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GENERALIZED THERMOELASTIC WAVES IN TRANSVERSELY ISOTROPIC PLATES

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Analysis for the propagation of free waves in homogeneous transversely isotropic thermoelastic plates is developed in context of generalized theory of thermoelasticity. Secular equations in closed form and isolated mathematical conditions for symmetric and skew symmetric wave mode propagation in completely separate terms are derived. Material system of higher symmetry, such as isotropic and cubic, is contained implicitly in the analysis. It is shown that the purely transverse motion (SH mode) gets de-coupled, which is not affected by thermal variations, from rest of the motion of wave propagation and occurs along an in-plane axis of symmetry. The results for isotropic materials, coupled and uncoupled theories of thermoelasticity and at short wave length are also deduced as particular cases at various stages of this work. Finally, the numerical solution for transversely isotropic plates of solid helium and zinc materials is carried out and dispersion curves for symmetric and anti-symmetric modes are presented and illustrated graphically.

Key Words: Lamb Waves; Thermal Relaxation; Transversely Isotropic; Secular Equations; Solid Helium

1. INTRODUCTION

The coupling between thermal and strain fields gives rise to the coupled theory of thermo-elasticity Chadwick and Sneddon[1] discussed in detail the influence of volume and thermal changes coupled with each other in the form of plane harmonic waves. A list of Nowacki's papers on the coupled theory of thermo-elasticity can be found in his monumental books[2, 3]. Chadwick and Windle[4] studied the effect of heat conduction upon the propagation of Rayleigh waves in the semi-infinite elastic solid (*i*) when the surface of the solid is maintained at constant temperature and (*ii*) when the surface is thermally insulated. Chadwick and Atkin[5] corrected and extended the earlier work of Chadwick and Windle[4] by reconsidering the same problem. The governing equations of coupled thermoelasticity (CT) are of the wave type (hyperbolic) equations of motion and diffusion type (parabolic) equation for heat conduction. It is seen that a part of solution of energy extends to infinity implying that if a homogeneous isotropic elastic medium is subjected to thermal or mechanical disturbances the effect of both temperature and displacement fields are felt at an infinite distance from the source of disturbance. This shows that a part of disturbance has an infinite velocity of propagations which is physically impossible. Keeping this drawback in view some researchers such as Lord and Schulam[6] and Green and Lindsay[7] modified the Fourier law and constitutive relations so as to get an hyperbolic equation for heat conduction. These waves include the time needed for acceleration of heat flow and taken into account the coupling between temperature and strain fields for isotropic materials. Banerjee and Pao[8] investigated the propagation of plane harmonic waves in infinitely extended anisotropic solids taking into account the thermal relaxation time. Dhaliwal and Sherief[9] extended the generalized thermoelasticity[6] to anisotropic elastic bodies. Nayfeh and Nasser[10] discussed the propagation of surface waves in homogeneous isotropic solids in the context of coupled and generalized thermoelastic bodies. Noda *et al.*[11] derived a formulation of generalized thermoelasticity that combine both Lord-Schulman (LS) and Green-Lindsay (GL) theories of generalized thermoelasticity for one-dimensional problems.

Kozlov[12] has considered the influence of coupling between temperature and strain fields on the dynamic behaviour of rectangular plates. Massalas *et al.*[13] studied the influence of a constant heat flux on the static and dynamic response of simply supported and clamped circular plate with edge immovably constrained. Kumar[14] studied the coupled thermo-elastic waves in plates of thickness '*d*' subject to axially symmetric hydrostatic tension. The expression by using integral transform techniques has been derived for stresses and temperature fields at short and long time. Saxena and Dhaliwal[15] studied two-dimensional problems of axi-symmetric and plane strain cases in coupled thermoelastic wave propagation in an homogeneous isotropic plate. The propagation of plane harmonic waves in homogeneous transversely isotropic and anisotropic materials has also been studied in coupled theory of thermoelasticity by Chadwick and Seet[1], Chadwick[17] and in the context of generalized thermoelasticities[7, 9] by Sharma[18] and Sharma and Singh[19]. Chandrashekhariah[20] brought out a comprehensive review of literature on the subject. The propagation of free guided waves in anisotropic homogeneous plate has been studied in detail by the authors[21, 22]. These studies provide an interesting picture of the rich dispersion characteristics of the guided waves. While Abubakar[23] studied free vibrations of a transversely isotropic plate. Solie and Auld[24] considered elastic waves in free anisotropic plates. No study of plane waves hass been made or available in literature on anisotropic thermoelastic plates by considering the equations of generalized theory of dynamic thermoelasticity, which motivated the authors to carry out the present work.

In the present paper we have discussed the propagation on plate waves in an infinite homogeneous,. isotropic thermoelastic plate of thickness 2d in the context of various theories of thermoelasticity. The effect of stress free insulated or isothermal and rigidly fixed insulated or isothermal boundaries on the wave propagation has been studied in addition to the thermo-mechanical coupling phenomenon. The results for uncoupled thermo-elasticity have been deduced at appropriate stages. The phase velocities of purely transverse (SH) modes that get decoupled from rest of the motion have also been obtained. The frequency equations of wave propagation at short wave length are also derived. The theoretical results obtained for symmetric and skew symmetric modes of wave propagation in various cases have been verified numerically and illustrated graphically for plates of zinc and solid helium material.

