Indian J. pure appl. Math., 34(6) : 891-905, June 2003

EXISTENCE THEOREMS FOR SECOND ORDER THREE-POINT BOUNDARY VALUE PROBLEMS CONTAINING IMPULSES IN A BANACH SPACE

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(*Received* 30 *January* 2002; *accepted* 22 *November* 2002)

The present paper is devoted to the investigation of the existence of solutions of three-point boundary value problems for second order nonlinear impulsive integro-differential equations in a Banach space E

$$\begin{cases} -u'' = f(t, u, u', Tu, Su), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u'(t_k)), & k = 1, 2, \ldots, m \\ \Delta u'|_{t=t_k} = \overline{I}_k(u(t_k), u'(t_k)), & k = 1, 2, \ldots, m \\ u(0) = \alpha u(\eta) = u(1) \end{cases}$$

where $\alpha \in \mathbf{R} \setminus \{0, 1\}$, $\eta \in (0, 1)$. We employ the Leray-Schauder Continuation Theorem for condensing mappings to establish the existence theorems via the obtained *a priori* estimates completely different from those in the previous literature.

Key Words : Impulsive Integro-Differential Equation; Three-point Boundary Value Problem; Strict-Set Contraction; *a priori* Estimate

1. INTRODUCTION

The earliest literature in the study of multi-point boundary value problems is due to Barr and Sherman[1] published in 1973. Since 1991, concerning with second order three-point boundary value problems, many authors have published a large quantity of research results (see Gupta[2--5], Ma[6], etc.); all the right-hand side nonlinearities they used are real-valued functions, and the Wirtinger's inequality is mostly applied to obtain the *a priori* estimates for the derivatives of solutions. However, the Wirtinger's inequality need not be valid any more in abstract spaces, and the multiplication or division operation can not be performed directly between abstract elements. Thus discussing second order three-point boundary value problems in a Banach space is an interesting task. Meanwhile, rather abundant results have been achieved in the study of impulsive differential equations (see Lakshmikantham, Bainov, and Simeonov[7]). In this paper, we shall tackle three-point boundary value problems for second order nonlinear impulsive integro-differential equations in a Banach space, and so this will impose, to a great extent, much more difficulties in obtaining the *a priori* estimates. We employ the Leray-Schauder Continuation Theorem for condensing mappings to establish the existence theorem via the obtained *a priori* estimates completely different from those in the previous literature. The existence results essentially extend and develop some related results in the literature[2-7], etc.

The following Leray-Schauder Continuation Theorem for condensing mappings will play a crucial role in our existence arguments.

***Theorem** L-S (Guo[8]) — Let E be a Banach space and $A : E \to E$ a condensing mapping. Provided the set $\{x \mid x \in E, x = \lambda Ax, 0 < \lambda < 1\}$ is bounded in E. Then A possesses at least one fixed point in E*

Theorem L-S is valid for a strict-set contraction, because a strict-set contraction is also condensing.

2. Preliminaries and Lemmas

We shall consider in a Banach space E the following second order three-point boundary value problem (3pBVP)

$$\begin{cases} -u'' = f(t, u, u', Tu, Su), & t \in J, t \neq t_k, \\ \Delta u\,|_{t=t_k} = I_k(u(t_k), u'(t_k)), & \\ \Delta u'\,|_{t=t_k} = \bar{I}_k(u(t_k), u'(t_k)), & \\ u(0) = \alpha u(\eta) = u(1) & \end{cases} \qquad \dots (2.1)$$

where $\alpha \in \boldsymbol{R} \setminus \{0, 1\}$, $f \in C[J \times E \times E \times E \times E, E]$, $J = [0, 1]$,

$$0 < t_1 < \dots < t_n < \eta < t_{n'+1} < \dots < t_m < 1,$$

$$(Tu)(t) = \int_0^t g(t, s)\, u(s)\, ds, \quad (Su)(t) = \int_0^1 h(t, s)\, u(s)\, ds,$$

$$g \in C[D, \boldsymbol{R}], h \in C[J \times J, \boldsymbol{R}], D = \{(t, s) \in J \times J \mid t \geq s\},$$

$$I_k \in C[E \times E, E]\, \bar{I}_k \in C[E \times E, E]$$

for $k = 1, 2, \dots, m$, θ is the zero element in E. $\Delta u\,|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e.

$$\Delta u\,|_{t=t_k} = u(t_k^+) - u(t_k^-),$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ respectively, and $\Delta u'\,|_{t=t_k}$ has a similar meaning for $u'(t)$. Set $PC[J, E] = \{x : J \to E \mid x(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k, x(t_k^+)$ exists for $k = 1, 2, \dots, m\}$, $PC^1[J, E] = \{x : J \to E \mid x(t)$ is continuously differentiable at $t \neq t_k$ and left continuous at $t = t_k, x(t_k^+), x'(t_k^+)$ and $x'(t_k^-)$ exist for $k = 1, 2, \dots, m\}$. It is noted that $PC[J, E]$ and $PC^1[J, E]$ are Banach spaces when equipped with the norms, $\|x\|_{PC} = \sup_{t \in J} \|x(t)\|$, and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$, respectively. Put $J' = J \setminus \{t_1, \dots, t_m\}$, $J_0 = [0, t_1]$, and $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \dots, m$ $(t_{m+1} = 1)$. We call u a solution of the boundary value problems (3pBVP) if $u \in PC^1[J, E] \cap C^2[J', E]$ satisfies (2.1).

