q}} \right] f^p(t)\, dt \qquad \text{... (12)}$$

(*ii*) For $b = \infty$, we have

$$\int_0^\infty x^{-s} \left(\int_0^x f(t)\, dt \right)^p dx$$

$$< \left(\frac{p(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \frac{\left(q\left(1-\frac{1}{r}\right)\right)^p}{(s-p)\left(1-\frac{1}{r}\right)q+1} \int_0^\infty t^{-s+p} f^p(t)\, dt. \qquad \text{... (13)}$$

PROOF : For case (i), $b \in (0, \infty)$, by (6) we have

$$\int_0^b x^{-s} \left(\int_0^x f(t)\, dt \right)^p dx$$

$$\leq \left(\frac{pq(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \left(1-\frac{1}{r}\right)^{p-1} \int_0^b x^{p-s-1} - \frac{p}{p+q}$$

$$\int_0^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\,dx$$

$$= \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1} \left(1-\frac{1}{r}\right)^{p-1} \int_0^b \left(\int_t^b x^{p-s-1-\frac{p}{p+q}}\,dx\right)$$

$$\times \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt$$

$$= \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1} \frac{\left(1-\frac{1}{r}\right)^{p-1}}{p-s-\frac{1}{(1-1/r)q}}$$

$$\times \int_0^b \left(b^{p-s-\frac{1}{(1-1/r)q}} - t^{p-s-\frac{1}{(1-1/r)q}}\right) \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt$$

$$= \left(\frac{p(p-1)}{(p+q)(p-1)-p}\right)^{p-1} \frac{\left(q\left(1-\frac{1}{r}\right)\right)^p}{(s-p)\left(1-\frac{1}{r}\right)q+1}$$

$$\int_0^b \left[1-\left(\frac{t}{b}\right)^{s-p+\frac{1}{(1-1/r)q}}\right] t^{-s+p} f^p(t)\,dt$$

This proves (11).

For case *(ii)*, $b=\infty$, by (7) we have

$$\int_0^\infty x^{-s}\left(\int_0^x f(t)\,dt\right)^p dx$$

$$< \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1} \left(1-\frac{1}{r}\right)^{p-1} \int_0^\infty x^{-s+p-1-\frac{p}{p+q}} \int_0^x \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt\,dx$$

$$= \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1} \left(1-\frac{1}{r}\right)^{p-1} \int_0^\infty \left(\int_t^\infty x^{-s+p-1-\frac{p}{p+q}}\,dx\right)$$

$$\times \frac{1}{t^{(1-1/r)q}} f^p(t)\,dt$$

$$= \left(\frac{p(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \frac{\left(q\left(1-\frac{1}{r}\right)\right)^p}{(s-p)\left(1-\frac{1}{r}\right)q+1} \int_0^\infty t^{-s+p} f^p(t)\, dt$$

This proves (13) and the proof of the theorem is complete.

Remark 3.1 : If we let $r \to \infty$ in Lemma 2.2 and Theorems 2.1 and 2.2, 2.3 (*ii*), then our results reduce to the corresponding Lemma 2.2, and Theorems 2.1 and 2.2, 2.3 (*ii*) obtained in (Ref. 2).

Remark 3.2 : If we let $s = p$ and $r \to \infty$ (13) reduces to the classical Hardy's inequality (1). Thus (13) is an improvement of (1).

Remark 3.3 : If we let $r \to \infty$ in (13), then we shall obtain the following useful inequality:

$$\int_0^\infty x^{-s} \left(\int_0^x f(t)\, dt \right)^p dx < \frac{q^p}{(s-p)\,q+1} \int_0^\infty t^{-s+p} f^p(t)\, dt, s > 1. \quad \text{... (14)}$$

Remark 3.4 : If we let $s = p$ in (14), then we shall obtain (1) due to Hardy[1]

$$\int_0^\infty \left[\frac{1}{x} \int_0^x f(t)\, dt \right]^p dx < q^p \int_0^\infty f^p(t)\, dt.$$

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