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GLOBAL SOLUTIONS OF NONLINEAR SECOND ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS OF MIXED TYPE IN BANACH SPACES

XIAOYAN ZHANG

School of Mathematics and System Science, Shandong University, Jinan, Shandong 250 100, *People's Republic of China*

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In this paper, we use a new comparison result and the Mönch's fixed point theorem to study the existence of global solutions of initial value problems for nonlinear second order impulsive integro-differential equations of mixed type in Banach spaces. As applications, the existence of global solutions of two classes of mixed boundary value problems for fourth order impulsive equations is given.

Key Words : Impulsive Integro-Differential Equation of Mixed Type; Initial Value Problem; Banach Space; Measure of Noncompactness

1. INTRODUCTION

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years (see [1]). In this paper, we consider the existence of global solutions of the initial value problem (IVP) for nonlinear second order impulsive integro-differential equation of mixed type in a Banach space $(E \|\cdot\|)$

$$\begin{cases} u'' = f(t, u, u', Tu, Su), & t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u'(t_k)), & \\ \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k), u'(t_k)) & (k = 1, 2, ..., m), \\ u(0) = u_0, \ u'(0) = u_1, & \end{cases} \quad \text{... (1)}$$

where $f \in C[J \times E \times E \times E \times E, E]$, $J = [0, a]$ $(a > 0)$, $0 < t_1 < t_2 < \ldots < t_m < a$, $I_k, \bar{I}_k \in C[E \times E, E]$ $(k = 1, 2, ..., m)$, $u_0, u_1 \in E$, and

$$(Tu)(t) = \int_0^t k(t, s)\, u(s)\, ds, \ (Su)(t) = \int_0^a h(t, s)\, u(s)\, ds, \quad t \in J,$$

where $k \in C[D, R^1], h \in C[J \times J, R^1], D = \{(t, s) \in J \times J : t \geq s\}$, $R^1 = (-\infty, +\infty)$, R^+ denotes the set of nonnegative real numbers. $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e.,

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$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively, and $\Delta u'|_{t=t_k}$ has a similar meaning for $u'(t)$.

In [2, Section 3.1], the existence of extremal solutions of the second order IVP (1) in which f does not contain Tu has been discussed by using monotone iterative technique. In the special cases where f does not possess u' and Su, and $\Delta u|_{t=t_k} = L_k u'(u_k)$, $\Delta u'|_{t=t_k} = \overline{L}_k u(t_k)$ (L_k and $\overline{L}_k$ are constants), the existence of extremal solutions of IVP (1) has been investigated in a recent paper[3].

In this paper, we shall investigate the existence of global solutions of IVP (1) under the more extensive conditions by means of completely different method, that is, by establishing a new comparison result and using the Mönch's fixed point theorem.

As applications of our main results, under the exact conditions we obtain the global solutions in $PC^1[J, E] \cap C^4[J', E]$ of the following two classes of mixed boundary value problems (MBVP) for fourth order impulsive equations in Banach spaces.

$$\begin{cases} x^{(4)} = f(t, x'', x''', x), & 0 \le t \le 1, \quad t \ne t_k, \\ \Delta x''|_{t=t_k} = I_k(x''(t_k), x'''(t_k)), & \\ \Delta x'''|_{t=t_k} = \overline{I}_k(x''(t_k), x'''(t_k)) & (k = 1, 2, \ldots, m), \\ \alpha_1 x(0) + \alpha_2 x'(0) = \theta, & \\ \beta_1 x(1) + \beta_2 x'(1) = \theta, & \\ x''(0) = x_0, x'''(0) = x_1, & \end{cases} \qquad \ldots \text{(I)}$$

$$\begin{cases} x^{(4)} = f(t, x'', x''', x', x), & 0 \le t \le 1, \quad t \ne t_k, \\ \Delta x'''|_{t=t_k} = \overline{I}_k(x''(t_k), x'''(t_k)) & (k = 1, 2, \ldots, m), \\ x'(0) = \theta, & \\ \beta_1 x(1) + \beta_2 x'(1) = \theta, & \\ x''(0) = x_0, x'''(0) = x_1. & \end{cases} \qquad \ldots \text{(II)}$$

The results in this paper generalize and improve the related results in [2, 3], and the method of our proof is also different from those used in [2, 3] in essence.

2. Preliminaries and Lemmas

Let $PC[J, E]$ = {u : u is a map from J into E such that $u(t)$ is continuous at $t \ne t_k$, left continuous at $t = t_k$ and $u(t_k^+)$ exists for $k = 1, 2, ..., m\}$ and $PC^1[J, E] = \{u \in PC[J, E] : u'(t)$ is continuous at $t \neq t_k$ and $u'(t_k^-), u'(t_k^+)$ exist for $k = 1, 2, ..., m\}$. Evidently, $PC[J, E]$ is a Banach space with norm $\| u \|_{PC} = \sup_{t \in J} \| u(t) \|$. For $u \in PC^1[J, E]$, by virtue of the mean value theorem, it is easy to see that

$$u'_-(t_k) = \lim_{h \to 0^+} h^{-1} [u(t_k) - u(t_k - h)] = u'(t_k^-).$$

Throughout this paper, $u'(t_k)$ is understood as $u'_-(t_k)$. It is clear that $PC^1[J, E]$ is also a Banach space with norm

$$\| u \|_{PC^1} = \max \left\{ \| u \|_{PC}, \| u' \|_{PC} \right\},$$

where $\| u \|_C = \sup_{t \in J} \| u(t) \|$, $\| u' \|_{PC} = \sup_{t \in J} \| u'(t) \|$. Let $J' = J \setminus \{t_1, t_2, ..., t_m\}$. A map $u \in PC^1[J, E] \cap C^2[J', E]$ is called a solution of IVP (1) if it satisfies (1).

In the following, let $k_0 = (\max \{| k(t, s) : (t, s) \in D\}$, $h_0 = \max \{| h(t, s) : (t, s) \in J \times J\}$, and $J_0 = [0, t_1], J_1 = (t_1, t_2], ..., J_i = (t_i, t_{i+1}], ..., J_m (t_m, a]$.

