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SOME PROPERTIES OF THE CLIFFORD TORUS AS ROTATION SURFACES

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In this article, we study rotation surfaces in the 4-dimensional Euclidean space $\mathbb{E}^4$. Also, we obtain the complete classification theorems for the flat rotation surfaces with pointwise 1-type Gausss map and an equation in terms of the mean curvature vector. In fact, we chracterize the flat rotation surfaces of finite type immersion with Gauss map and the mean curvature vector filed, namely the pointwise 1-type Gauss map and some algebraic equations in terms of the Gauss map and the mean curvature vector field related to the Laplacian of the surface with respect to the induced metric.

Key Words : Rotation Surface; Gauss Map; Finite Type; Pointwise 1-Type Gauss Map; Clifford Torus

1. INTRODUCTION

A Riemannian submaniforld M of an m-dimensional Euclidean space $\mathbb{E}^m$ is said to be of finite type if its position vector fled x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + \sum_{i=1}^{k} x_i$ where x_0 is a constant map, $x_1, ..., x_k$ non-constant maps such that $\Delta x_i = \lambda_i, x_i, \lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If $\lambda_1, \lambda_2, ..., \lambda_k$ are different, then M is said to be of k-type. Similarly, a smooth map ϕ of an n-dimensional Riemannian submanifold M of $\mathbb{E}^m$ is said to be of finite type if ϕ is a finite sum of $\mathbb{E}^m$-valued eiguenfunctions of Δ. As is seen in the recent publication, it is a useful tool in the study of submanifolds. Also, it is the natural extension of minimal submanifolds on which many mathematicians have devoted in the last years. The first results on this subject have been collected in book [7]; for a recent survey, see [8]. Many works were done to characterize or classify submanifolds in terms of finite type. In a framework of the theory of finite type, Chen and Piccinni [10] made a general study on submanifolds of Euclidean spaces with finite type Gauss map. Several geometers also studied submanifolds of Euclidean spaces or pseudo-Euclidean spaces with finite type Gauss map.

It is interesting to ask which submanifolds with finite type Gauss map are themselves of finite type. In [3] Baikoussis, Chen and Verstraelen classified ruled surfaces with finite type Gauss map in an m-dimensional Euclidean space $\mathbb{E}^m$. Furthermore, in[17] the author with Kim investigated the finite type Gauss map of the ruled surfaces with non-null base curve in an m-dimensional Minkowski space $\mathbb{E}_1^m$, and in [16] Kim, Kim and the author completely classified rules surfaces with 1-type Gauss map in an m-dimensional Minkowski spaces $\mathbb{E}_1^m$.

If a submanifold M of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. However, the Laplacian of the Gauss map of several important surfaces such as a catenoid and a right cone in $\mathbb{E}^3$ [9], and a helicoid of the 1st, 2nd and 3rd kind, a conjugate of Enneper's surface of the 2nd kind and a B-scroll in $\mathbb{E}^3_1$ [18] take a somewhat different form; namely,

$$\Delta G = f(G + C) \quad \text{... (1.1)}$$

for some function f and some constant vector C.

A submanifold M is said to be *pointwise 1-type Gauss map* if its Gauss map satisfies (1.1) for some smooth function f on M and constant vector C. A pointwise 1-type Gauss map is called *proper* if the function f defined by (1.1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of *the first kind* if the vector C in (1.1) is the zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of *the second kind.*

In[18] the present author with Kim completely classified all possible ruled surfaces with pointwise 1-type Gauss map of the first kind in Minkowski 3-space $\mathbb{E}^3_1$ and obtained a new characterization of minimal ruled surfaces. Also, Chen, Choi and Kim[9] recently investigated surfaces of revolution with pointwise 1-type Gauss map in $\mathbb{E}^3$.