2. Formulation of the Problem and its Solution

We consider an infinite homogeneous, transversely isotropic, thermally conducting elastic plate of thickness 2d initially at uniform temperature T_0. We take origin of the co-ordinate system (x_1, x_2, x_3) on the middle surface of the plate. The x_1, x_2-plane is chosen to coincide with the middle surface and the x_3-axis normal to it along the thickness as shown in Fig. 1. The surfaces $x_3 = \pm d$ are assumed to be (*i*) stress free insulated or isothermal and (*ii*) rigidly fixed insulated or isothermal, boundaries. The basic governing equations for homogeneous anisotropic thermoelasticity in the absence of body forces and heat sources, are given by

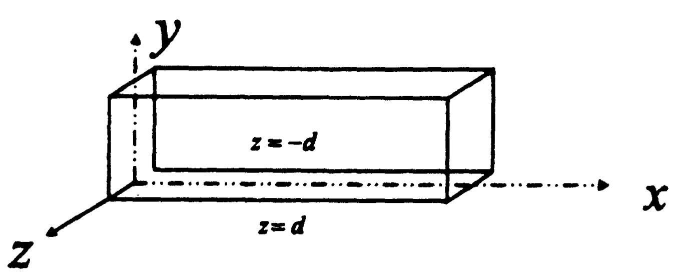

FIG. 1. Geometry of the problem

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad \text{... (1)}$$

$$K_{ij} T_{,ij} - \rho C_e (\dot{T} + t_0 \ddot{T}) = T_0 \beta_{ij} (\dot{u}_{i,j} + t_0 \delta_{1k} \ddot{u}_{i,j}) \quad \text{... (2)}$$

$$\sigma_{ij} = c_{ijkl} e_{kl} - \beta_{ij} (T + t_1 \delta_{1k} \dot{T}),\ \beta_{ij} = c_{ijkl} \alpha_{kl} \quad \text{... (3)}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector $T(x_1, x_2, x_3, t)$ is the temperature change, c_{ijkl} are isothermal elastic parameters; K_{ij} is thermal conductivity, ρ and C_e are respectively, the density and specific heat at constant strain and α_{kl} is the linear thermal expansion tensor. The dot notation denotes time differentiation and δ_{ij} is the Kronecker's delta. Here $k = 1$ for Lord-Schulman (LS) theory and $k = 2$ for Green-Lindsay (GL) theory. We take $x_1 x_3$-plane as the plane of incidence and we assume that the solutions are explicitly independent of x_2 but implicit dependence is there so that the transverse component u_2 of displacement is non-vanishing in eqs. (1) and (2). In view of this the governing eqs. (1) and (2) in the non-dimensional form can be rewritten as

$$u_{1,11} + c_3 u_{3,13} + c_2 u_{1,33} - (T + \delta_{2k} t_1 \dot{T})_{,1} = \ddot{u}_1 \quad \text{... (4)}$$

$$c_3 u_{1,13} + c_2 u_{3,11} + c_1 u_{3,33} - \bar{\beta} (T + \delta_{2k} t_1 \dot{T})_{,3} = \ddot{u}_3 \quad \text{... (5)}$$

$$c_2 u_{2,11} + c_4 u_{2,33} = \ddot{u}_2 \quad \text{... (6)}$$

$$T_{,11} + \bar{K} T_{,33} - (\dot{T} + t_0 \ddot{T}) = \varepsilon_1 [\dot{u}_{1,1} + \bar{\beta} \dot{u}_{3,3} + \delta_{1k} t_0 (\ddot{u}_{1,1} + \bar{\beta} \ddot{u}_{3,3})] \quad \text{... (7)}$$

where comma notation is used for spatial derivatives and we have defined the quantities

$$x_i' = \omega_1^* x_i/v_p,\ t' = \omega_1^* t,\ u_i' = \rho \omega_1^* v_p, u_i/\beta_1 T_0,\ T' = T/T_0,\ t_1' = \omega_1^* t_1,\ t_0' = \omega_1^* t_0$$

$$c_1 = \frac{c_{33}}{c_{11}},\ c_2 = \frac{c_{44}}{c_{11}},\ c_3 = \frac{c_{13} + c_{44}}{c_{11}},\ c_4 = \frac{c_{11} - c_{23}}{2c_{11}} = \frac{c_{66}}{c_{11}},\ \bar{K} = \frac{K_3}{K_1},\ \bar{\beta} = \beta_3/\beta_1$$

$$d' = \omega_1^* d/v_p,\ h' = h\, \omega_1^*/v_p,\ \xi' = \xi\, v_p/\omega_1^*,\ \sigma_{ij}' = \sigma_{ij}/\beta_1 T_0,\ \omega' = \omega/\omega_1^* \quad \text{... (8)}$$

$$\varepsilon_1^2 = \beta_1' T_0/\rho C_e c_{11},\ \omega_1^* = \frac{C_e c_{11}}{K_1},\ c' = c/v_p,\ v_p = \sqrt{c_{11}/\rho},\ i = 1, 2, 3$$

Here ε is the thermoelastic coupling constant, ω_1^* the characteristic frequency of the medium along x_1-axis and v_p the longitudinal wave velocity in the medium. The primes have been suppressed for convenience.

