*Indian J. pure appl. Math.*, 34(6) : 853-857, June 2003


# ON A HEIGHT RELATED TO THE *abc* CONJECTURE

ARTŪRAS DUBICKAS

*Department of Mathematics and Informatics, Vilnius University, Naugarduko* 24, *Vilnius* 2600, *Lithuania*
*E-mail: arturas.dubickas@maf.vu.lt*

(*Received* 3 *December* 2001; *accepted* 22 *November* 2002)

We prove that there are only finitely many algebraic numbers whose certain height is bounded by an absolute constant. This answers in the affirmative a question posed by Browkin who showed that this height is related to the *abc* conjecture for algebraic numbers. Main tool in the proof is a result of Langevin on equidistribution of arguments of conjugates of an algebraic number having small Mahler measure.

**Key Words : The Mahler Measure; Heights of Algebraic Numbers; *abc* Conjecture**

## 1. INTRODUCTION

Let $K$ be a number field, i.e. a finite extension of the field of rational numbers $\mathbb{Q}$. The absolute height of a number $\alpha \in K$ is defined by $H_K(\alpha) = \prod_v \max\left\{1, \|\alpha\|_v\right\}$, where the product is taken over the places $v$ of $K$ and the $v$-adic valuations are normalised in such a way that the product of all $\|\alpha\|_v$ is equal to 1 (see, e.g., [3] or [21]). It is well known that the quantity $h(\alpha) = (1/[K:\mathbb{Q}]) \log H_K(\alpha)$ is independent of $K$ ([13], [20]). It is called the *Weil logarithmic height* of $\alpha$ and is equal to $(1/d) \log M(\alpha)$, where $d$ is the degree of $\alpha$ over $\mathbb{Q}$ and

$$M(\alpha) = a(\alpha) \prod_{j=1}^{d} \max\left\{1, |\alpha_j|\right\}$$

is the *Mahler measure* of $\alpha$. Here, $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the conjugates of $\alpha$ over $\mathbb{Q}$, and $a(\alpha)$ is the leading coefficient of the minimal polynomial of $\alpha$ over $\mathbb{Q}$.

The logarithmic height of a point $(x_0 : x_1 : x_2)$ of the projective plain $\mathbb{P}^2(K)$ is given by the formulae

$$h(x_0 : x_1 : x_2) = \frac{1}{[K:\mathbb{Q}]} \sum_v \log \max\left\{\|x_0\|_v, \|x_1\|_v, \|x_2\|_v\right\}$$

so that the Weil logarithmic height of $\alpha \in \overline{\mathbb{Q}}$ is the projective height of $(1:\alpha) \in \mathbb{P}^1(K)$. If $(x_0 : x_1 : x_2)$ is a rational point represented by a triple of integers $(x_0, x_1, x_2)$ with no common divisor, then $h(x_0 : x_1 : x_2) = \log \max\{|x_0|, |x_1|, |x_2|\}$. For an arbitrary $\alpha \in \overline{\mathbb{Q}}$, set

$$h_0(\alpha) = h(\alpha : 1-\alpha : -1) = \frac{1}{[K:\mathbb{Q}]} \sum_v \log \max\left\{1, \|\alpha\|_v, \|1-\alpha\|_v\right\}.$$

Note that the point $(\alpha : 1-\alpha : -1)$ lies on the curve $x_0 + x_1 + x_2 = 0$. Zagier[21] showed that $h_0(\alpha) \geq \log c_1 = 0.1911...$ for all $\alpha \in \overline{\mathbb{Q}}$ except for four points $\alpha \in \{0, 1, (1 \pm \sqrt{-3}/2\}$ for which $h_0(\alpha)$ vanishes. Here, $c_1$ is the larger real root of the equation $z^6 - z^4 - 1 = 0$.

Given an integer $q$, $|q| \geq 2$, the radical of $q$, rad $(q)$, is, by definition, the product of distinct primes dividing $q$. The classical Masser-Oesterle's *abc*-conjecture states that for any positive number $\varepsilon$ there are only finitely many nonzero relatively prime integers $a$, $b$, $c$ such that $a + b + c = 0$ and

$$L(a, b, c) = \frac{\log \max\{|a|, |b|, |c|\}}{\log rad\,(abc)} \geq 1 + \varepsilon .$$

Recently, Browkin[6] defined the radical of an algebraic number $\alpha \in K \backslash \{0, 1\}$ (with respect to $K$) $rad_K(\alpha)$. Setting

$$L(\alpha) = \frac{[K : \mathbb{Q}]\, h_0(\alpha)}{\log rad_K(\alpha)},$$

he showed that $L(\alpha)$ is independent of the field $K$, so that one can always take $K = \mathbb{Q}(\alpha)$. The classical *abc* - conjecture was reformulated to algebraic numbers as follows : is it true that for every positive number $\varepsilon$ there are only finitely many algebraic numbers $\alpha \neq 0, 1$ such that $L(\alpha) \geq 1 + \varepsilon$?

Among other things, Browkin[6] asked the following.