For any $B \subset PC^1[J, E]$, we denote $B' = \{u' : u \in B\} \subset PC[J, E]$, $B_i = \{u | J_i : u \in B\}$, $B'_i = \left\{ u' | J_i : u \in B \right\}$ $(i = 0, 1, ..., m)$. For any $t \in J$, let $B(t) = \{u(t) : u \in B\}$, $B'(t) = \{u'(t) : u \in B\}$, $(TB)(t) = \{(Tu)(t) : u \in B\}$, $(SB)(t) = \{(Su)(t) : u \in B\}$. In the same way, we can define $(TB)'(t)$ and $(SB)'(t)$.

Let α denotes the Kuratowski measure of noncompactness in E, the definitions and properties of which may be found in[2, 6].

We shall need the following lemmas :

Lemma 1[4] — Let $B_1, B_2 \subset PC^1[J, E]$ be two countable subsets satisfying $\overline{B}_1 = \overline{co}(\{x_0\} \cup B_2)$ for some $x_0 \in PC^1[J, E]$, then

(1) $\overline{B_1(t)} = \overline{co}(\{x_0(t)\} \cup B_2(t))$, $t \in J$;

(2) $B'_1(t) = \overline{co}(\{x'_0(t)\} \cup B'_2(t))$, $t \in J$.

Lemma 2[5] — Let B be a countable set of strongly measurable functions $u : J \to E$, and there exists a $m \in L[J, R^+]$ such that $\| u(t) \| \leq m(t)$ for almost all $t \in J$ and $u \in B$. Then $\alpha(B(t)) \in L[J, R^+]$ and

$$\alpha\left(\left\{\int_0^t u(s)\,ds : u \in B\right\}\right) \le 2\int_0^t \alpha(B(s))\,ds,\quad t \in J.$$

Lemma 3[2] — If $B \subset PC^1[J, E]$ is bounded and the elements of B are equicontinuous on each J_i $(i = 0, 1, \ldots, m)$, then $\alpha(\{u(t) : u \in B_i\})$ is continuous on $t \in J_i$ and

$$\alpha\left(\int_J B(s)\,ds\right) \le \int_J \alpha(B(s))\,ds.$$

The following comparison result plays an important role in this paper.

Lemma 4 — Assume that $m \in C[J_i, R^+]$ $(i = 0, 1, \ldots, m)$ satisfies

$$m(t) \le \int_0^t b(s)\,m(s)\,ds + \int_0^a c(s)\,m(s)\,ds + \sum_{0<t_k<t} M_k\,m(t_k),\quad t \in J, \qquad \ldots (2)$$

where $b, c \in C[J, R^+]$ and $M_k \ge 0$ are constants. Then $m(t) \equiv 0$ for $t \in J$ provided for any $t \in J$ one of the following two conditions holds :

$$(i)\quad c(t)\left[\left(e^{\int_0^{t_1} b(s)\,ds} - 1\right) + (1 + M_1)\left(e^{\int_0^{t} b(s)\,ds} - e^{\int_0^{t_1} b(s)\,ds}\right) + \ldots + \prod_{k=1}^{m}(1 + M_k)\left(e^{\int_0^{a_1} b(s)\,ds}\; e^{\int_0^{t_m} b(s)\,ds}\right)\right] < b(t).$$

$$(ii)\quad (b(t) + c(t))\left[t_1 + (t_2 - t_1)(1 + M_1) + \ldots + (a - t_m)\prod_{k=1}^{m}(1 + M_k)\right] < 1.$$

PROOF : Suppose first that (i) holds. When $t \in J_0 = [0, t_1]$, by (2), we have

$$m(t) \le \int_0^t b(s)\,m(s)\,ds + \int_0^a c(s)\,m(s)\,ds. \qquad \ldots (3)$$

Let $p_0(t) = \int_0^t b(s)\,m(s)\,ds$. Then $p_0(t) \in C^1[J_0, R^+]$ and $p_0(0) = 0, p_0'(t) = b(t)\,m(t)$ for $t \in J_0$. By (3), we have

$$p_0(t) \le b(t)\,p_0(t) + b(t)\int_0^a c(s)\,m(s)\,ds,\quad t \in J_0,$$

then the above inequality reduces to

$$\left(p_0(t)\, e^{-\int_0^t b(s)\,ds}\right)' \le e^{-\int_0^t b(s)\,ds}\, b(t) \int_0^a c(s)\, m(s)\, ds, \quad t \in J_0.$$

It follows by integrating each side from 0 to t_1 that

$$p_0(t_1)\, e^{-\int_0^{t_1} b(s)\,ds} \le \left(1 - e^{-\int_0^{t_1} b(s)\,ds}\right) \int_0^a c(s)\, m(s)\, ds,$$

i.e.,

$$\int_0^{t_1} b(s)\, m(s)\, ds \le \left(e^{\int_0^{t_1} b(s)\,ds} - 1\right) \int_0^a c(s)\, m(s)\, ds. \qquad \dots (4)$$

When $t \in J_1 = (t_1, t_2]$, from (2), (3) and (4), we have

$$m(t) \le \int_0^t b(s)\, m(s)\, ds + \int_0^a c(s)\, m(s)\, ds + M_1\, m(t_1)$$

$$\le \left(\int_0^{t_1} b(s)\, m(s)\, ds + \int_{t_1}^t b(s)\, m(s)\, ds\right) + \int_0^a c(s)\, m(s)\, ds$$

$$+ M_1 \left(\int_0^{t_1} b(s)\, m(s)\, ds + \int_0^a c(s)\, m(s)\, ds\right)$$

$$= \int_{t_1}^t b(s)\, m(s)\, ds + (1 + M_1) \int_0^{t_1} b(s)\, m(s)\, ds + (1 + M_1) \int_0^a c(s)\, m(s)\, ds$$

$$\le \int_{t_1}^t b(s)\, m(s)\, ds + (1 + M_1)\, e^{\int_0^{t_1} b(s)\,ds} \int_0^a c(s)\, m(s)\, ds. \qquad \dots (5)$$