Following the ideas of[14], one can also study surfaces in $\mathbb{E}^3$ for which trhe Gauss map G satisfies the condition of the form

$$\Delta G = AG, \ A \in \mathbb{R}^{3\times 3}. \quad \text{... (1.2)}$$

Concerning this condition, Dillen, Pas and Verstraelen[13] studied surfaces of revolution in $\mathbb{E}^3$ such that its Gauss map G satisfies the condition (1.2). Baikoussis and Blair[2] proved that the ruled surfaces in $\mathbb{E}^3$ satisfying the condition (1.2) are circular cylinders or plans. Baikoussis and Verstraelen[4, 5, 6] studied the helicoidal surfaces, the translation surfaces and the spiral surfaces in $\mathbb{E}^3$ satisfying the condition (1.2). Also, for the Lorentz version, Choi[11, 12] completely classified the surfaces of revolution and the ruled surfaces with non-null base curve satisfying the condition (1.2) in Minkowski 3-space $\mathbb{E}^3_1$. Furthermore, Alias, Ferrandez, Lucas and Merono[1] studied the ruled surfacs with null ruling satisfying the condition (1.2) in $\mathbb{E}^3_1$. The author[21] recently classified translation surfaces satisfying the condition (1.2) in $\mathbb{E}^3_1$.

Rotation surface were studied in[19] by Vraneanu as surface in Euclidean 4-space $\mathbb{E}^4$ which are defined by the following equations with respect to an orthonormal system of coordinates

$$x(s, t) = (x_1(s, t), x_2(s, t), x_3(s, t), x_4(s, t))$$

given by

$$x_1 = u(s) \cos s \cos t, \quad x_2 = u(s) \cos s \sin t,$$

$$x_3 = u(s) \sin s \cos t, \quad x_4 = u(s) \sin s \sin t \qquad \text{... (1.3)}$$

where $u = u(s)$ is a smooth function

On the other hand, Houh[15] and Yoon[20] have studied the flat rotation surfaces of finite type and finite type Gauss map in $\mathbb{E}^4$, respectively.

Theorem **A** [15] — *Let M be a flat rotation surface in Euclidean 4-space $\mathbb{E}^4$. Then M is of finite type if and only if M is a Clifford torus, i.e., it is the product of two plane circles with the same radius.*

Theorem **B** [20] — *Let M be a flat rotation surface in Euclidean 4-space $\mathbb{E}^4$. Then M is of finite type Gauss map if and only if M is a Clifford torus, i.e., it is the product of two plane circles with the same radius.*

In § 2 we recall some basic formulas and gauss map of submanifolds in Euclidean spaces. In § 3 we investigate the flat rotation surfaces with pointwise 1-type Gauss map in Euclidean 4-space $\mathbb{E}^4$. In the last two sections we are concerned with the flat rotation surfaces satisfying the partial differential equations $\Delta G = AG$ and $\Delta H = AH$ for some real matrix A, where H is the mean curvature vector field of the surfaces.

2. Preliminaries

Let $x : M \to \mathbb{E}^m$ be an isometric immersion from an n-dimensional connected Riemannian manifold M into an m-dimensional Euclidean space $\mathbb{E}^m$. Let be the Levi-Civita connection of $\mathbb{E}^m$ and ∇ the induced connection on M. Let $e_1, ..., e_n, e_{n+1}, ..., e_m$ be an adapted local orthonormal frame in $\mathbb{E}^m$ such that $e_1, e_2, ..., e_n$ are tangent to M and $e_{n+1}, e_{n+2}, ..., e_m$ normal to M we shall make use of the following convection on the ranges of indices : $1 \le i, j, k, \ldots \le n, \quad n+1 \le s, t, ..., \le m$, $1 \le A, B, \ldots \le m$.