In order to obtain the existence principles, we now state and prove the next two lemmas.

Lemma 1 — $u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of (3pBVP) if and only if $u \in PC^1[J, E]$ is a solution of the following impulsive integral equation (IE)

$$u(t) = \frac{\alpha\eta + (1-\alpha)t}{1-\alpha}\left[\int_0^1 (1-s) f(s, u, u', Tu, Su)\, ds - \sum_{k=1}^{m} I_k(u(t_k), u'(t_k)) + (1-t_k)\bar{I}_k(u(t_k), u'(t_k))]\right.$$

$$-\frac{\alpha}{1-\alpha}\left[\int_0^\eta (\eta - s) f(s, u, u', Tu, Su)\, ds - \sum_{k=1}^{n} [I_k(u(t_k), u'(t_k))\right. \qquad \text{... (2.2)}$$

$$-(\eta - t_k)\bar{I}_k(u(t_k), u'(t_k))]] - \int_0^t (t-s) f(s, u, u', Tu, Su)\, ds$$

$$+ \sum_{0<t_k<t} [I_k(u(t_k), u'(t_k)) + (t-t_k)\bar{I}_k(u(t_k), u'(t_k))] =: (\lambda u)(t).$$

PROOF : To prove "only if". Let $u \in PC^1[J, E] \cap C^2[J', E]$ be a solution of (3pBVP), then integrating on $[0, t]$ the first equation in (2.1) gives

$$u'(t) = u'(0) - \int_0^t f(s, u, u', Tu, Su)\, ds + \sum_{0<t_k<t} \bar{I}_k(u(t_k), u'(t_k))$$

Integrating the above eq. on $[0, t]$ yields

$$u(t) = u(0) + tu'(0) - \int_0^t (t-s) f(s, u, u', Tu, Su)\, ds$$

$$+ \sum_{0<t_k<t} [I_k(u(t_k), u'(t_k)) + (t-t_k)\bar{I}_k(u(t_k), u'(t_k))] \qquad \text{... (2.3)}$$

Hence,

$$u(\eta) = u(0) + \eta u'(0) - \int_0^\eta (\eta - s) f(s, u, u', Tu, Su)\, ds$$

$$+ \sum_{k=1}^{n} [I_k(u(t_k), u'(t_k)) + (\eta - t_k)\bar{I}_k(u(t_k), u'(t_k))] \qquad \text{... (2.4)}$$

$$u(1) = u(0) + u'(0) - \int_0^1 (1-s) f(s, u, u', Tu, Su)\, ds$$

$$+ \sum_{k=1}^{m} [I_k (u (t_k), u' (t_k)) + (1 - t_k) \overline{I}_k (u (t_k), u' (t_k))] \qquad \text{... (2.5)}$$

Observing the boundary conditions, we have from (2.4) and (2.5) that

$$u (0) = \frac{\alpha}{1-\alpha} \left\{ \eta \left[\int_0^1 (1-s) f (s, u, u', Tu, Su)\, ds - \sum_{k=1}^{m} [I_k (u (t_k), u' (t_k)) \right.\right.$$

$$\left. + (1 - t_k) \overline{I}_k (u (t_k), u' (t_k))] \right] - \int_0^{\eta} (\eta - s) f (s, u, u', Tu, su)\, ds$$

$$\left. + \sum_{k=1}^{n} [I_k (u (t_k), u' (t_k)) + (\eta - t_k) \overline{I}_k (u (t_k), u' (t_k))] \right\}$$

$$u' (0) = \int_0^1 (1-s) f (s, u, u', Tu, Su)\, ds$$

$$- \sum_{k=1}^{m} [I_k (u (t_k), u' (t_k)) + (1 - t_k) \overline{I}_k (u (t_k), u' (t_k))]$$

And thus substituting the above two eqs. into (2.3) leads to (2.2).

To prove "if". Assume $u \in PC^1 [J, E]$ is a solution of (2.2), then it is obvious that

$$u (0) = \alpha u (\eta) = u (1),$$

and $$\Delta u |_{t = t_k} = I_k (u (t_k), u' (t_k))$$

A differentiation of (2.2) with respect to t gives

$$u' (t) = \int_t^1 f (s, u, u', Tu, Su)\, ds - \int_0^1 s f (s, u, u', Tu, Su)\, ds$$

$$- \sum_{k=1}^{m} [I_k (u (t_k), u' (t_k)) + (1 - t_k) \overline{I}_k (u (t_k), u' (t_k))] \qquad \text{... (2.6)}$$

$$+ \sum_{0 < t_k < t} \overline{I}_k (u (t_k), u' (t_k))$$

Therefore, $$\Delta u' |_{t = t_k} = \overline{I}_k (u (t_k), u' (t_k)),$$

Again differentiating (2.6) with respect to t shows

$$u'' = - f (t, u, u', Tu, Su),\ t \in J,\ t \neq t_k.$$

This completes the proof of Lemma 1.