The non-dimensional mechanical and thermal boundary conditions at $x_3 = \pm d$ are given by

$$\sigma_{33} = 0, \sigma_{13} = 0, \sigma_{23} = 0, T_{,3} + hT = 0 \qquad \text{... (9)}$$

for stress free boundary and

$$u_1 + u_2 = u_3 = 0, T_{,3} + hT = 0 \qquad \text{... (10)}$$

for rigidity fixed boundaries. Here h is the surface heat transfer coefficient so that $h \to 0$ corresponds to thermally insulated boundaries and $h \to \infty$ refers to isothermal one.

We assume solutions of the form

$$(u_1, u_2, u_3, T) = (1, S, V, W)\, U \exp\ \{i\xi (x_1 \sin\theta + m\, x_3 - ct)\} \qquad \text{... (11)}$$

where ξ is the wave number, ω is the angular frequency and $c = \omega/\xi$ is the phase velocity of the wave. Here θ is the angle of inclination of wave normal with the axes of symmetry (x_3 - axis), m is still an unknown parameter S, V, W are respectively the amplitude ratios of the displacements u_2, u_3 and temperature T to that of displacement u_1. The use of eq. (11) in eqs. (4) to (7) leads to a system of four coupled equations that has a non-trivial solution if the determinant of the coefficients of $[1, S, V, W]^T$ vanishes. This leads to the following polynomial equations

$$c_4 m^2 + c_2 s^2 - c^2 = 0 \qquad \text{... (12)}$$

and

$$m^6 + Am^4 + Bm^2 + C = 0 \qquad \text{... (13)}$$

where

$$A = \frac{Ps^2 - Jc^2}{c_1 c_2} + \frac{s^2 - \tau_0 c^2}{\overline{K}} - \varepsilon_1 \overline{\beta} \tau_1 \tau_0' c^2/\overline{K} c_1$$

$$B = \frac{(s^2 - c^2)(c_2 s^2 - c^2)}{c_1 c_2} + \frac{(Ps^2 - Jc^2)(s^2 - \tau_0 c^2)}{\overline{K} c_1 c_2} - \frac{\varepsilon_1 \tau_0' \tau_1 c^2}{\overline{K} c_1 c_2} [(c_1 - 2c_3 \overline{\beta} - \overline{\beta}^2) s^2 - \overline{\beta}^2 c^2]$$

$$C = \frac{(c_2 s^2 - c^2)}{\overline{K} c_1 c_2} [(s^2 - c^2)(s^2 - \tau_0 c^2) - \varepsilon_1 \tau_1' c^2 s^2] \qquad \text{... (14)}$$

$$P = c_1 + c_2^2 - c_3^2, J = c_1 + c_2, s = \sin\theta$$

$$\tau_0 = \tau_0 + i\,\omega^{-1}, \tau_0' = t_0 \delta_{1k} + i\,\omega^{-1}, \tau_1 = t_1 \delta_{2k} + i\,\omega^{-1}. \qquad \text{... (15)}$$

The eq. (12) corresponds to purely transverse (SH) modes that decoupled from rest of the motion and are independent of thermal effects. This equation provides us

$$m_7 = \sqrt{((c^2 - c_2 s^2)/c_4)} = -m_8 \qquad \text{... (16)}$$

The eq. (13) corresponds to the coupled longitudinal (02), shear vertical (SV) and thermal (T-mode) in plane motion. This equation being cubic in m^2 admits six solutions for m with property

$m_2 = -m_1, m_4 = -m_3$ and $m_6 = -m_5$. For each $m_q, q = 1, 2, 3, 4, 5, 6$, the amplitude ratios V and W can be expressed as

$$V_q = \begin{cases} \dfrac{m_q a_q}{\sin\theta}, & q = 1, 3 \\ \dfrac{-(c_2 m_q^2 + \sin^2\theta - c^2 + c, c\sin\theta, Sq.)}{c_3 m_q \sin\theta}, & q = 5 \end{cases} \quad \text{... (17.1)}$$

$$W_q = \begin{cases} -((c_2 + c_3 a_q) m_q^2 + \sin^2\theta - c^2)/\tau_1 c \sin\theta, q = 1, 3 \\ [c_1 c_2 m_q^4 + (P\sin^2\theta - Jc^2) m_q^2 + \\ (c_2 \sin^2\theta - c^2)(\sin^2\theta - c^2)]/\beta\, \tau_1 c \sin\theta\, [c_2 m_2^2 + (1 - c_3/\beta)\sin^2\theta - c^2], q = 5 \end{cases} \quad \text{... (17.2)}$$

where $$a_q = \beta \frac{[c_2 m_q^2 + (1 - c_3/\beta)\sin^2\theta - c^2]}{[(c_1 - c_3\beta) m_q^2 + c_2 \sin^2\theta - c^2]}, q = 1, 3 \quad \text{... (18)}$$