Let ω_A be the dual 1-form e_A defined by $\omega_A(e_B) = \delta_{AB}$. The connection forms ω_A^B are defined by

$$de_A = \sum_B \omega_A^B e_B, \quad \omega_A^B + \omega_B^A + 0. \qquad \text{... (2.1)}$$

Then, the structure equations of $\mathbb{E}^m$ are obtained as follows :

$$\nabla_{e_i} e_j = \omega_j^k (e_i)\, e_k + h_{ij}^s e_s, \qquad \text{... (2.2)}$$

$$\nabla_{e_i} e_s = -h_{ij}^s e_j + \omega_s^t (e_i)\, e_t, \quad D_{e_s} e_t = \omega_s^t (e_i)\, e_t, \qquad \text{... (2.3)}$$

where D is the normal connection and h_{ij}^s the coefficiènts of the second fundamental form h. The mean curvature vectr H of M in $\mathbb{E}^m$ is defined by

$$H = \frac{1}{n} \sum_{s=n+1}^{m} \sum_{i=1}^{n} h_{ii}^s e_s. \qquad \text{... (2.4)}$$

For any real function f on M the Laplacian Δf of f is given by

$$\Delta f = -\sum_i (\nabla_{e_i}\nabla_{e_i} f - \nabla_{\nabla_{e_i} e_i} f). \quad \text{... (2.5)}$$

Let us now define the Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n \mathbb{E}^m$, where $G(n, m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$ and $\wedge^n \mathbb{E}^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. In a natural way, we can identify $\wedge^n \mathbb{E}^m$ with some Euclidean space $\mathbb{E}^N$ where $N = \binom{m}{n}$. The map $G : M \to G(n,m) \subset \mathbb{E}^N$ defined by $G(p)(e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p)$ is called the *Gauss map* of M that is a smooth map which carries a point p in M into the oriented n-plane in $\mathbb{E}^m$ obtained from the parallel translation of the tangent space of M at p in $\mathbb{E}^m$.

3. ROTATION SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

In this section we investigate the flat rotation surfaces in Euclidean 4-space $\mathbb{E}^4$ satisfying the following condition

$$\Delta G = f(G + C) \quad \text{... (3.1)}$$

for some function f and some constant vector C.

Let M be a rotation surface in $\mathbb{E}^4$ defined by (1.3). We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M as are given by the following (cf. [15]) :

$$e_1 = (-\cos s \sin t, \cos s \cos t - \sin s \sin t, \sin s \cos t),$$

$$e_2 = \frac{1}{A_1}(A_2 \cos t, A_2 \sin t, A_3 \cos t, A_3 \sin t),$$

$$e_3 = \frac{1}{A_1}(-A_3 \cos t, -A_3 \sin t, A_2 \cos t, A_2 \sin t),$$

$$e_4 = (-\sin s \sin t, \sin s \cos t, \cos s \sin t - \cos s \cos t), \quad \text{... (3.2)}$$

where we have put $A_1 = \sqrt{u^2 + u'^2}, A_2 = u' \cos s - u \sin s, A_3 = u' \sin s + u \cos s$. Then we have

$$e_1 = \frac{1}{u}\frac{\partial}{\partial t}, e_2 = \frac{1}{A_1}\frac{\partial}{\partial s} \quad \text{... (3.3)}$$

and

$$\omega^1 = u dt, \omega^2 = A_1 ds \quad \text{... (3.4)}$$

As is introduced in section 2, the Gauss map G of M is given by $G = e_1 \wedge e_2$. Using (2.2), (2.3), (3.2) and (3.3) we can obtain the coefficients of the second fundamental form h and the connection forms ω_B^A are as following :

$$h^3_{11} = \frac{1}{\sqrt{u^2 + u'^2}} = \alpha,\ h^3_{12} = h^3_{21} = 0,$$

$$h^3_{22} = \frac{2u'^2 - u\,u'' + u^2}{(u^{2+}\ u'^2)^{\frac{3}{2}}} = \beta,$$

$$h^4_{11} = h^4_{22} = 0,\ h^4_{12} = h^4_{21} = -\alpha. \qquad \text{... (3.5)}$$