Lemma 2 — Assume for any $r > 0$, f is uniformly continuous on $J \times B_r \times B_r \times B_r \times B_r$ ($B_r = \{ \| x \in E \mid \| x \| \le r \}$), I_k and $\bar{I}_k$ are bounded on $B_r \times B_r$ for each $k = 1, ..., m$. Assume furthermore that

(H_1) there exist nonnegative constants L_i ($i = 1, 2, 3, 4$), M_{kj}, N_{kj} ($j = 1, 2$; $k = 1, ..., m$), such that

$$\begin{cases} \alpha(f(t, U_1, U_2, U_3, U_4)) \le \sum_{i=1}^{4} L_i \, \alpha(U_i), & t \in J,\ U_i \subset E \text{ bounded } (i = 1, 2, 3, 4); \\ \alpha(I_k(V_1, V_2)) \le \sum_{j=1}^{2} M_{kj} \, \alpha(V_j), & V_j \subset E \text{ bounded;} \\ \alpha(\bar{I}_k(V_1, V_2)) \le \sum_{j=1}^{2} N_{kj} \, \alpha(V_j), & V_j \subset E \text{ bounded } (j = 1, 2), \end{cases}$$

and $\delta = \max\{\delta_1, \delta_2\} < 1$, ... (2.7)

where

$$\delta_1 = 2\left[\left| \frac{\alpha}{1-\alpha} \right| \eta(1+\eta) + 2 \right](L_1 + L_2 + GL_3 + HL_4) + \left(\left| \frac{\alpha}{1-\alpha} \right| \eta + 2 \right) \sum_{k=1}^{m} [M_{k1} + M_{k2} + (1 - t_k)(N_{k1} + N_{k2})] + \left| \frac{\alpha}{1-\alpha} \right| \sum_{k=1}^{n} [M_{k1} + M_{k2} + (\eta - t_k)(N_{k1} + N_{k2})]$$

$$\delta_2 = 4(L_1 + L_2 + GL_3 + HL_4) + \sum_{k=1}^{m} [M_{k1} + M_{k2} + (2 - t_k)(N_{k1} + N_{k2})]$$

$$G = \max_{(t,s) \in D} | g(t,s) |, \quad H = \max_{(t,s) \in J \times J} | h(t,s) |.$$

Then the operator $\mathcal{A}$, defined by the right-hand side of (2.2), is a strict-set contraction mapping $PC^1[J, E]$ into itself.

PROOF : Clearly, the operator $\mathcal{A} : PC^1[J, E] \to PC^1[J, E]$ is bounded and continuous.

Let the bounded set $B \subset PC^1[J, E]$ be arbitrarily given, then by Theorem 1.4.1 and Remark 1.4.1[9], it follows that

$$\alpha((TB)(J)) = \alpha\left(\left\{ \int_0^t g(t,s)\, u(s)\, ds : t \in J,\ u \in B \right\} \right)$$

$$= \alpha(\overline{co}\{tg(t,s)u(s) : s \in [0,t], t \in J, u \in B\})$$

$$= \alpha(\{tg(t,s)u(s) : s \in [0,t], t \in J, u \in B\})$$

$$= \left(\sup_{s \in [0,t], t \in J} |tg(t,s)|\right) \alpha(\{u(s) : s \in [0,t], t \in J, u \in B\})$$

$$= \left(\sup_{s \in [0,t], t \in J} |tg(t,s)|\right) \alpha(\{u(s) : s \in J, u \in B\})$$

$$\leq G\,\alpha(B(J)), \qquad \dots (2.8)$$

We can also show by a similar argument that

$$\alpha((SB)(J)) \leq H\,\alpha(B(J)); \qquad \dots (2.9)$$

In view of Lemma 1.4.1 and Remark 1.4.1[9], it follows from (2.8) and (2.9) that

$$\alpha\left(\left\{\int_0^t (t-s) f(s,u,u',Tu,Su)\,ds : u \in B\right\}\right)$$

$$= t\,\alpha(\overline{co}\{(t-s) f(s,u,u',Tu,Su) : s \in [0,t], u \in B\})$$

$$= t\,\alpha(\{(t-s) f(s,u,u',Tu,Su) : s \in [0,t], u \in B\})$$

$$\leq t\,\alpha(\{(t-s) : s \in [0,t]\} \cdot \{f(s,u,u',Tu,Su) : s \in [0,t], u \in B\})$$

$$\leq t\left(\sup_{s \in [0,t]} |t-s|\right) \cdot \alpha(f[0,t] \times B([0,t]) \times B'([0,t]) \times (TB)([0,t]) \times (SB)([0,t])))$$

$$\leq t^2\,\alpha(f(J \times B(J) \times B'(J) \times (TB)(J) \times (SB)(J)))$$

$$= t^2 \sup_{t \in J} \alpha(f(t, B(J), B'(J), (TB)(J), (SB)(J)))$$

$$\leq t^2 (L_1\,\alpha(B(J)) + L_2\,\alpha(B'(J)) + L_3\,\alpha((TB)(J)) + L_4\,\alpha((SB)(J)))$$

$$\leq L_1\,\alpha(B(J)) + L_2\,\alpha(B'(J)) + GL_3\,\alpha(B(J)) + HL_4\ (B(J)) \qquad \dots (2.10)$$