Combining eqs. (17) with stress – strain-tremperature relations (3) we rewrite the formal solution for the displacements, temperature, stresses and temperature gradient as

$$(u_1, u_3, T) = \sum_{q=1}^{6} (1, V_q, W_q)\, U_q \exp[i\xi(x_1 \sin\theta + m_q x_3 - ct) \quad \text{... (19)}$$

$$(u_2, \sigma_{23}) = \sum_{q=7}^{8} (1, \xi c_2 m_q)\, S_q\, U_q \exp[i\xi(x_1 \sin\theta + m_q x_3 - ct)] \quad \text{... (20)}$$

$$(\sigma_{33}, \sigma_{13}, T_{,3}) = \sum_{q=1}^{6} i\xi(D_{1q}, D_{2q}, D_{3q})\, U_q \exp[i\xi(x_1 \sin\theta + m_q x_3 - ct)] \quad \text{... (21)}$$

where $$D_{1q} = (c_3 - c_2)\sin\theta + c_1 m_q V_q + \tau_1 c W_q$$

$$D_{2q} = c_2 m_q + c_2 \sin\theta\, V_q, \quad D_{3q} = m_q W_q, q = 1, 2, 3, 4, 5, 6 \quad \text{... (22)}$$

3. Derivation of the Secular Equations

By invoking the stress free and thermal boundary conditions (9) at plate surfaces $x_3 = \pm d$, we obtain a system of eight simultaneous linear equations in amplitudes U_q, q = 1,2, 3, 4, 5, 6, 7, 8 which admits a non-trivial solution if the determinant of the coefficients of these unknown amplitudes vanishes. This requirement leads to the characteristic equation for the propagation of modified guided thermoelastic waves in such a plate. We refer such waves as plate waves rather than Lamb waves whose properties were originally derived by Lamb in 1917 for isotopic elastic solids. The characteristic equation for the thermoelastic plate waves in this case, after applying lengthy algebraic reductions and manipulations, leads to the following equations

$$\sin(2\xi m_7 d) = 0 \quad \text{... (23)}$$

$$\left(\frac{\tan(\gamma m_1)}{\tan(\gamma m_5)}\right)^{\pm 1} - \frac{D_{13}\,G_3}{D_{11}\,G_1}\left[\frac{\tan(\gamma m_3)}{\tan(\gamma m_5)}\right]^{\pm 1} = \frac{-D_{15}\,G_5}{D_{11}\,G_1} \quad \text{... (24)}$$

where $\gamma = \xi\, d,$

$$G_1 = D_{23}\,D_{35} - D_{33}\,D_{25},\ G_3 = D_{21}\,D_{35} - D_{31}\,D_{25},\ G_5 = D_{21}\,D_{33} - D_{31}\,D_{23} \quad \text{... (25)}$$

for stress free insulated boundaries of the plate and

$$G_1 = D_{23}\,W_5 - D_{25}\,W_3,\ G_3 = D_{21}\,W_5 - D_{25}\,W_1,\ G_5 = D_{21}\,W_3 - D_{23}\,W_1 \quad \text{... (26)}$$

for stress free isothermal boundaries of the plate. Here the superscripts + 1 and -1 refers to skew symmetric and symmetric modes respectively.

Again invoking rigidly fixed and thermal boundary conditions (10) at the plate surfaces $x_3 = \pm d,$ we arrive at the following secular equations

$$\sin(2\,\xi\, d m_7) = 0 \quad \text{... (27)}$$

$$\left[\frac{\tan(\gamma m_1)}{\tan(\gamma m_5)}\right]^{\pm 1} + \frac{G_3'}{G_1'}\left[\frac{\tan(\gamma m_3)}{\tan(\gamma m_5)}\right]^{\pm 1} = -\frac{G_5'}{G_1'} \quad \text{... (28)}$$

where

$$G_1' = m_5\,V_3 W_5 - m_3\,V_5\,W_3,\ G_3' = m_5\,V_1\,W_5 - m_1\,V_5\,W_1,$$

$$G_5' = m_3\,V_1\,W_3 - m_1\,V_3\,W_1 \quad \text{... (29)}$$

for rigidly fixed, thermally insulated plate and

$$G_1' = V_3\,W_5 - V_5\,W_3,\ G_2' = V_1\,W_5 - V_5\,W_1,\ G_5' = V_1\,W_3 - V_3\,W_1 \quad \text{... (30)}$$

for rigidly fixed isothermal plate. The eqs. (24) and (28) are the characteristic equations for the symmetric and anti-symmetric modes of waves propagating along an in-plane axis of symmetry of a transversely isotropic plate. Eq. (23) and (27) are the characteristic equations of a horizontally polarized SH-wave in the plate that is clearly independent of thermal variations and thermal relaxation time. The phase velocities of various modes of propagation corresponding to eqs. (23) and (27) are given by

$$c = \pm\sqrt{\frac{c_{44}}{\rho}}\left[\frac{p^2\,\pi^2}{4\,\xi^2\,d^2} + \frac{c_{66}}{c_{44}}\sin^2\theta\right]^{\frac{1}{2}},\ p = 0, 1, 2, 3, 4\ \ldots \quad \text{... (31)}$$