$$\omega^3_1 = \alpha\,\omega^1,\ \omega^3_2 = \beta\,\omega^2,\ \omega^4_1 = -\alpha\,\omega^2,\ \omega^4_2 = -\alpha\,\omega^1, \qquad \text{... (3.6)}$$

$$\omega^1_2 = \omega^3_4 = \alpha\,\kappa\,\omega^1, \qquad \text{... (3.7)}$$

where we have put $\kappa = \frac{u'}{u}$. The Gaussian curvature is obtained by

$$K = \det(h^3_{ij}) + \det(h^4_{ij}) = \frac{u'^2 - u\,u''}{(u^2 + u'^2)^2}. \qquad \text{... (3.8)}$$

Moreover, combining (2.2), (3.5), (3.6) and (3.7) we have

$$\tilde{\nabla}_{e_1} e_1 = \alpha(-\kappa e_2 + e_3),\ \ \tilde{\nabla}_{e_1} e_2 = \alpha(\kappa e_1 - e_4),$$

$$\tilde{\nabla}_{e_1} e_3 = -\alpha(e_1 + \kappa e_4),\ \tilde{\nabla}_{e_1} e_4 = \alpha(e_2 + \kappa e_3),$$

$$\tilde{\nabla}_{e_2} e_1 = -\alpha\,e_4,\ \tilde{\nabla}_{e_2} e_2 = \beta\,e_3,$$

$$\tilde{\nabla}_{e_2} e_3 = -\beta\,e_2,\ \tilde{\nabla}_{e_2} e_4 = \alpha\,e_1,$$

$$\nabla_{e_1} e_1 = -\alpha\,\kappa\,e_2,\ \ \nabla_{e_2} e_2 = 0. \qquad \text{... (3.9)}$$

If a rotation surface M is flat, then (3.8) implies

$$u\,u'' = u'^2. \qquad \text{... (3.10)}$$

Thus $u = ce^{ks}$ for some constants $c \neq 0$ and k. If necessary, by an appropriate homothetic transformation we may assume that $c = 1$. Thus we have

$$u = e^{ks},\ \ \alpha = \beta = \frac{e^{-ks}}{\sqrt{1 + k^2}}. \qquad \text{... (3.11)}$$

If $k = 0$, then $u = 1$. In this case, the surface is a Clifford torus, that is, it is the product of two plane circles with the same radius.

Now, we assume that $k \neq 0$. By using (2.5), (3.9), (3.11) and straight-foreword computation, the Laplacian ΔG of the Gauss map G can be expressed as

$$\Delta G = 4\,\alpha^2\,e_1 \wedge e_2 + 2k\,\alpha^2\,e_1 \wedge e_3 - 2k\,\alpha^2\,e_2 \wedge e_4. \qquad \text{... (3.12)}$$

We suppose that the rotation surface M is of pointwise 1-type Gauss map in $\mathbb{E}^4$. From (3.1) and (3.12)

$$4\alpha^2 = f + f\langle C, e_1 \wedge e_2 \rangle, \qquad \text{... (3.13)}$$

$$2k\alpha^2 = f + f\langle C, e_1 \wedge e_3 \rangle, \qquad \text{... (3.14)}$$

$$-2k\alpha^2 = f\langle C, e_2 \wedge e_4 \rangle. \qquad \text{... (3.15)}$$

We then note that a smooth function f is non-zero. In fact, if f vanishes on some open subset $\mathcal{U}$, then by (3.1) and (3.12) α is zero on $\mathcal{U}$, which is a contradiction. Let $\mathcal{W}$ be the open subset of M consisting of points where f is non-zero. We now consider the matters on $\mathcal{W}$ for a while. Then, we obtain from (3.12)

$$\langle C, e_1 \wedge e_4 \rangle = 0, \langle C, e_2 \wedge e_3 \rangle = 0, \langle C, e_3 \wedge e_4 \rangle = 0. \qquad \text{... (3.16)}$$