Similarly,

$$\alpha\left(\left\{\int_0^\eta (\eta-s) f(s,u,u',Tu,Su)\,ds : u \in B\right\}\right)$$

$$\leq \eta^2 (L_1\,\alpha(B(J)) + L_2\,\alpha(B'(J)) + GL_3\,\alpha(B(J)) + HL_4\,\alpha(B(J))) \qquad \dots (2.11)$$

$$\alpha\left(\left\{\int_0^1 (1-s) f(s,u,u',Tu,Su)\,ds : u \in B\right\}\right)$$

$$\leq L_1\,\alpha(B(J)) + L_2\,\alpha(B'(J)) + GL_3\,\alpha(B(J)) + HL_4\,\alpha(B(J)) \qquad \dots (2.12)$$

$$\alpha\left(\left\{\int_0^1 f(s, u, u', Tu, Su)\, ds : u \in B\right\}\right)$$

$$\le L_1\, \alpha(B(J)) + L_2\, \alpha(B'(J)) + GL_3\, \alpha(B(J)) + HL_4\, \alpha(B(J)) \quad \text{... (2.13)}$$

$$\alpha\left(\left\{\int_0^1 sf(s, u, u', Tu, Su)\, ds : u \in B\right\}\right)$$

$$\le L_1\, \alpha(B(J)) + L_2\, \alpha(B'(J)) + GL_3\, \alpha(B(J)) + HL_4\, \alpha(B(J)) \quad \text{... (2.14)}$$

In addition, we easily find that (similar to the proof of Theorem 2.1.1[10]

$$\alpha(B(J)) \le 2\, \alpha_{PC^1}(B),\ \alpha(B'(J)) \le 2\, \alpha_{PC^1}(B), \quad \text{... (2.15)}$$

and

$$\alpha(B(t_k)) \le \alpha_{PC^1}(B),\ \alpha(B'(t_k)) \le \alpha_{PC^1}(B),\ k = 1, ..., m, \quad \text{... (2.16)}$$

Hence it follows from (2.2), and (2.10) - (2.16) that for any $t \in J$,

$$\alpha((\mathcal{A}B)(t)) \le \left(\left|\frac{\alpha}{1-\alpha}\right| \eta(1+\eta) + 1 + t^2\right)[L_1\, \alpha(B(J)) + L_2\, \alpha(B'(J))$$

$$+\ GL_3\, \alpha(B(J)) + HL_4\, \alpha(B(J))]$$

$$+ \left|\frac{\alpha}{1-\alpha}\right| (n+2) \sum_{k=1}^{n} \left\{M_{k1}\, \alpha(B(t_k)) + M_{k2}(B'(t_k)) + (1-t_k)\right.$$

$$\left.[N_{k1}\, \alpha(B(t_k)) + M_{k2}\, \alpha(B'(t_k))]\right\}$$

$$+ \sum_{k=1}^{n} \left\{M_{k1}\, \alpha(B(t_k)) + M_{k2}\, \alpha(B'(t_k)) + (1-t_k)\right.$$

$$\left.[N_{k1}\, \alpha(B(t_k)) + N_{k2}\, \alpha(B'(t_k))]\right\} \le \delta_1\, \alpha_{PC^1}(B), \quad \text{... (2.17)}$$

Analogously, we see that for any $t \in J$,

$$\alpha((\mathcal{A}B)'(t)) \le \delta_2\, \alpha_{PC^1}(B). \quad \text{... (2.18)}$$

On the other hand, by the uniform continuity of f and the boundedness of $B \subset PC^1[J, E]$, we can easily show that $\mathcal{A}''(B)$ is uniformly bounded and thus $\mathcal{A}'(B)$ is equicontinuous on each J_k for $k = 0, 1, ..., m$. Therefore by Lemma 4.3.11[10], it follows from (2.17) and (2.18) that

$$\alpha_{PC^1}(\mathcal{A}(B)) \le \delta\, \alpha_{PC^1}(B).$$

where $\alpha(\cdot)$ and $\alpha_{PC^1}(\cdot)$ denote the Kuratowski's measure of noncompactness in the Banach spaces E and $PC^1[J, E]$, respectively. This completes the proof of Lemma 2. □

3. Existence Theorems

Theorem 1 — *Assume all the conditions in Lemma 2 are satisfied, and*

$$\sum_{k=1}^{n} a_k < \frac{|1-\alpha|}{|\alpha|+|1-\alpha|}.$$ *Assume furthermore that*