This depends on the thickness of plate and direction of propagation of the wave. If we take $\Omega = 2\,\omega\, d/\pi\, V_H,\ \bar{\xi} = 2\,\xi\, d/\pi,\ V_H^2 = c_{44}/\rho,$ the relation (31) leads to

$$\bar{\xi} = \pm\left[c_{44}\,(\Omega^2 - p^2)/c_{66}\right]^{\frac{1}{2}} \mathit{cosec}\ \theta,\ p = 0, 1, 2, 3 \quad \text{... (32)}$$

where, p odd gives the anti-symmetric and p even the symmetric modes. The group velocity given by $c_g = d\,\omega/d\,\xi$ or, in non-dimensional form, by $\bar{C}_g = d\,\Omega/d\,\bar{\xi}$ also follows from relation (32). For

$p = 0$, the relation (32) gives us $\xi = \Omega (c_{44}/c_{66})^{\frac{1}{2}} \operatorname{cosec} \theta$. Thus we see that all modes of propagation except the first mode are dispersive. We also see that for $\Omega > p$, ξ is real and the spectrum consists of a family of hyperbolas (p = 1, 2, 3...). The imaginary wave number corresponds to a non-propagating spatially varying disturbance. Finally we note that for a given frequency there is only a finite number of propagating SH modes. This suggests it would not be possible to form an arbitrary stress distribution by a Fourier superposition of propagating modes. However, if the imaginary branches are included, an infinite mode set is obtained and the formation of an arbitrary stress distribution becomes possible. Also for large wave number ($\xi \to \infty$) the SH wave branches are asymptotic to $\Omega = (c_{66}/c_{44})^{1/2} \sin \theta \xi$ or equal to shear wave velocity at high frequencies and short wave lengths.

For isotropic materials, we have

$$c_1 = 1, c_2 = \delta^2, c_3 = 1 - \delta^2, \bar{\beta} = 1, \bar{K} = 1, \delta^2 = \mu/(\lambda + 2\mu), \text{ so that}$$

$$V_q = \begin{cases} \alpha_q \; q = 1, 2, 3, 4 \\ -\alpha_q^{-1}, q = 5, 6 \end{cases}, \quad W_q = \begin{cases} i\,\xi\,(\alpha_q^2 + 1 \;\; c^2), q = 1, 2, 3, 4 \\ 0, \qquad\qquad q = 5, 6 \end{cases} \qquad \text{... (33)}$$

The secular eqs. (24) reduce to

$$\left[\frac{\tan(\gamma\alpha_1)}{\tan(\gamma\alpha_5)}\right]^{\pm 1} - \frac{\alpha_1(\alpha_1^2 + 1 - c^2)}{\alpha_3(\alpha_3^2 + 1 - c^2)}\left[\frac{\tan(\gamma\alpha_3)}{\tan(\gamma\alpha_5)}\right]^{\pm 1} = \frac{-4\,\alpha_1\,\alpha_5\,(\alpha_3^2 - \alpha_1^2)}{(\alpha_5^2 - 1)^2\,(\alpha_3^2 + 1 - c^2)} \qquad \text{... (34)}$$

which is the same as obtained and discussed by Sharma *et al.*[25] in case of homogeneous isotropic thermoelastic plate with stress free thermally insulated plate. The result in the context of coupled thermoelasticity (CT) can be obtained by setting $t_0 = 0 = t_1$ in the above analysis.

4. Uncoupled Thermoelasticity

In case of uncoupled thermoelasticity (UCT), the thermo-mechanical coupling constant vanishes i.e. $\varepsilon_1 = 0$, consequently the secular equations (24) in this case reduce to

$$\frac{\tan(\gamma\alpha_1)}{\tan(\gamma\alpha_5)} = \left[\frac{D'_{15}\,D'_{21}}{D'_{11}\,D'_{25}}\right]^{\pm 1} \qquad \text{... (35)}$$

where $$D'_{1q} = (c_3 - c_2)\sin\theta + c_1\,m_q\,V_q,\; D'_{2q} = c_2\,(m_q + \sin\theta\,V_q),\; q = 1, 5 \qquad \text{... (36)}$$

$$W_1 = 0 = W_5,\; W_3 = i\,\xi/\sin\theta \qquad \text{... (37)}$$

$$m_1^2 + m_5^2 = \frac{-(P\sin^2\theta - Jc^2)}{c_1\,c_2},\; m_1^2\,m_5^2 = \frac{(\sin^2\theta - c^2)(c_2\sin^2\theta - c^2)}{c_1\,c_2}, \qquad \text{... (38)}$$