Let $p_1(t) = \int_{t_1}^t b(s)\, m(s)\, ds$. Then $p_1(t) \in C^1[[t_1, t_2], R^+]$ and $p_1(t_1) = 0, p_1'(t) = b(t)\, m(t)$ for $t \in J_1$. Therefore, we have by (5)

$$p_1'(t) \le b(t)\, p_1(t) + (1 + M_1)\, e^{\int_0^{t_1} b(s)\,ds}\, b(t) \int_0^a c(s)\, m(s)\, ds.$$

Similar to the proof of inequality (4), we get

$$\int_{t_1}^{t_2} b(s)\, m(s)\, ds \leq (1 + M_1)\left(e^{\int_0^{t_2} b(s)\, ds} - e^{\int_0^{t_1} b(s)\, ds} \right) \int_0^a c(s)\, m(s)\, ds.$$

By repeating above process until $t \in J_m$, we have

$$\int_{t_1}^{t_m} b(s)\, m(s)\, ds \leq \prod_{k=1}^{m} (1 + M_k)\left(e^{\int_0^{a} b(s)\, ds} - e^{\int_0^{t_m} b(s)\, ds} \right) \int_0^a c(s)\, m(s)\, ds.$$

Adding together, we find

$$\int_0^a \left\{ b(s) - \left[\left(e^{\int_0^{t_1} b(s)\, ds} - 1 \right) + (1 + M_1)\left(e^{\int_0^{t_2} b(s)\, ds} - e^{\int_0^{t_1} b(s)\, ds} \right) \right.\right.$$

$$\left. + \dots + \left\{ \prod_{k=1}^{m} (1 + M_k)\left(e^{\int_0^{a} b(s)\, ds} - e^{\int_0^{t_m} b(s)\, ds} \right) \right] c(s) \right\} m(s)\, ds \leq 0.$$

From (i), we infer that $m(t) \equiv 0$, $\forall t \in J$.

Let us suppose now that (ii) holds. When $t \in J_0$, by (2), we have

$$m(t) \leq \int_0^t b(s)\, m(s)\, ds + \int_0^a c(s)\, m(s)\, ds \leq \int_0^a b(s) + c(s))\, m(s)\, ds, \qquad \dots (6)$$

and, after integrating each side, we get

$$\int_0^{t_1} m(t)\, dt \leq t_1 \int_0^a (b(s) + c(s))\, m(s)\, ds.$$

When $t \in J_1$, (2) and (6) imply that

$$m(t) \leq \int_0^t b(s)\, m(s)\, ds + \int_0^a c(s)\, m(s)\, ds + M_1\, m(t_1)$$

$$\leq \int_0^a (b(s) + c(s))\, m(s)\, ds + M_1 \int_0^a (b(s) + c(s))\, m(s)\, ds$$

$$= (1 + M_1) \int_0^a (b(s) + c(s))\, m(s)\, ds,$$

and so $$\int_{t_1}^{t_2} m(t)\,dt \le (t_2 - t_1)(1 + M_1) \int_0^a (b(s) + c(s))\, m(s)\, ds.$$

By repeating above process until $t \in J_m$, we have

$$\int_{t_m}^{a} m(t)\,dt \le (a - t_m) \prod_{k=1}^{m} (1 + M_k) \int_0^a (b(s) + c(s))\, m(s)\, ds$$

Hence, we get by adding together

$$\int_0^a m(t)\,dt \le \Big[t_1 + (t_2 - t_1)(1 + M_1) + \dots + (a - t_m) \prod_{k=1}^{m} (1 + M_k)\Big] \int_0^a (b(s) + c(s))\, m(s)\, ds,$$

which implies by virtue of (ii) that $m(t) \equiv 0$, $\forall t \in J$. The lemma is proved.

Lemma 5[2] — If $B \subset PC^1[J, E]$ is bounded and the elements of B' are equicontinuous on each J_i $(i = 0, 1, \dots, m)$, then

$$\alpha_{PC^1}(B) = \max\left\{ \sup_{t \in J} \alpha(B(t)),\ \sup_{t \in J} \alpha(B'(t)) \right\}.$$

Lemma 6[6] — Let X be a Banach space, $K \subset X$ closed and convex, $F : K \to K$ continuous and such that for some $x_0 \in K$ condition

if $B \subset K$ is countable and $\overline{B} = \overline{co}\left(\{x_0\} \cup F(B)\right)$ then $\overline{B}$ is compact is satisfied. Then F has a fixed point in K.

3. Main Results

Before giving the main results, we first introduce some denotions. For any $R > 0$, set

$$M_1(R) = \sup\left\{ \| f(t, u(t), u'(t), (Tu)(t), (Su)(t)) \| : t \in J, \| u \|_{PC^1} \le R \right\},$$

$$M_2(R) = \sup\left\{ \sum_{k=1}^{m} \| I_k(u(t_k), u'(t_k)) \| : \| u \|_{PC^1} \le R \right\},$$

$$M_3(R) = \sup\left\{ \sum_{k=1}^{m} \| \overline{I}_k(u(t_k), u'(t_k)) \| : \| u \|_{PC^1} \le R \right\}.$$

***Theorem* 1** — *Suppose that* $f \in C[J \times E \times E \times E \times E, E]$ *and* $I_k, \overline{I}_k \in C[E \times E, E]$ $(k = 1, 2, \dots, m)$ *satisfy the following conditions :*

(H_1) $\lim_{R \to \infty} \sup \dfrac{M_1(R) + a^{-2} M_2(R) + a^{-1} M_3(R)}{R} < \dfrac{1}{a \cdot \max\{a, 1\}}$;