(H_2) *there exist functions* $p_i, q \in C[J, R_+] \cap L^1(J)$ $(i = 1, 2, 3, 4)$, *and nonnegative constants* $a_k, b_k, c_k, \bar{a}_k, \bar{b}_k$, and $\bar{c}_k$ $(k = 1, \ldots, m)$ *such that*

$$\|f(t, u_1, u_2, u_3, u_4)\| \le \sum_{i=1}^{4} p_i(t)\|u_i\| + q(t), \quad t \in J, u_i \in E\ (i = 1, 2, 3, 4);$$

$$\|I_k(u, v)\| \le a_k\|u\| + b_k\|v\| + c_k, \quad u, v \in E;$$

$$\|\bar{I}_k(u, v)\| \le \bar{a}_k\|u\| + b_k\|v\| + \bar{c}_k, \quad u, v \in E\ (k = 1, \ldots, m)$$

and

$$\beta = \int_0^1 (1+t)\left[\left|\frac{\alpha}{1-\alpha}\right| \eta\,(p_1(t) + Gtp_3(t) + Hp_4(t)) + tp_1(t) + p_2(t) + \frac{Gt^2}{2}p_3(t) + \frac{H}{2}p_4(t)\right]dt$$

$$+ \left|\frac{\alpha}{1-\alpha}\right| \int_0^1 (1+t)\,[(p_1(t) + Gtp_3(t) + Hp_4(t)]\,dt \sum_{k=1}^{n} (a_k P_k + b_k)$$

$$+ \sum_{k=1}^{m} \left\{\left[\int_{t_k}^{1} (1+t)(p_1(t) + G(t-t_k)\,p_3(t))\,dt + H\int_0^1 (1+t)\,p_4(t)\,dt\,(1-t_k) + 1\right]\right.$$

$$\left.\cdot\,(a_k P_k + b_k) + (2-t_k)(\bar{a}_k P_k + \bar{b}_k)\right\}$$

$$< 1. \qquad \ldots (3.1)$$

where

$$P_{k+1} = t_{k+1} + \left|\frac{\alpha}{1-\alpha}\right|\left(\eta + \frac{\Delta_1}{\Delta} + \sum_{i=1}^{n} b_i\right)\prod_{i=1}^{k}(1+a_i)$$

$$+ \sum_{i=1}^{k}\prod_{j=i+1}^{k}(a_i t_i + b_i)(1+a_j), \; k = 0, 1, \ldots, m-1.$$

$$\Delta_1 = \left(\sum_{k=1}^{n} a_k\right)\left|\frac{\alpha}{1-\alpha}\right|\left(\eta + \sum_{k=1}^{n} b_k\right) + \sum_{k=1}^{n} a_k b_k$$

$$+ \sum_{i=1}^{n-1} \prod_{k=i+1}^{n} a_k b_i ,$$

$$\Delta = 1 - \left(\sum_{k=1}^{n} a_k \right) \left(\left| \frac{\alpha}{1-\alpha} \right| + 1 \right).$$

(Take $\sum_{i=1}^{0} (a_i t_i + b_i) = 0,\ \prod_{j=k+1}^{k} (1 + a_j) = 1$

Then the boundary value problem (3pBVP) has at least one solution in

$$PC^1 [J, E] \cap C^2 [J', E].$$

PROOF : By Lemma 2, we see that $\mathcal{A}: PC^1 [J, E] \to PC^1 [J, E]$ is a strict-set contraction. Again by Lemma 1 and Theorem L-S, it suffices to show that

$$\left\{ u \in PC^1 [J, E] \mid u = \lambda \mathcal{A} u, \lambda \in (0, 1) \right\}$$

is, *a priori*, bounded in $PC^1 [J, E]$ by a positive constant independent of $\lambda \in (0, 1)$. Suppose any $u \in PC^1 [J, E]$ satisfies $u = \lambda \mathcal{A} u$ for some $\lambda \in (0, 1)$. Then it follows that

$$u'(t) = \lambda \int_t^1 f(s, u, u', Tu, Su)\, ds - \lambda \int_0^1 s f(s, u, u', Tu, Su)\, ds$$

$$- \lambda \sum_{k=1}^{m} [I_k (u(t_k), u'(t_k)) + (1 - t_k) \overline{I}_k (u(t_k), u'(t_k))] \qquad \text{... (3.2)}$$

$$+ \lambda \sum_{0 < t_k < t} \overline{I}_k (u(t_k), u'(t_k))$$

We shall next obtain the *a priori* estimates for $\| u(t_k) \|$ for $k = 1, 2, \ldots, m$. By virtue of

$$u(t) = \frac{\alpha}{1-\alpha} \left\{ \int_0^\eta u'(s)\, ds + \sum_{k=1}^{n} I_k (u(t_k), u'(t_k)) \right\} + \int_0^t u'(s)\, ds$$

$$+ \sum_{0 < t_k < t} I_k (u(t_k), u'(t_k)),$$

we show that

$$\| u(t) \| \leq \left| \frac{\alpha}{1-\alpha} \right| \left(\eta \| u' \|_{PC} + \sum_{k=1}^{n} \| I_k \| \right) + t \| u' \|_{PC}$$

$$+ \sum_{0<t_k<t} \| I_k(u(t_k), u'(t_k)) \|, \qquad \dots (3.3)$$

Hence,

$$\| u(t_1) \| \le \left| \frac{\alpha}{1-\alpha} \right| \left(\eta \| u' \|_{PC} + \sum_{k=1}^{n} \| I_k \| \right) + t_1 \| u' \|_{PC}, \qquad \dots (3.3)_1$$

$$\| u(t_2) \| \le \left| \frac{\alpha}{1-\alpha} \right| \left(\eta \| u' \|_{PC} + \sum_{k=1}^{n} \| I_k \| \right) + t_2 \| u' \|_{PC} + \| I_1 \|, \qquad \dots (3.3)_2$$