$$m_3^2 = (\sin^2\theta - \tau_0\,c^2)/\bar{K}$$

The thermal motion separates out from rest of the motion in this case. In case of isotropic materials eq. (34) becomes

$$\frac{\tan(\gamma\alpha_1)}{\tan(\gamma\alpha_5)} = \left[\frac{-4\,\alpha_1\,\alpha_5}{(\alpha_5^2-1)^2}\right]^{\pm 1} \qquad \text{... (39)}$$

where $\alpha_1^2 = c^2 - 1$, $\alpha_5^2 = c^2/\delta^2 - 1$. The transcendental eq. (39) is the Rayleign-Lamb equation and was discussing by Graff[26] and Achenbach[27] in case of homogeneous isotropic elastic plates with stress free boundaries.

5. Waves at Short Wavelength

Some information on the asymptotic behaviour is obtainable by letting $\xi \to \infty$. If we take $\xi > \omega/v_H$, $v_H^2 = C_{44}/\rho$ it follows that $\xi > \omega$ and $c < v_H$. Then we replace m_1, m_3 and m_5 in the secular equation by im_1', im_3' and im_5'. Hence for $\xi \to \infty$, $\frac{\tanh(\gamma m_1)}{\tanh(\gamma m_5)} \to 1, \frac{\tanh(\gamma m_3)}{\tanh(\gamma m_5)} \to 1$, so that the secular eqs. (24) and (28) respectively reduce to

$$D_{21}\,G_1 - D_{23}\,G_3 + D_{25}\,G_5 = 0 \qquad \text{... (40)}$$

$$V_1\,G_1' - V_3\,G_3' + V_5\,G_5' = 0 \qquad \text{... (41)}$$

for symmetric and anti-symmetric cases. In case of uncoupled thermoelasticity the eq. (34) reduces to

$$D_{15}'\,D_{21}' - D_{11}'\,D_{25}' = 0 \qquad \text{... (42)}$$

The eqs. (40) and (42) are merely Rayleigh surface wave equation. The Rayleigh waves enter here since for such small wavelengths, the finite thickness plate appears as semi-infinite medium. Hence vibration energy is transmitted along the surface of the plate.

6. Numerical Results and Discussion

With the view of illustrating theoretical results obtained in the preceding sections, we now present some numerical results. The materials chosen for this purpose are single crystals of solid helium and zinc, the physical data for which is given [28] in Table I.

The plate thickness (d) is taken as unity and the angle of inclination of the wave normal with axis of symmetry have been considered between 0 and $\pi/2$ for the purpose of numerical calculation. The complex roots of eq. (13) has been computed with th help of reduced Cardan's method, which are used in various relevant relations. The secular eq. (24) is solved for the phase velocity by using iteration method. The sequence of iteration is made to converge after sampling it over about 100 sample values in order to achieve the desired level of accuracy (seven decimal places here). The phase velocity of first few modes (symmetric and skew symmetric) have been computed and represented graphically in figures 2 and 3 respectively, in case of solid Helium and zinc material plates for the thermal relaxation time $t_0 = 0$ and $t_0 = t_1 = 0.02$, in coupled thermoelastcity (CT), generalized (LS and GL) theories of thermoelasticity, along the direction of wave propagation making angle $\theta = 30^0, 45^0, 60^0, 75^0$ and 90^0 with axis of symmetry.

The non-dimensional phase velocity of lowest (acoustic) skew symmetric mode is observed to increase from zero value at vanishing wave number to become closer to the Rayleigh wave velocity at higher wave numbers in all the directions of wave propagation. The velocity of the

TABLE I : Physical data for single crystals of zinc and solid helium

Quantity	Units	Helium	Zinc
ρ	Kgm^{-3}	0.1910	$7.14^2 \times 10^3$
T_0	K	0.8500	296
c_{11}	Nm^{-2}	$0{,}44040 \times 10^{10}$	1.628×10^{11}
c_{12}	Nm^{-2}	0.2120×10^{10}	0.362×10^{11}
c_{13}	Nm^{-2}	0.01050×10^{10}	0.508×10^{11}
c_{33}	Nm^{-2}	0.5530×10^{10}	0.627×10^{11}
c_{44}	Nm^{-2}	0.1245×10^{10}	0.385×10^{11}
β_1	$Nm^{-2} deg^{-1}$	2.3620×10^{10}	5.75×10^6
β_3	$Nm^{-2} deg^{-1}$	2.641×10^6	5.07×10^6
C_e	$Jkg^{-1}\ deg^{-1}$	1.4770×10^5	3.9×10^2
K_1	$Wm^{-1}\ deg^{-1}$	0.3000×10^2	1.24×10^2
K_3	$Wm^{-1}\ deg^{-1}$	0.2000×10^1	1.24×10^2
ε_1	---	0.04162	0.0221
ω_1^*	s^{-1}	1.9890×10^{13}	5.01×10^{11}