(H_2) for any countable equicontinuous on each J_i ($i = 0, 1, \ldots, m$) and bounded subset $B \subset PC^1[J, E]$ and $t \in J$,

$$\alpha(f(t, B(t), B'(t), (TB)(t)(SB)(t))) \leq L_1(t) \max\{\alpha(B(t)), \alpha(B'(t))\}$$

$$+ L_2(t)\,\alpha((TB)(t)) + L_3(t)\,\alpha((SB)(t)),$$

$$\alpha(I_k(B(t_k), B'(t_k))) \leq \sigma_k \max\{\alpha(B(t_k)), \alpha(B'(t_k))\},$$

$$\alpha(\bar{I}_k(B(t_k), B'(t_k))) \leq \bar{\sigma}_k \max\{\alpha(B(t_k)), \alpha(B'(t_k))\},\ k = 1, 2, \ldots, m,$$

where $L_i \in C[J, R^+]$ ($i = 1, 2, 3$) and $\sigma_k, \bar{\sigma}_k \geq 0$ ($k = 1, 2, \ldots, m$) are constants which satisfy (i) or (ii) of Lemma 4, here $a^* = \max\{a, 1\}$, and

$$b(t) = 2(L_1(t) + ak_0 L_2(t))\, a^*,\ c(t) = 2ah_0 L_3(t)\, a^*,\ t \in J,$$

$$M_k = \max\left\{\sigma_k + (a - t_k)\bar{\sigma}_k, \bar{\sigma}_k\right\},\ k = 1, 2, \ldots, m.$$

Then IVP (1) has at least a solution in $PC^1[J, E] \cap C^2[J', E]$.

PROOF : It is clear that $u \in PC^1[J, E] \cap C^2[J' E]$ is a solution of IVP (1) if and only if u is a solution of the following impulsive integral equation

$$u(t) = u_0 + tu_1 + \int_0^t (t - s) f(s, u(s), u'(s), (Tu)(s)(Su)(s))\, ds$$

$$+ \sum_{0 < t_k < t} [I_k(u(t_k), u'(t_k)) + (t - t_k)\bar{I}_k(u(t_k), u'(t_k))],\quad t \in J.$$

Define operator A by

$$A = A_1 + A_2, \qquad \ldots (7)$$

where
$$(A_1 u)(t) = u_0 + tu_1 + \int_0^t (t - s) f(s, u(s)\, u'(s), (Tu)(s), (Su)(s))\, ds,$$

$$(A_2 u)(t) = \sum_{0 < t_k < t} [I_k(u(t_k), u'(t_k)) + (t - t_k)\bar{I}_k(u(t_k), u'(t_k))],$$

then A is a continuous operator from $PC^1[J, E]$ into $PC^1[J, E]$. In fact, for any sequences $\{u_n\} \subset PC^1[J, E]$ and $u \in PC^1[J, E]$ with

$$\| u_n - u \|_{PC^1} \to 0 \text{ as } n \to \infty. \qquad \ldots (8)$$

We denote $t_0 = 0, t_{m+1} = a, I_k = [t_{k-1}, t_k], 1 \le k \le m+1$ and define on I_k :

$$[u_n]_k(t) = \begin{cases} u_n(t), & t \in (t_{k-1}, t_k] \\ u_n(t_{k-1}+0), & t = t_{k-1}, \end{cases} \quad [u_n']_k(t) = \begin{cases} u_n'(t), & t \in (t_{k-1}, t_k] \\ u_n'(t_{k-1}+0), & t = t_{k-1}, \end{cases}$$

$$[u]_k(t) = \begin{cases} u(t), & t \in (t_{k-1}, t_k] \\ u(t_{k-1}+0), & t = t_{k-1}, \end{cases} \quad [u'](t) = \begin{cases} u'(t), & t \in (t_{k-1}, t_k] \\ u'(t_{k-1}+0), & t = t_{k-1}, \end{cases}$$

then $[u_n]_k, [u_n']_k, [u]_k, [u']_k, f(t, [u_n]_k, [u_n']_k, Tu_n, Su_n), f(t, [u]_k, [u']_k, Tu, Su) \in C[I_k, E]$. By the continuity of f. For any $t_0 \in I_k$ and $\varepsilon > 0$, there exists $\delta_1(t_0, \varepsilon) > 0$ such that for $t \in I_k$ and $|t - t_0 < \delta_1 \cdot \| x - [u]_k(t_0)\| < \delta_1, \|y - [u']_k(t_0)\| < \delta_1 \; \|z - Tu(t_0)\| < \delta_1, \|w - Su(t_0)| < \delta_1$, we have

$$\| f(t, x, y, z, w) - f(t_0, [u]_k(t_0), [u']_k(t_0), Tu(t_0), Su(t_0)) \| < \frac{\varepsilon}{2a^2}. \qquad \text{... (9)}$$

On the other hand,

$$\| [u_n]_k(t) - [u]_k(t_0) | \le \| u_n - u \| PC + \| [u]_k(t) - [u]_k(t_0) \|, \qquad \text{... (10)}$$

$$\| [u_n']_k(t) - [u']_k(t_0) | \le \| u_n' - u' \| PC + \| [u']_k(t) - [u']_k(t_0) \|, \qquad \text{... (11)}$$

$$\| Tu_n(t) - Tu(t_0) | \le ak_0 \| u_n - u \| PC + \| Tu(t) - Tu(t_0) \|, \qquad \text{... (12)}$$

$$\| Su_n(t) - Su(t_0) | \le ah_0 \| u_n - u \| PC + \| Su(t) - Su(t_0) \|. \qquad \text{... (13)}$$

By the continuity of $[u]_k, [u']_k$, Tu, Su and (8), (10)-(13), there exist $0 < \delta_2 (< \delta_1)$ and $n^* (\delta_2, \varepsilon) \ge 0$ such that for $t \in I_k$ and $|t - t_0 < \delta_2, n \ge n^*$, we have

$$\| [u_n]_k(t) - [u]_k(t_0) | < \delta_1, \quad \| [u_n']_k(t) - [u']_k(t_0) | < \delta_1,$$

$$\| Tu_n(t) - Tu(t_0) | < \delta_1, \quad \| Su_n(t) - Su(t_0) | < \delta_1,$$

so, we have by (9)

$$\| f(t, [u_n]_k(t), [u_n']_k(t), Tu_n(t), Su_n(t)) - f(t_0, [u]_k(t_0), [u']_k(t_0), Tu(t_0), Su(t_0)) \| < \frac{\varepsilon}{2a^2}. \qquad \text{... (14)}$$