....

$$\| u(t_n) \| \le \left| \frac{\alpha}{1-\alpha} \right| \left(\eta \| u' \|_{PC} + \sum_{k\ 1}^{n} \| I_k \| \right) + t_n \| u' \|_{PC} + \| I_1 \| + \dots + \| I_{n-1} \|, \qquad \dots (3.3)_n$$

Multiplying $(3.3)_k$ by a_k for each $k = 1, \dots, n$, and adding them together yield

$$\sum_{k=1}^{n} a_k \| u(t_k) \|$$

$$\le \left(\sum_{k=1}^{n} a_k \right) \left| \frac{\alpha}{1-\alpha} \right| \left[\eta \| u' \|_{PC} + \sum_{k=1}^{n} (a_k \| u(t_k) \| + b_k \| u' \|_{PC} + c_k) \right] \qquad \dots (3.4)$$

$$+ \left(\sum_{k=1}^{n} a_k t_k \right) \| u' \|_{PC} + \sum_{i'=1}^{n-1} \sum_{k=i+1}^{n} a_k (a_i \| u(t_i) \| + b_i \| u' \|_{PC} + c_i),$$

Moreover,

$$\sum_{i=1}^{n-1} \sum_{k=i+1}^{n} a_k a_i \| u(t_i) \| \le \left(\sum_{k=1}^{n} a_k \right) \sum_{k=1}^{n} a_k \| u(t_k) \|, \qquad \dots (3.5)$$

and thus after a rearrangement, (3.4) and (3.5) imply that

$$\sum_{k=1}^{n} a_k \| u(t_k) \| \le \frac{1}{\Delta} [\Delta_1 \cdot \| u' \|_{PC} + \Delta_2], \qquad \dots (3.6)$$

where $$\Delta_2 = \left(\sum_{k=1}^{n} a_k \right) \left| \frac{\alpha}{1-\alpha} \right| \sum_{k=1}^{n} c_k + \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} a_k c_i.$$

Take $t = t_{k+1}$ in (3.3), and substitute (3.6) into it, we obtain using (H_2) that

$$\| u(t_{k+1}) \|$$

$$\le \left[t_{k+1} + \left| \frac{\alpha}{1-\alpha} \right| \left(\eta + \frac{\Delta_1}{\Delta} + \sum_{k=1}^{n} b_k \right) + \sum_{i=1}^{k} b_i \right] \cdot \| u' \|_{PC} \quad \text{... (3.7)}$$

$$+ \sum_{i=1}^{k} (a_i \| u(t_i) \| + c_i) + \left| \frac{\alpha}{1-\alpha} \right| \left(\frac{\Delta_2}{\Delta} + \sum_{k=1}^{n} c_k \right)$$

In view of (3.7), we can prove by induction that

$$\| u(t_{k+1}) \|$$

$$\le \left[t_{k+1} + \left| \frac{\alpha}{1-\alpha} \right| \left(\eta + \frac{\Delta_1}{\Delta} + \sum_{k=1}^{n} b_k \right) \prod_{i=1}^{k} (1 + a_i) \right.$$

$$\left. + \sum_{i=1}^{k} \prod_{j=i+1}^{k} (a_i t_i + b_i)(1 + a_j) \cdot \| u' \|_{PC} \right.$$

$$+ \left| \frac{\alpha}{1-\alpha} \right| \left(\frac{\Delta_2}{\Delta} + \sum_{k=1}^{n} c_k \right) \prod_{i=1}^{k} (1 + a_i),$$

$$+ \sum_{i=1}^{k} \prod_{j=i+1}^{k} c_i (1 + a_j) =: P_{k+1} \| u' \|_{PC} + Q_k$$

$k = 0, 1, ..., m - 1$; (Let $Q_0 = 0$). ... (3.8)

Moreover, we can also estimate the following integrals.

$$\int_0^1 p_1(s) \| u(s) \| ds$$

$$\le \int_0^1 p_1(s) \left(\left| \frac{\alpha}{1-\alpha} \right| \eta + s \right) ds \| u' \|_{PC}$$

$$+ \left| \frac{\alpha}{1-\alpha} \right| \int_0^1 p_1(s) ds \sum_{k=1}^{n} \| I_k(u(t_k), u'(t_k) \| + \sum_{k=1}^{m}$$

$$\int_{t_k}^1 p_1(s) ds \| I_k(u(t_k), u'(t_k)) \|, \quad \text{... (3.9)}$$

$$\int_0^1 p_3(s) \| (Tu)(s) \| ds$$

$$\leq G\left[\int_0^1 sp_3(s)\left(\left|\frac{\alpha}{1-\alpha}\right|\eta+\frac{s}{2}\right)ds\,\|u'\|_{PC}\right.$$