acoustic symmetric mode decreases from a value greater than unity towards the Rayleigh wave velocity with the increase of wave number in all the considered directions of propagation. The phase velocity of higher (optical) modes of wave propagation, symmetric and skew symmetric, attain quite large values at vanishing wave number which sharply decrease to become steady and becomes asymptotically closer to shear wave velocity with increasing values of the wave number. The velocity of optical modes of wave propagation is observed to develop at a rate, which is approximately n-times, the magnitude of first higher mode ($n = 1$). These numerically computed results are found to be quite in agreement with the corresponding results and their trends are similar to those discussed by Graff[22], Achenbach[23] and Sharma *et al.*[25]. in case of homogeneous isotropic elastic and thermoelastic plates except the modifications due to thermo-mechanical couplings, thermal relaxation and anisotropic effects of the medium. The thermal relaxation effect is observed to be quite significant in case of single crystal plate of solid helium as compared to the zinc plate. None of the mode is found to exist for $\theta = 0^0$ as whole of the energy will be transmitted along the axis of symmetry of the plate without disturbing it, in this case. The magnitude of the phase velocities of the symmetric and skew symmetric modes is observed to increase at the angle of incidence θ progresses from 0^0 to $\pi/2$, except in the case of acoustic symmetric modes propagating in the neighborhood of the axis of symmetry viz. $\theta = 0^0$. At higher wave numbers the phase velocities of acoustic modes become asymptotically closer to the Rayleigh wave speed because a finite thickness plate appears to be a half space in such situations and the vibration energy is mainly transmitted through the surface of the plate. The phase velocity of higher (optical) models tends asymptotically to shear wave speed at light wave numbers in both the materials. The various modes of propagation are observed to have negligible effect of thermal relaxation time at high temperature in case of zinc material but these modes exhibit significant effect at low temperature in solid helium plate.