Since $f(t, [u]_k, [u']_k, Tu, Su) \in C[I_k, E]$, there exists $0 < \delta (< \delta_2)$ such that for $t \in I_k$ and $|t - t_0| < \delta$, we have

$$\| f(t, [u]_k(t), [u']_k(t), Tu(t), Su(t)) - f(t_0, [u]_k(t_0), [u']_k(t_0), Tu(t_0), Su(t_0)) \| < \frac{\varepsilon}{2a^2}. \qquad \text{... (15)}$$

Hence, for $n \ge n^*$ and $t \in I_k, |t - t_0| < \delta$, from (14) and (15), we get

$$\| f(t, [u_n]_k(t), [u_n']_k(t), Tu_n(t), Su_n(t))$$

$$-f(t, [u]_k(t), [u']_k(t), Tu(t), Su(t)) \| < \frac{\varepsilon}{a^2}.$$

By the compactness of interval I_k, there exist $N_k > 0$ such that for $n \geq N_k$ and $t \in I_k$, we have the above inequality. Let $N_0 = \max\{N_1, N_2, \ldots, N_{m+1}\}$, then for $n \geq N_0$ and $t \in J$, it follows from the definition of A_1 that

$$\| A_1 u_n(t) - A_1 u(t) \|$$

$$\leq a \int_0^t \| f(s, u_n(s), u_n'(s), Tu_n(s), Su_n(s)) - f(s, u(s), u'(s), Tu(s), Su(s)) \| ds$$

$$\leq a \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \| f(s, u_n(s), u_n'(s), Tu_n(s), Su_n(s))$$

$$-f(s, u(s), u'(s), Tu(s), Su(s)) \| ds$$

$$= a \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \| f(s, [u_n]_k(s), [u_n']_k(s), Tu_n(s), Su_n(s))$$

$$-f(s, [u]_k(s), [u']_k(s), Tu(s), Su(s) \| ds < \varepsilon,$$

i.e., A_1 is continuous. Using the similar method, it is easy to show that A_2 is also continuous. Therefore A is a continuous operator from $PC^1[J, E]$ into $PC^1[J, E]$.

By (H_1), there exist real number $r \in (0, (a \cdot \max\{a, 1\})^{-1})$ and $R_0 > 0$ such that

$$M_1(R) + a^{-2} M_2(R) + a^{-1} M_3(R) < rR \qquad \ldots (16)$$

for $R \geq R_0$. Let $R^* = \max\{R_0, [\| u_0 \| + (a+1) \| u_1 \|][1 - ra \cdot \max(a, 1)]^{-1}\}$ and

$$F = \{u \in PC^1[J, E] : \| u \|_{PC^1} \leq R^*, \ \| u(t') - u(t'') \| \leq \| u_1 \| + arR^*) \| t' - t'' \|,$$

$$t', T'' \in J_i, i = 0, 1, \ldots, m\},$$

then it is easy to see that F is equicontinuous on each $J_i\,(i + 0, 1, \ldots, m)$ and bounded convex subset in $PC^1[J, E]$. For any $u \in F$, we have by (7) and (16)

$$\| Au \|_{PC} \leq \| u_0 \| + a \| u_1 \| + a^2 M_1(R^*) + M_2(R^*) + aM_3(R^*)$$

$$\leq \| u_0 \| + a \| u_1 \| + {}^2 rR^2,$$

and for $t \in J, t \neq t_k$,

$$(Au)'(t) = u_1 + \int_0^t f(s, u(s), u'(s), (Tu)(s), (Su)(s))\, ds + \sum_{0 < t_k < t} \overline{I}_k (u(t_k), u'(t_k))$$

implies that

$$\| (Au)' \|_{PC} \le \| u_1 \| + aM_1 (R^*) + M_3 (R^*) \le \| u_1 \| + arR^*, \qquad \dots (17)$$

hence $\| Au \|_{PC^1} \le [\| u_0 \| + (a+1) \| u_1 \|] + ra \cdot \max \{a, 1\}\ R^* \le R^*$. By the mean value theorem, we have

$$(Au)(t') - (Au)(t'') \in (t' - t'')\, \overline{co}\ (\{(Au)'(t) : t'' < t < t'\}),\ t', t'' \in J_i,\ i = 0, 1, \dots, m,$$

therefore, from (17), we get

$$\| (Au)(t') - (Au)(t'') | \le (\| u_1 \| + arR^* \| t' - t'', t', t'' \in J_i,\ i = 0, 1, \dots, m.$$

Hence A is a continuous operator from F into F.

Let $B \subset F$ be any countable subset satisfying $\overline{B} = \overline{co}\left(\left\{x_0\right\} \cup A(B)\right)$ for some $x_0 \in F$. We now prove that B is relatively compact in $PC^1 [J, E]$. Firstly, by Lemma 1, we get

$$\overline{B(t)} = \overline{co}\,(\{x_0(t)\} \cup A(B)(t)), \overline{B'(t)} = \overline{co}\ (\{x_0'(t)\} \cup (AB)'(t)),\quad t \in J,$$

Then by the properties of measure of noncompactness, it is easy to see that

$$\alpha (B(t)) = \alpha (\overline{B(t)}) = \alpha (A(B)(t)),\quad t \in J, \qquad \dots (18)$$

$$\alpha (B'(t)) = \alpha (\overline{B'(t)}) = \alpha (B)'(t)),\quad t \in J. \qquad \dots (19)$$