$$+\left|\frac{\alpha}{1-\alpha}\right|\int_0^1 sp_3(s)\,ds\sum_{k=1}^{n}\|I_k u(t_k), u'(t_k))\| \qquad \ldots (3.10)$$

$$\left.+\sum_{k=1}^{m}\int_{t_k}^{1}(s-t_k)p_3(s)\,ds\,\|I_k(u(t_k), u'(t_k))\|\right],$$

and
$$\int_0^1 p_4(s)\,|(Su)(s)|\,ds$$

$$\leq H\int_0^1 p_4(s)\,ds\left[\left(\left|\frac{\alpha}{1-\alpha}\right|\eta+\frac{1}{2}\right)\|u'\|_{PC}\right.$$

$$\left.+\left|\frac{\alpha}{1-\alpha}\right|\sum_{k=1}^{n}\|I_k(u(t_k), u'(t_k))\|+\sum_{k=1}^{m}(1-t_k)\|I_k(u(t_k), u'(t_k))\|\right], \qquad \ldots (3.11)$$

Therefore, provided, $t\in J$ (we let, without loss of generality, $t\in(t_k, t_{k+1}]$), then Combining (3.2), (3.8)-(3.11) and applying (H_2), we have after rearranging that

$$\|u'(t)\|\leq\lambda\int_t^1\|f(s,u,u',Tu,Su)\|\,ds+\lambda\int_0^1 s\,\|f(s,u,u',Tu,Su)\|\,ds$$

$$+\lambda\sum_{k=1}^{m}[\|I_k(u(t_k), u'(t_k))\|+(1-t_k)\|\bar{I}_k(u(t_k), u'(t_k))\|]$$

$$\leq\int_0^1(1+s)\|f(s,u,u',Tu,Su))\|\,ds+\sum_{k=1}^{m}[\|I_k(u(t_k), u'(t_k))\|$$

$$+(2-t_k)\|\bar{I}_k(u(t_k), u'(t_k))\|]$$

$$\leq\left[\int_0^1 p_1(s)\|u(s)\|\,ds+\int_0^1 p_2(s)\,ds\,\|u'\|_{PC}+\int_0^1 p_3(s)\,ds\right.$$

$$\int_0^s|g(s,r)\|\,|u(r)\|\,dr$$

$$+ \int_0^1 p_4(s)\, ds \int_0^1 |h(s,r)| \, \| u(r) \| \, dr \Bigg] + \sum_{k=1}^{m} [\| I_k(u(t_k), u'(t_k)) \|$$

$$+ (2 - t_k) \| \bar{I}_k (u(t_k), u'(t_k)) \|]$$

$$\leq \Bigg\{ \int_0^1 (1+s) \Bigg[\left| \frac{\alpha}{1-\alpha} \right| \eta (p_1(s) + Gsp_3(s) + Hp_4(s)) + sp_1(s) + p_2(s)$$

$$+ \frac{Gs^2}{2} p_3(s) + \frac{H}{2} p_4(s) \Bigg] ds$$

$$+ \left| \frac{\alpha}{1-\alpha} \right| \int_0^1 (1+s) [(p_1(s) + Gsp_3(s) + Hp_4(s)] \, ds \sum_{k=1}^{n} (a_k P_k + b_k)$$

$$+ \sum_{k=1}^{m} \Bigg\{ \Bigg[\int_{t_k}^1 (1+s)(p_1(s) + G(s - t_k) p_3(s)) \, ds + H$$

$$\int_0^1 (1+s) p_4(s) \, ds \, (1 - t_k) + 1 \Bigg] (a_k P_k + b_k) + (2 - t_k)(\bar{a}_k P_k + \bar{b}_k) \Bigg\} \Bigg\} \cdot \| u' \|_{PC}$$

$$+ \Bigg\{ \left| \frac{\alpha}{1-\alpha} \right| \int_0^1 (1+s) [(p_1(s) + Gsp_3(s) + Hp_4(s)] \, ds \sum_{k=1}^{n} (a_k Q_{k-1} + c_k)$$

$$+ \int_0^1 (1+s) q(s) \, ds + \sum_{k=1}^{m} \Bigg\{ \Bigg[\int_{t_k}^1 (1+s)(p_1(s) + G(s - t_k) p_3(s)) \, ds$$

$$+ H \int_0^1 (1+s) p_4(s) \, ds \, (1 - t_k) + 1 \Bigg] (a_k Q_{k-1} + c_k) + (2 - t_k)(\bar{a}_k Q_{k-1} + \bar{c}_k) \Bigg\} \Bigg\}$$

$$=: \beta \| u' \|_{PC} + M_0$$

Thus, $$\| u' \|_{PC} \leq \frac{M_0}{1 - \beta}.$$

hence, $$\| u \|_{PC} \leq \left| \frac{\alpha}{1-\alpha} \right| \Bigg\{ \eta \| u' \|_{PC} + \sum_{k=1}^{n} [a_k \| u(t_k) \| + b_k \| u'(t_k) \| + c_k]$$