By differentiating (3.16) with respect to s and using the first and third equation in (3.16) and (3.9), we get

$$k\langle C, e_1 \wedge e_3 \rangle + \langle C, e_1 \wedge e_2 \rangle - k\langle C, e_2 \wedge e_4 \rangle = 0, \qquad \text{... (3.17)}$$

and

$$\langle C, e_2 \wedge e_4 \rangle + \langle C, e_1 \wedge e_3 \rangle = 0, \qquad \text{... (3.18)}$$

which imply

$$\langle C, e_1 \wedge e_2 \rangle - 2k\langle C, e_2 \wedge e_4 \rangle = 0. \qquad \text{... (3.19)}$$

Combining (3.13), (3.15) and (3.19) we then have

$$f = 4k^2\alpha^2 + 4\alpha^2, \qquad \text{... (3.20)}$$

this is, a smooth function f depends only on s.

On the other hand, by (3.3), (3.9) and (3.15) we get

$$2k\alpha^2 f' = (4k\alpha\alpha' - 4\alpha^2 + f)f, \qquad \text{... (3.21)}$$

where the prime denotes th edifferentiation with respect to s. From (3.20) and (3.21) we thus have

$$k^2(1 + k^2) = 0, \qquad \text{... (3.22)}$$

which gives $k = 0$. Hence, $u(s) = 1$ and $\alpha(s) = 1$. In this case we can also show that a smooth function f is a constant and $C = 0$. Therefore the Gauss map is of 1-type by (3.12), that is, $\Delta G = 4G$. Consequently, from (1.3) the surface M is the Clifford tours, i.e., it is the product of two plane circles with the same radius.

Thus we have

***Theorem* 3.1** — *Let M be a flat rotation surface in Euclidean 4-space $\mathbb{E}^4$. Then, the following are equivalent.*

1. *The Gauss map G on M is of pointwise 1-type.*
2. *The Gauss map G on M is of 1-type.*
3. *M is a Clifford torus, that is, it is the product of two plane circle with the same radius.*

4. Rotation Surfaces Satisfying $\Delta G = AG$

In this section we study the flat rotation surfaces M in Euclidean 4-space $\mathbb{E}^4$ satisfying the following condition

$$\Delta G = AG, \; A \in \mathbb{R}^{6\times 6}. \qquad \text{... (4.1)}$$

We may assume that the rotation surface M is parametrized by (1.3). From (4.1) and (3.12) we have

$$A\,(e_1 \wedge e_2) = 4\,\alpha^2\, e_1 \wedge e_2 + 2k\,\alpha^2\, e_1 \wedge e_3 - 2k\,\alpha^2\, e_2 \wedge e_4. \qquad \text{... (4.2)}$$

Since α is a nonzero function, by (3.3), (3.9) and (4.2) we obtain

$$A\,(e_2 \wedge e_3) + A\,(e_1 \wedge e_4) = (4k^2+4)\,\alpha^2\, e_1 \wedge e_4 + (4k^2+4)\,\alpha^2\, e_2 \wedge e_3. \qquad \text{... (4.3)}$$

On the other hand, the eq. (3.9) gives

$$\nabla_{e_2} A\,(e_2 \wedge e_3) = 0 \text{ and } \nabla_{e_2} A\,(e_1 \wedge e_4) = 0,$$

which imply

$$(k^2+1)\,\alpha^2\,\alpha' = 0 \qquad \text{... (4.4)}$$

with the aid of (3.3), (3.9) and (4.3). Thus, α must be a constant. By (3.11) we have $k = 0$ and then $u(s) = 1$ and $\alpha = \beta = 1$. Therefore, the surface M is of 1-type Gauss map by (3.12), that is, $\Delta G = AG$. By Theorem 3.1 the surface M is the Clifford torus.

In conclusion, we have

***Theorem* 4.1** — *Let M be a flat rotation surface in Euclidean 4-space* $\mathbb{E}^4$. *The Gauss map G of M satisfies a partial differentialequation*

$$\Delta G = AG, \; A \in \mathbb{R}^{6\times 6}$$

if and only if M is a Clifford torus.