By applying Lemma 2 to (7) and employing (H_2), we find

$$\alpha (A(B)(t)) \le \alpha \left(\left\{ \int_0^t (t-s) f(s, u(s), u'(s), (Tu)(s), (Su)(s))\, ds : u \in B \right\} \right)$$

$$+ \sum_{0 < t_k < t} [\alpha (I_k (B(t_k), B'(t_k))) + (a - t_k)\, \alpha (\overline{I}_k (B(t_k), B'(t_{kl}))))]$$

$$\le 2a \int_0^t \alpha (f(s, B(s), B'(s), (TB)(s), (SB)(s)))\, ds$$

$$+ \sum_{0 < t_k < t} [\sigma_k + (a - t_k)\, \overline{\sigma}_k] \max \left\{ \alpha (B)(t_k)), \alpha (B)'(t_k)) \right\}. \qquad \dots (20)$$

In addition, by (H_2) and Lemma 3, we have

$$\alpha(f(s, B(s), B'(s), (TB)(s), (SB)(s)) \le L_1(s)\ \max\{ \alpha(B(s)), \alpha(B'(s))\}$$

$$+ \ k_0 L_2(s) \int_0^s \alpha(B(\tau))d\tau$$

$$+ \ h_0 L_3(s) \int_0^a \alpha(B(\tau))d\tau. \qquad \text{... (21)}$$

Substituting (21) into (20), altering integral sequence and by (18)-(19), we get

$$\alpha(A(B)(t)) \leq 2a \int_0^t L_1(s) \ \max\{\alpha(A(B)(s)), \alpha((AB)'(s))\}ds$$

$$+ \ 2ak_0 \int_0^t (t-s)L_2(s) \ \alpha(A(B)(s))ds + 2ah_0 t \int_0^a L_3(s)\alpha(A(B)(s))ds$$

$$+ \sum_{0 < t_k < t} [\sigma_k + (a - t_k) \ \overline{\sigma}_k] \ \max\{\alpha(A(B)(t_k)), \alpha((AB)'(t_k))\}. \qquad \text{... (22)}$$

Using the same method as above, we also get

$$\alpha((AB)'(t)) \leq 2 \int_0^t L_1(s) \max\{\alpha(A(B)(s)), \alpha((AB)'(s))\} ds$$

$$+ \ 2k_0 \int_0^t (t-s) L_2(s) \ \alpha(A(B)(s)) \ ds + 2h_0 \, t \int_0^a L_3(s) \ \alpha(A(B)(s)) \ ds$$

$$+ \sum_{0 < t_k < t} \overline{\sigma}_k \max\{\alpha(A(B)(t_k)), \alpha((AB)'(t_k))\}. \qquad \text{... (23)}$$

Since $B \subset F$, then we have $A(B) \subset F$ is a bounded subsset in $PC^1[J, E]$, and the elements of $A(B)$ and $(AB)'$ are all equicontinuous on each J_i $(i = 0, 1, ..., m)$. It follows from Lemma 3 that $m(t) = \max\{\alpha(A(B)(t)), \alpha((AB)'(t))\} \in C[J_i, R^+]$ $(i = 0, 1, ..., m)$, and by (22) and (23), we have

$$m(t) \leq \int_0^t b(s)m(s) \, ds + \int_0^a c(s) \, m(s) \, ds + \sum_{0 < t_k < t} M_k \, m(t_k), \quad t \in J,$$

where $b(s) = 2(L_1(s) + ak_0 L_2(s)) \max a, 1, c(s) = 2ah_0 L_3(s) \max a, 1, \quad s \in J,$

$$M_k = \max\{\sigma_k + (a - t_k) \ \overline{\sigma}_k, \overline{\sigma}_k\}, \ k = 1, 2, ..., m,$$

which implies by virtue of Lemma 4 that $m(t) \equiv 0$ for $t \in J$, i.e., $\alpha(A(B)(t)) = \alpha((AB)'(t)) = 0$, $\forall \ t \in J$. Hence, by Lemma 5, we have

$$\alpha_{PC^1}(B) = \alpha_{PC^1}(A(B)) = \max\left\{\sup_{t\in J}\alpha(A(B)(t)),\ \sup_{t\in J}\alpha((AB)'(t))\right\} = 0,$$

that is, B is relatively compact in $PC^1[J, E]$. It follows from Lemma 6 that A has at least a fixed point in F, i.e., IVP (1) has a solution in F. Thus the proof is complete.

Remark 1 : In condition (H_2), we use $L_i \in C[J, R^+]$ $(i = 1, 2, 3)$ instead of non-negative constants L_i $(i = 1, 2, 3)$ in [2, 3, 7]. In fact, we not only further weaken the compactness-type conditions used in [2, 3, 7] by means of the new comparison result, but also cross out completely the stronger assumptions "for any $r > 0$, $f(t, x, y)$ be bounded and uniformly continuous in t on $J \times B_r \times B_r$, where $B_r = \{x \in E : \|x\| \le r\}$" in[2]. Therefore, Theorem 1 in this paper generalize and improve the related results in [2, 3], and our conclusions cannot be obtained by the methods used in[2, 3].

Remark 2 : When removing the impulsive terms and $L_i \in [J, R^+]$ $(i = 1, 2, 3)$ are non-negative constants, the conclusions of Lemma 4 include and extend Lemma 2 (comparison result) which is important in[7]. From condition (H_2), we can see that when f does not contain integral operator Su, Theorem 1 holds for any $L_1(t) + ak_0 L_2(t) > 0$ $(t \in J)$ and $\sigma_k, \overline{\sigma}_k \ge 0$ $(k = 1, 2, \ldots, m)$.

Remark 3 : In the same way, by using the new comparison result we may discuss the IVP of the nth order impulsive integro-differential equation in E

$$\begin{cases} u^{(n)} = f(t, u, u', \ldots, u^{(n-1)}, Tu, Su), & t \in J, t \ne t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u'(t_k), \ldots, u^{(n-1)}(t_k)), & \\ \Delta u'|_{t=t_k} = \overline{I}_k(u(t_k), u'(t_k), \ldots, u^{(n-1)}(t_k)), & \\ \ldots & \\ \Delta u^{(n-1)}|_{t=t_k} = I_k^*(u(t_k), u'(t_k), \ldots, u^{(n-1)}(t_k)) \quad (k = 1, 2, \ldots, m), & \\ u(0) = u_0, u'(0) = u_1, \ldots, u^{(n-1)}(0) = u_{n-1} & \end{cases}$$

and obtain similar results.