$$- \| u' \|_{PC} + \sum_{k=1}^{m} [a_k \| u(t_k) \| + b_k \| u'(t_k) \| + c_k]$$

$$\le \left[\left| \frac{\alpha}{1-\alpha} \right| \left(\eta + \sum_{k=1}^{n} (a_k P_k + b_k) \right) + \sum_{k=1}^{m} (a_{k'} p_k + b_k)t + 1 \right] \| u' \|_{PC}$$

$$+ \left| \frac{\alpha}{1-\alpha} \right| \sum_{k=1}^{n} (a_k Q_{k-1} + c_k) + \sum_{k=1}^{m} (a_k Q_{k-1} + c_k)$$

$$\le \left[\left| \frac{\alpha}{1-\alpha} \right| \left(\eta + \sum_{k=1}^{n} (a_k P_k + b_k) \right) + \sum_{k=1}^{m} (a_{k'} p_k + b_k)t + 1 \right] \frac{M_0}{1-\beta}$$

$$+ \left| \frac{\alpha}{1-\alpha} \right| \sum_{k=1}^{n} (a_k Q_{k-1} + c_k) + \sum_{k=1}^{m} (a_k Q_{k-1} + c_k) =: M$$

Consequently, we conclude $\| u \|_{PC^1} \le M$. The proof of Theorem 1 is complete. □

The following Theorem 2 could be treated as an immediate consequence of Theorem 1, so we only present it without proof.

***Theorem* 2** — *Assume for any $r > 0$, f is uniformly continuous on $J \times B_r \times B_r \times B_r \times B_r$, I_k, and $\bar{I}_k$ are bounded on $B_r \times B_r$ for each $k = 1, \ldots, m$. Suppose furthermore that*

(H_3) there exist nonnegative constants $L_i\,(i = 1, 2, 3, 4)$, M_{kj}, N_{kj} $(j = 1, 2;\ k = 1, \ldots, m)$, such that $\sum_{k=1}^{n} M_{k1} < \frac{|1-\alpha|}{|\alpha| + |1-\alpha|}$; moreover,

$$\begin{cases} \| f(t, u_1, u_2, u_3, u_4) - f(t, v_1, v_2, v_3, v_4) \| \le \sum_{i=1}^{4} L_i \| u_i - v_i \|, & t \in J, u_i, v_i \in E; \\ \| I_k(u_1, u_2) - I_k(v_1, v_2) \| \le \sum_{j=1}^{2} M_{kj} \| u_j - v_j \|, & u_j, v_j \in E; \\ \| \bar{I}_k(u_1, u_2) - \bar{I}_k(v_1, v_2) \| \le \sum_{j=1}^{2} N_{kj} \| u_j - v_j \|, & u_j, v_j \in E\,(j = 1, 2), \end{cases}$$

$$\delta = \max \{ \delta_1, \delta_2 \} < 1,$$

and

$$\left| \frac{\alpha}{1-\alpha} \right| (9L_1 + 5GL_3 + 9HL_4) \left[\eta + \sum_{k'=1}^{n} (M_{k1} P_k + M_{k2}) \right]$$

$$+ 5L_1 + 9L_2 + 1.75GL_3 + 4.5HL_4 + \sum_{k=1}^{m} \{[3(3 - 2t_k - t_k^2) L_1$$

$$+ (5 - 3t_k + 3t_k^2 + t_k^3) GL_3 + 9(1 - t_k) HL_4 + 6] (M_{k1} P_k + M_{k2})$$

$$+ 6(2-t_k)(N_{k1}P_k + N_{k2})\} < 6$$

where
$$P_{k+1} = t_{k+1} + \left(\eta + \frac{\Delta_1}{\Delta} + \sum_{i='1}^{n} M_{i2}\right) \prod_{i=1}^{k} (1+M_{i1})$$

$$+ \sum_{i=1}^{k} \prod_{j=i+1}^{k} (M_{i1} t_i + M_{i2})(1+M_{j1}),\ k = 0, 1, \ldots, m-1.$$

$$\Delta_1 = \left(\sum_{k=1}^{n} M_{k1}\right) \left|\frac{\alpha}{1-\alpha}\right| \left(\eta + \sum_{k=1}^{n} M_{k2}\right) + \sum_{k=1}^{n} M_{k1} M_{k2}$$

$$+ \sum_{i=1}^{n-1} \prod_{k=i+1}^{n} M_{k1} M_{i2},$$

$$\Delta = 1 - \left(\sum_{k=1}^{n} M_{k1}\right)\left(\left|\frac{\alpha}{1-\alpha}\right| + 1\right).$$

Then the boundary value problem (3pBVP) possesses at least one solution in

$$PC^1[J, E] \cap C^2[J', E].$$

Remark 1. We can also deal with, obtaining the similar a priori estimates, the second order three-point boundary value problems with the boundary conditions in (2.1) replaced by $\alpha u(0) = u(\eta = u(1)$ or $u(0) = u(\eta) = \alpha u(1)$.

Remark 2. The obtained existence results extend and generalize some related ones in the previous Literature 2-7 etc.

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