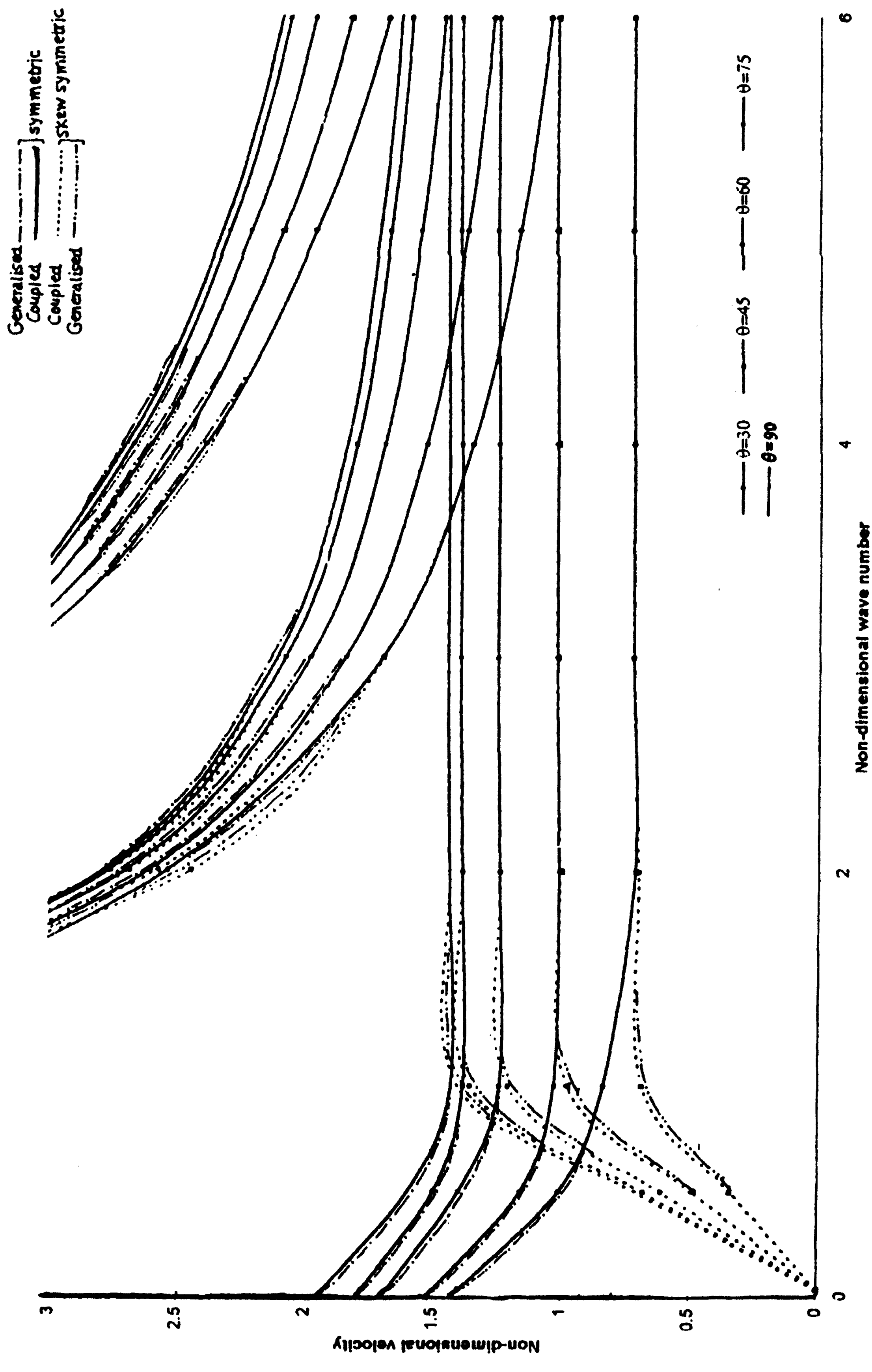

FIG. 2. Dispersion curves for symmetric and skew symmetric modes in a zinc plate

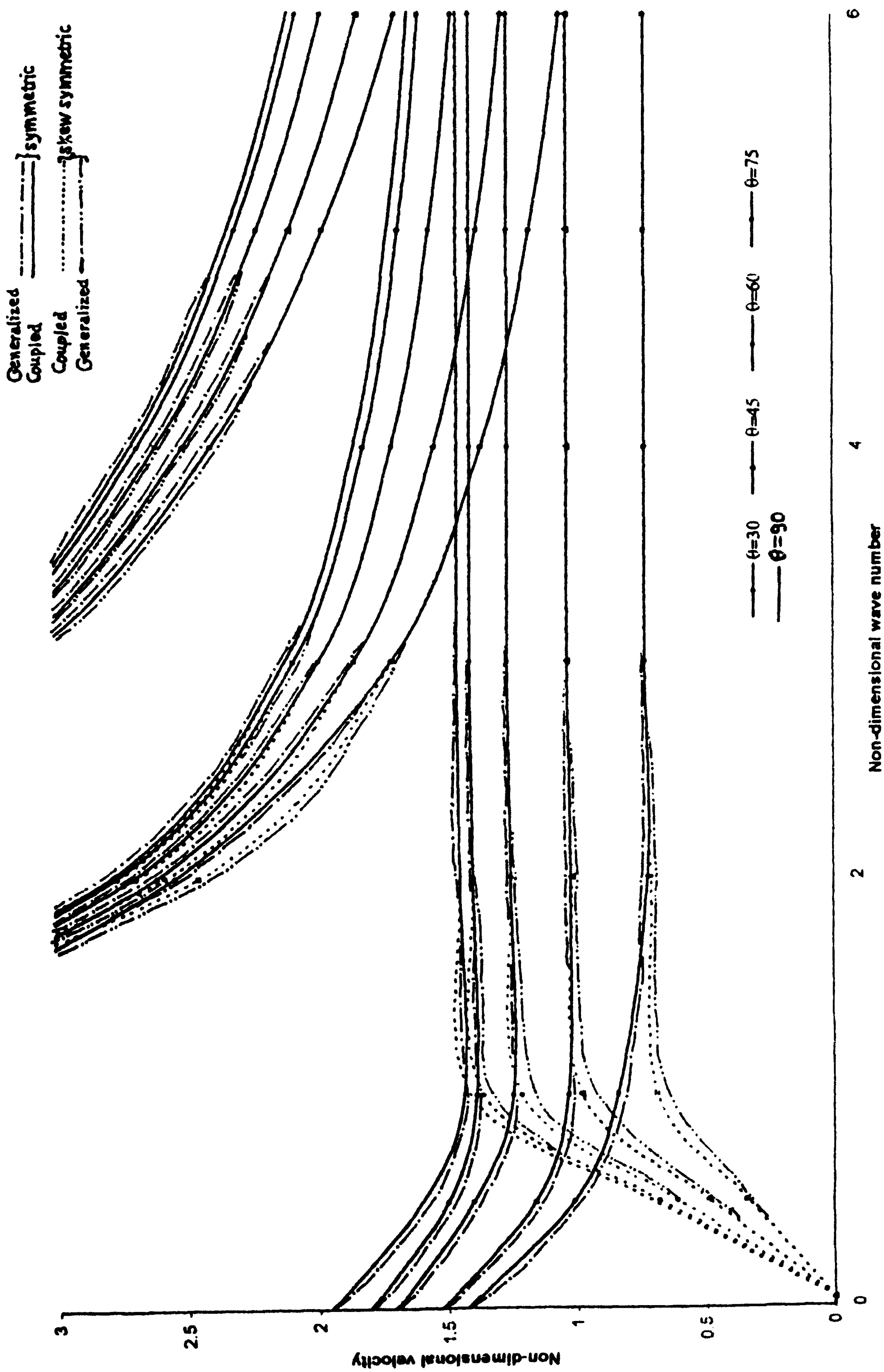

FIG. 3. Dispersion curves for symmetric and skew symmetric modes in a Helium Crystal plate

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