*Question* 1 — Is for every $X \geq 1$ the set of all algebraic numbers $\alpha$ satisfying $\exp\{[\mathbb{Q}(\alpha) : \mathbb{Q}]\, h_0(\alpha)\} \leq X$ finite?

The aim of this note is to answer this question in the affirmative. Let $\mathbb{C}$ be the set of complex numbers, and let $f : \mathbb{C} \to \mathbb{C}$ be a fixed function. Write

$$M_f(\alpha) = \prod_{j=1}^{d} \max\{1, |f(\alpha_j)|\}.$$

For a real $X$, let $A_f(X)$ be the set of algebraic numbers $\alpha$ such that $M(\alpha) \leq X$ and $M_f(\alpha) \leq X$.

***Theorem*** — *Let $f : \mathbb{C} \to \mathbb{C}$ be a fixed function which is analytic in an open set containing the unit circle $U = \{z \in \mathbb{C} : |z| \leq 1\}$, and let $X \geq 1$. Then the set $A_f(X)$ is finite if and only if $f(U) \setminus U \neq 0$.*

The proof of the theorem is given in the next section. In Section 3, we will explain why this theorem contains the positive answer to Question 1 and give some further comments.

## 2. Proof of the Theorem

The following lemma due Langevin[14] will be most useful in the proof.

*Lemma* — Let $V$ be an open set containing a point of modulus 1. Then there are $v = v(V) > 1$ and $d(V)$ such that if $\alpha$ is an algebraic number of degree $d \geq d(V)$ which has no conjugates lying in $V$, then $M(\alpha) > v^d$.

Assume that $f(U) \setminus U = 0$. It follows that $f(U) \subset U$, thus for every root of unity $\zeta$, we have that $M(\zeta) = M_f(\zeta) = 1$. The set $A_f(1)$ is therefore infinite, because so is the set of all roots of unity.

Alternatively, if $f(U) \setminus U \neq 0$, then there is a number $\xi > 1$ and a neighbourhood of a point on the unit circle $\psi$, say $V = V(\psi)$, such that $f(V) \subset \{z \in \mathbb{C} : |z| > \xi\}$. Fix a positive integer $m$ so that $\xi^m > X$. If the set $V$ contains at least $m$ conjugates of $\alpha$, then $M_f(\alpha) > \xi^m > X$, and so $\alpha \notin A_f(X)$. It follows that every $\alpha \in A_f(X)$ has at most $m$ conjugates lying in $V$.

Let $V_1, V_2, \dots, V_{m+1}$ be open sets each containing a point on the unit circle such that $V_1 \cup \dots \cup V_{m+1} \subset V$ and $V_i \cap V_j = 0$ for all $e \leq i \neq j \leq m+1$. By the lemma, there are $v_1 = v_1(V_1) > 1, \dots, v_{m+1} = v_{m+1}(V_{m+1}) > 1$ and $d(V_1), \dots, d(V_{m+1})$ with the following property: if $\alpha$ is of degree $d \geq D_0$ and it has no conjugates in at least one $V_i$, then $M(\alpha) > v^d$. Here, $D_0 = \max\{d(V_1), \dots, d(V_{m+1})\}$ and $v = \min\{v_1, \dots, v_{m+1}\}$. Let $D$ be a positive integer greater than $\max\{D_0, (\log X)/(\log v)\}$. We claim that every $\alpha \in A_f(X)$ is of degree at most $D$. Indeed, assume that some $\alpha \in \overline{\mathbb{Q}}$ of degree $d > D$ belongs to $A_f(X)$. Of course, either every set $V_i$, $i = 1, 2, \dots, m+1$, contains a conjugate of $\alpha$, or at least one of these sets, say $V_l$, is free of conjugates of $\alpha$. The first case is impossible, for otherwise, at least $m + 1$ conjugates of $\alpha$ lie in $V$. In the second case, we deduce that $M(\alpha) > v_l^d \geq v^d > v^D > X$, which is also impossible.

It follows that every $\alpha \in A_f(X)$ is of degree at most $D$. However, the set

$$B(D, X) = \{\alpha \in \overline{\mathbb{Q}} : \deg \alpha \leq D, M(\alpha) \leq X\}$$

is finite. (See, for instance, the paper of the author and Koniagin[11] for an upper bound in terms of $D$ and $X$). This finishes the proof, because $A_f(X) \subset B(D, X)$.

## 3. REMARKS

1. Since $\max\{1, \|\alpha\|_v, \|1-\alpha\|_v\} \leq \max\{1, \|\alpha\|_v\} \max\{1, \|1-\alpha\|_v\}$, we have that

$$\exp\{[\mathbb{Q}(\alpha) : \mathbb{Q}]\, h_0(\alpha)\} \leq M(\alpha)\, M(1-\alpha).$$

Furthermore, if $f$ is a polynomial with rational coefficients, then

$$M(\beta) = a(\beta)\, M_f(\alpha) \geq M_f(\alpha),$$

where $\beta = f(\alpha)$ and $a(\beta)$ is the leading coefficient of the minimal polynomial of $\beta$. The set considered in Question 1 is therefore contained in the set $A_f(X)$ with the function $f(z) = 1 - z$ which maps $U$ to $\{z \in \mathbb{C} : |z-1| \leq 1\}$.