4. Application to MBVP for Fourth Order Impulsive Equations

As applications of theorem 1, we obtain the global solutions of two classes of mixed boundary value problems for fourth order impulsive equations.

Firstly, we consider the MBVP (I) in Banach space E. Suppose

$$\alpha_1^2 + \alpha_2^2 > 0, \beta_1^2 + \beta_2^2 > 0, \Delta\ \alpha_1\beta_1 + \alpha_1\beta_2 - \alpha_2\beta_1 \ne 0. \text{ Let } x'' = u,$$

then we have

$$x(t) = \int_0^1 h(t, s)\, u(s)\, ds \equiv (Su)(t), \quad t \in J = [0, 1],$$

where $h(t, s)$ is Green function of boundary value problem

$$\begin{cases} x'' = \theta, \\ \alpha_1 x(0) + \alpha_2 x'(0) = \theta, \\ \beta_1 x(1) + \beta_2 x'(1) = \theta. \end{cases}$$

Then MBVP (I) can be regarded as an IVP of the following second order integro-differential equation

$$\begin{cases} u'' = f(t, u, u', Su), & t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u'(t_k)), \\ \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k), u'(t_k)) & (k = 1, 2, ..., m), \\ u(0) = x_0, \ u'(0) = x_1. \end{cases}$$

The conclusions of theorem 2 follow from theorem 1.

***Theorem* 2** — *If $f \in C[J \times E \times E \times E, E]$ and $I_k, \bar{I}_k \in C[E \times E, E]$ $(k = 1, 2, ..., m)$ satisfy the following conditions :*

$$(C_1) \ \lim_{R \to \infty} \sup \frac{M_1(R) + M_2(R) + M_3(R)}{R} < 1,$$

where $M_1(R) = \sup \{\| f(t, u(t), u'(t), (Su)(t)) \| : \| u \|_{PC} \le R\}$, $M_2(R)$ and $M_3(R)$ is defined as above;

(C_2) for any countable equicontinuous on each J_i $(i = 0, 1, ..., m)$ and bounded subset $B \subset PC^1[J, E]$ and $t \in J$,

$$\alpha(f(t, B(t), B'(t), (SB)(t))) \le L_1(t) \max\{\alpha(B(t)), \alpha(B'(t))\} + L_3(t)\, \alpha((SB)(t)),$$

$$\alpha(I_k(B(t_k), B'(t_k))) \le \sigma_k \max\{\alpha(B(t_k)), \alpha(B'(t_k))\},$$

$$\alpha(\bar{I}_k(B(t_k), B'(t_k))) \le \bar{\sigma}_k \max\{\alpha(B(t_k)), \alpha(B'(t_k))\}, \ k = 1, 2, ..., m$$

where $L_i \in C[J, R^+]$ $(i=1,3)$ and $\sigma_k, \bar{\sigma}_k \ge 0$ $(k = 1, 2, ..., m)$ are constants which for any $t \in J$ satisfy one of the following conditions:

$$(1) \ h_0 L_3(t) \left[\left(e^{\int^{t_1} 2L_1(s)\, ds} - 1 \right) + (1 + M_1) \left(e^{\int^{t_2} 2L_1(s)\, ds} - e^{\int^{t_1} 2L_1(s)\, ds} \right) \right.$$

$$+ \dots + \prod_{k=1}^{m} (1+M_k) \left(\int_{e0}^{1} 2L_1(s)\, ds \; \int_{-e0}^{t_m} 2L_1(s)\, ds \right) \Bigg] < L_1(t);$$

$$(2)\ 2\,(L_1(t) + h_0 L_3(t)) \left[t_1 + (t_2 - t_1)(1+M_1) + \dots + (1-t_m) \prod_{k=1}^{m} (1+M_k) \right] < 1;$$

in which $M_k = \max \left\{ \sigma_k + (1-t_k)\,\bar{\sigma}_k, \bar{\sigma}_k \right\}$, $k = 1, 2, \dots, m$.

Then MBVP (I) has at least a global solution in $PC^1[J, E] \cap C^4[J', E]$.

Secondly, we consider the MBVP(II) in Banach space E. Suppose

$$\beta_1 \neq 0, \alpha_1 = 0, \alpha_2 = 1, \Delta = -\beta_1 \neq 0. \text{ Let } x'' = u,$$

then we have

$$x(t) = \int_0^t u(s)\, ds \equiv (Tu)(t),\ k(ts) \equiv 1, 0 \le S \le t \le 1.$$

$$x^2(t) = \int_0^1 h(t, s)\, u(s)\, ds \equiv (Su)(t), \quad t \in J = [0, 1],$$

where $h(t, s)$ is defined as above. Then MBVP (II) can be regarded as an IVP of the following second order integro-differential equation

$$\begin{cases} u'' = f(t, u, u', Tu, Su), & t \in J,\ t \neq t_k, \\ \Delta u' |_{t=t_k} = \bar{I}_k(u(t_k), u'(t_k)) & (k = 1, 2, \dots, m), \\ u(0) = x_0,\ u'(0) = x_1. \end{cases}$$

From Theorem 1, we can obtain the conclusions of Theorem 3.