5. Rotating Surfaces Satisfying $\Delta H = AH$

In this section we look into the flat rotation surfaces in Euclidean 4-space $\mathbb{E}^4$ satisfying the following condition

$$\Delta H = AH, \; A \in \mathbb{R}^{4\times 4}. \qquad \text{... (5.1)}$$

Let M be a rotation surface parametrized by (1.3).We assume that the rotation surface M is flat. Then, from (2.4), (3.5) and (3.11) the mean curvature vector H of M is give by

$$H = \frac{1}{2} \sum_{s=3}^{4} \sum_{i=1}^{2} h_{ii}^{s}\, e_s = \alpha\, e_3. \qquad \text{... (5.2)}$$

By a straightforward computation, the Laplacian ΔH of the mean curvature vector H with the help of (2.5) and (5.2) turns out to be

$$\Delta H = -2k\,\alpha^3\, e_2 + 2\,\alpha^3\, e_3. \qquad \text{... (5.3)}$$

Since α is a non-zero function, from (5.1), (5.2) and (5.3) we obtain

$$Ae_3 = -2k\,\alpha^2\, e_2 + 2\,\alpha^2\, e_3. \qquad \text{... (5.4)}$$

From this together with (3.3) and (3.9) we also have

$$Ae_2 = (4k\,\alpha\,\alpha' + 2\,\alpha^2)\, e_2 + (2k\,\alpha^2 - 4\,\alpha\,\alpha')\, e_3. \qquad \text{... (5.5)}$$

And, the eqs. (5.5), (3.3) and (3.9) yield

$$Ae_3 = (4k\,\alpha\,\alpha'' + 4k\,\alpha'^2 + 8\,\alpha\,\alpha' - 2\,k\,\alpha^2)\, e_2$$
$$+ \; (8\,k\,\alpha\,\alpha' + 2\,\alpha^2 - 4\,\alpha'^2 - 4\,\alpha\,\alpha'')\, e_3. \qquad \text{... (5.6)}$$

Thus, by combining (5.4) and (5.6) we have

$$\begin{cases} k\,\alpha\,\alpha'' + k\,\alpha'^2 + 2\,\alpha\,\alpha' = 0, \\ 2\,k\,\alpha\,\alpha' - \alpha'^2 - \alpha\,\alpha'' = 0, \end{cases} \qquad \text{... (5.7)}$$

which imply

$$(k^2 + 1)\,\alpha\,\alpha' = 0. \qquad \text{... (5.8)}$$

From which a function α is constant. If we make use of (3.11) again, then we obtain $k = 0$, $u(s) = 1$. Thus the eq. (5.3) gives $\Delta H = 2H$, and it is easily seen that the surface M is of 1-type. Furthermore, when $u(s) = 1$, the surface is the Clifdford tours, i.e., it is the product of two plane circles with the same radius.

Consequently, we have the following

***Theorem* 5.1** — *Let M be a flat rotation surface in* $\mathbb{E}^4$. *Then, the following are equivalent:*

1. M is a Clifford torus

2. M is of 1-type

3. The mean curvature vector H satisfies $\Delta H = AH$ *for some real matrix A.*

Combining Theorem A, Theorem B and the our Theorems 3.1, 4.1, 5.1 we have the following

***Theorem* 5.2** *(Characterization)* — *Let M be a flat rotation surface in Euclidean 4-space* $\mathbb{E}^4$. *Then the following are equivalent :*

1. M is of finite type.

2. M is of 1-type.

3. M has finite type Gauss map.

4. M has 1-type Gauss map.

5. M has pointwise 1-type Gauss map.

6. The gauss map G of M satisfies a partial differential equation $\Delta H = AH$ *for some real matrix A.*

8. M is a Clifford tours, i.e., it is the product of two plane circles with the same radius.

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