The smallest value for $(M(\alpha)M(1-\alpha))^{1/d}$ was found by Zagier[21]. This is at least $((\sqrt{5}+1)/2)^{1/2} = 1.2720\dots$ except for $\alpha \in \{0, 1, (1 \pm \sqrt{-3})/2\}$ for which it is equal to 1. See also [3], [7], [8], [17], [19] for further work on similar problems.

2. In a particular case, namely, if instead of all algebraic numbers only those which belong to a fixed number field $K$ are considered, Question 1 was answered affirmatively in[6]. (The method was different from that of this paper). A respective result was fully answered for the radical of $\alpha$: it was shown in[6] that for every $X$ the set of algebraic numbers whose radicals are at most $X$ is finite.

3. The largest known value for $L(\alpha)$, where $\alpha \in \bar{\mathbb{Q}}$,

$$L((\sqrt{2}-1)^4) = 2.5431 \ldots$$

was found by Broberg[5] (see also[6]). The largest known value for $L(a, b, c)$ in the classical case was discovered by Reyssat: the identity $2 + 3^{10} \cdot 109 - 23^5 = 0$ gives the value 1.6299 .... (See the recent review of Mazur[15] about the applications of the *abc*- conjecture to Fermat type problems).

4. The constants $(\sqrt{2}-1)^4$ and $((\sqrt{5}+1)/2)^{1/2}$ appeared earlier under different circumstances. It is not at all surprising that the latter constant features in Schinzel's lower bound for $M(\alpha)^{1/d}$, where $\alpha$ is a totally real algebraic number of degree $d$[18]. The appearance of the former constant, $(\sqrt{2}-1)^4$, also as the maximum of a function in certain integral which Beukers[2] constructed for the proof of irrationality of the Riemann zeta function at 3 may seen somewhat mysterious.

5. The paper of Erdös and Turan[12] on equidistribution of roots of a polynomial of small length in angles with vertex at the origin was published in 1950. Since then, many new results on equidistribution were obtained. See, for instance, [1], [9], [10], [16] and also [4], [22], [23] for much more general results on other arithmetic surfaces.

6. It would be of interest to find other natural heights $h^*$ with the same property as proved in the theorem, namely, such that for every $X$ there are only finitely many algebraic numbers $\alpha$ satisfying $h^*(\alpha) \le X$ (without any restriction on the degree of $\alpha$).

ACKNOWLEDGEMENT

I thank to Professor A. Schinzel who first asked me Question 1 at a conference in Palanga (September 2001) and to Professor J. Browkin who sent me his preprint[6]. This research was partially supported by the Lithuanian State Science and Studies Foundation.

REFERENCES

1. F. Amoroso and M. Mignotte, *Ann. l'Inst. Fourier, Grenoble*, **46** (1996), 1275-91.
2. F. Beukers, *Bull. London Math. Soc.*, **11** (1979), 268-72.
3. F. Beukers and D. Zagier, *Acta Arith.*, **79** (1997), 103-11.
4. Y. Bilu, *Duke Math. J.*, **89** (1997), 465-76.
5. N. Broberg, *Math. Comp.*, **69** (2000), 1707-10.
6. J. Browkin, The *abc- conjecture of algebraic numbers*, 2001, preprint.
7. C. Doche, *J. Theor. des Nombres de Bordeaux*, (to appear).
8. G. P. Dresden, *Math. Comp.*, **67** (1998), 815-20.
9. A. Dubickas, *Liet Matem. Rink.*, **38** (1998), 34-58.
10. A. Dubickas, *Liet. Matem. Rink.*, **38** (1998) 151-68.
11. A. Dubickas and S. V. Koniagin, *Acta Arith.*, **86** (1998), 325-42.
12. P. Erdos and P. Turan, *Ann. of Math.*, (2), **51** (1950), 105-19.
13. S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag, Berlin, New York, 1983.
14. M. Langevin, *C. R. Acad. Sci. Paris*, **301** (1985), 463-66.
15. B. Mazur, *Notices Amer. Math. Soc.*, **47** (2000), 195-202.
16. M. Mignotte, *Acta Arith.*, **54** (1989) 81-86.
17. G. Rhin and C. J. Smyth, *Acta Arith.*, **79** (1997), 239-47.

18. A. Schinzel, *Acta Arith.*, **24** (1973), 385-99.

19. W. Schmidt, *Proc. Amer. Math. Soc.*, **124** (1996), 3003-13.

20. M. Waldschmidt, *Diophantine Approximation on Linear Algebraic Groups*, Springer-Verlag, Berlin, New York, 2000.

21. D. Zagier, *Math. Comp.*, **61** (1993), 485-91

22. S. Zhang, *Ann. of Math.*, **136(2)** (1992), 569-87.

23. S. Zhang, *In* : *Documenta Math.* (G. Fisher and U. Rechmann, eds.), *Proc. of the International Congress of Mathemathicias*, Berlin 1998, August 18-27, Vol. II: *Invited Lectures*, 1998, 217-25.