***Theorem* 3** — *If $f \in C[J \times E \times E \times E \times E, E]$ and $\bar{I}_k \in C[E \times E, E]$ $(k = 1, 2, \dots, m)$ satisfy the following conditions :*

(D_1) $\limsup_{R \to \infty} \dfrac{M_1(R) + M_3(R)}{R} < 1$, where $M_i(R)$ $(i = 1, 3)$ are defined as above

(D_2) for any countable equicontinuous on each J_i $(i = 0, 1, \dots, m)$ and bounded subset $B \subset PC^1[J, E]$ and $t \in J$,

$$\alpha(f(t, B(t), B'(t), (TB)(t), (SB)(t))) \le L_1(t) \max \{\alpha(B(t)), \alpha(B'(t))\}$$
$$+ L_2(t)\, \alpha((TB)(t)) + L_3(t)\, \alpha((SB)(t)),$$

$$\alpha(\overline{I}_k(B(t_k), B'(t_k))) \le \overline{\sigma}_k \max\{\alpha(B(t_k)), \alpha(B'(t_k))\}, \ k = 1, 2, \ldots, m,$$

where $L_i \in C[J, R^+]$ $(i=1,2,3)$ and $\overline{\sigma}_k \ge 0$ $(k = 1, 2, \ldots, m)$ are constants which for any $t \in J$ satisfy one of the following conditions :

$$(1)\ h_0 L_3(t)\left[\left(e^{\int_0^t b(s)\,ds} - 1\right) + (1+\overline{\sigma}_1)\left(e^{\int_0^{t_2} b(s)\,ds} - e^{\int_0^{t_1} b(s)\,ds}\right) + \ldots + \prod_{k=1}^{m}(1+\overline{\sigma}_k)\left(e^{\int_0^{1} b(s)\,ds} - e^{\int_0^{t_m} b(s)\,ds}\right)\right] < L_1(t) + L_2(t) :$$

$$(2)\ 2(L_1(t) + L_2(t) + h_0 L_3(t))$$

$$\left[t_1 + (t_2 - t_1)(1+\overline{\sigma}_1) + \ldots + (1 - t_m)\prod_{k=1}^{m}(1+\overline{\sigma}_k)\right] < 1$$

in which $b(s) = 2(L_1(s) + L_2(s)),\ s \in J$.

Then MBVP (II) has at least a global solution in $PC^1[J, E] \cap C^4[J', E]$.

Remark 4 : In order to applications for convenience, we give the concrete sufficient condition of f in theorem 2 (in case of theorem 3, the condition is similar). If

$$\lim_{\|x\|+\|y\|+\|z\| \to \infty} \frac{\max_{t \in J}\|f(t,x,y,z)\| + \sum_{k=1}^{m}[\|I_k(x,y)\| + \|\overline{I}_k(x,y)\|]}{\|x\|+\|y\|+\|z\|} < \frac{1}{2+h_0}. \quad \ldots (24)$$

then condition (C_1) of Theorem 2 is satisfied. In fact, by (24), there exist a real number $R > 0$ such that for any $t \in J$ and $\|x\|+\|y\|+\|z\| \ge R$,

$$\|f(t,x,y,z)\| + \sum_{k=1}^{m}[\|I_k(x,y)\| + \|\overline{I}_k(x,y)\|] \le \overline{h}(\|x\|+\|y\|+\|z\|),$$

where $\overline{h} < \frac{1}{2+h_0}$. For $\forall t \in J$, $\forall x, y, z \in E$, it follow from $f \in C[J \times E \times E \times E, E]$ and $I_k, \overline{I}_k \in C[E \times E, E]$ $(k = 1, 2, \ldots, m)$ that

$$\|f(t,x,y,z)\| + \sum_{k=1}^{m}[\|I_k(x,y)\| + \|\overline{I}_k(x,y)\|] \le \overline{h}(\|x\|+\|y\|+\|z\|) + M.$$

where $$M = \max\left(\|f(t,x,y,z)\| + \sum_{k=1}^{m}[\|I_k(x,y)\| + \|\overline{I}_k(x,y)\|]\right.$$

$: t \in J, \| x \| + \| y \| + \| z \| \leq R \} < +\infty$, so for $\forall t \in J$, $\forall\ u \in PC^1 [J, E]$, we have

$$\| f(t, u(t), u'(t), (Su)(t)) \| + \sum_{k=1}^{m} [\| \overline{I}_k(Ut_k), u'(t) \|]$$

$$\| \overline{h} (\| u (t) \| + \| u' (t) \| + \| (Su)(t) \|) + M$$

$$\leq \overline{h} (2 + h_0) \| u \|_{PC}^1 + M.$$

Hence $$\lim_{R \to \infty} \sup \frac{M_1 (R) + M_2 (R) + M_3 (R)}{R} \leq \overline{h} (2 + h_0) < 1,$$ i.e., condition (C_1) holds.

On the other hand, if for $\forall t \in J$, $\forall x_i, y_i, z_i, u_i, \upsilon_i \in E$ $(i = 1, 2)$, we have

$$\| f (t, x_1, y_1, z_1) - f (t, x_2, y_2, z_2) \|$$

$$\leq L_1 (t) (\| x_1 - x_2 \| + \| y_1 - y_2 \|) + L_3 (t) \| z_1 - z_2 \|,$$

$$\| I_k (u_1, \upsilon_1) - I_k (u_2 \upsilon_2) \| \leq \sigma_k (\| u_1 - u_2 \| + \| \upsilon_1 - \upsilon_2 \|),$$

$$\| \overline{I}_k (u_1, \upsilon_1) - I_k (C_2, \upsilon_2) \| \leq \overline{\sigma}_k (\| u_1 - u_2 \| + \| \upsilon_1 - \upsilon_2 \|),\ k = 1, 2, ..., m$$

where $L_i \in C [J, R^+]$ $(i = 1, 3)$ and $\sigma_k, \overline{\sigma}_k \geq 0\ (k = 1, 2, ..., m)$ satisfy (1) or (2) in condition (C_2). then it is clear that condition (C_2) is satisfied. If $E = c_0 = \left\{ x = (x_1, ..., x_n, ...) : x_n \to 0 \right\}$, then by the diagonal method, it is easy to verify that $\alpha (f (t, B(t), B' (t), (SB)(t))) = 0$ for $\forall t \in J$ and $\alpha (I_k (B(t_k), B' (t_k))) = 0$, $\alpha (\overline{I}_k (B (t_k), B' (t_k))) = 0$ $(k = 1, 2, ..., m)$. In this case, condition (C_2) holds for any $L_1 (t), L_3 (t)\ (t \in J)$ and $\sigma_k, \overline{\sigma}_k \geq 0\ (k = 1, 2, .., m)$.

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