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BOUNDS ON THE GREATEST EIGENVALUE OF GRAPHS

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The eigenvalues of a graph are the eigenvalues of its adjacency matrix. This paper presents some upper bounds on the greatest eigenvalue of graphs and lower bound on the greatest eigenvalue of trees.

Key Words : Graph; Adjacency Matrix; Spectral Radius

1. INTRODUCTION

Let G (V, E) be a simple graph with vertex set $\{\upsilon_1, \upsilon_2, ..., \upsilon_n\}$. For $\upsilon_i \in V$, the degree of υ_i and the average of the degrees of the vertices adjacent to υ_i are denoted by d_i and m_i respectively. Let $A(G)$ be the adjacency matrix of G and $A(G) = (a_{ij})$ is defined to be $n \times n$ matrix (a_{ij}), where

$$a_{ij} = \begin{cases} 1 & \text{if } \upsilon_i \upsilon_j \in E, \\ 0 & \textit{otherwise}. \end{cases}$$

It follows immediately that if G is a simple graph, then $A(G)$ is a symmetric (0, 1) matrix in which every diagonal entry is zero.

Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as

$$\lambda_1 (A(G)) \geq \lambda_2 (A(G)) \geq .... \geq \lambda_n (A(G)).$$

Denote $\lambda_i (A(G))$ simply by $\lambda_i (G)$. The sequence of n eigenvalues is called the spectrum of G.

2. UPPER BOUND FOR SPECTRAL RADIUS

The largest eigenvalue $\lambda_1 (G)$ is often called the spectral radius of G. We now give some known upper bounds for the spectral radius $\lambda_1 (G)$.

Let G be a simple graph with n vertices and e edges.

(1) (Brualdi and Hoffman[2]). if $e = \binom{k}{2}$, then

$$\lambda_1(G) \leq k-1, \qquad \text{... (1)}$$

where the equality holds iff G is a disjoint union of the complete graph K_k and some isolated vertices.

(1) (Stanley[9]).

$$\lambda_1(G) \leq (-1+\sqrt{1+8e})/2, \qquad \text{... (2)}$$

where the equality occurs iff $e = \binom{k}{2}$ and G is a disjoint union of the complete graph K_k and some isolated vertices.

(2) (Hong[7]). If G is a connected graph, then

$$\lambda_1(G) \leq \sqrt{2e-n+1}, \qquad \text{... (3)}$$

where the equality holds iff G is one of the following graphs :

(a) the star $K_{1,n-1}$;

(b) the complete graph K_n.

(3) (Berman and Zhang[1]). If G is a connected graph, then

$$\lambda_1(G) \leq \max\left\{\sqrt{d_i d_j} : \upsilon_i \upsilon_j \in E\right\}, \qquad \text{... (4)}$$

where the equality holds if and only if G is a regular or a bipartite semiregular graph.

Hong[8] has pointed out that the upper bound in (3) is an improvement on the upper bound in (2) while the upper bound (1) is a special case of the upper bound (2), but (2), (3) and (4) are only applicable for connected graphs. Now we will give some new upper bounds for connected graphs and simple graphs.

It is a result of Perron-Frobenius in matrix theory (see[5], Page no. 66) which states that a non-negative matrix B always has a non-negative characteristic vale r such that the modulii of all the characteristic values of B do not exceed r. To this 'maximal' characteristic value r there corresponds a non-negative characteristic vector

$$B\,Y = r\,Y\;(Y \geq 0,\, Y \neq 0).$$

***Theorem* 2.1** — *Let G be a connected graph and* $\lambda_1(G)$ *be the spectral radius of A*(G). *Then*

$$\lambda_1(G) \leq \max\left\{\frac{TT_i}{T_i} : 1 \leq i \leq n\right\}, \qquad \text{... (5)}$$

where

$$T_i = \sum_j \left\{d_j : \upsilon_i \upsilon_j \in E\right\} = d_i\, m_i,$$

$$T\,T_i = \sum_j \left\{T_j : \upsilon_i \upsilon_j \in E\right\} = \sum_j \left\{d_j\, m_j : \upsilon_i \upsilon_j \in E\right\}$$

and the degree of the vertex υ_i, average degrees vertex υ_i, the average of the degrees of the vertices adjacent to υ_i are d_i and m_i respectively.

PROOF : Let us consider the matrix $M(G)^{-1} A(G) M(G)$, where $M(G)$ is the diagonal matrix with diagonal elements $d_i m_i$, $i = 1, 2, \ldots, n$.

Now the (i, j)th element of $M(G)^{-1} A(G) M(G)$ is

$$\begin{cases} \dfrac{d_j m_j}{d_i m_i} & \text{if } v_i v_j \in E, \\ 0 & \textit{otherwise.} \end{cases}$$

By applying Gerŝsgorin's theorem to the rows of $M(G)^{-1} A(G) M(G)$, we get

$$\lambda_1(G) \le \max \left\{ \sum_j \left\{ \frac{d_j m_j}{d_i m_i} : v_i v_j \in E \right\} : 1 \le i \le n \right\} = \max \left\{ \frac{TT_i}{T_i} : 1 \le i \le n \right\}.$$

***Theorem* 2.2** — *Let G be a connected graph and $\lambda_1(G)$ be the spectral radius of $A(G)$. Then*

$$\lambda_1(G) \le \max \left\{ \sqrt{\frac{TT_i}{d_i}} : 1 \le i \le n \right\}, \qquad \ldots (6)$$

where $TT_i = \sum_j \left\{ d_j m_j : v_i v_j \in E \right\}$

and the degree of the vertex $v_i \in V$ and the average of the degrees of the vertices adjacent to $v_i \in V$ are d_i and m_i respectively.

PROOF : Let us consider the matrix $D(G)^{-1} A(G) D(G)$, where $D(G)$ is the diagonal matrix of vertex degrees.

Now the (i, j)th element of $D(G)^{-1} A(G) D(G)$ is

$$\begin{cases} \dfrac{d_j}{d_i} & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue $\lambda_1(G)$ of $D(G)^{-1} A(G) D(G)$.

We can assume that one eigencomponent (say x_i) be equal to 1 and the other eigencomponents be less than or equal to 1, that is, $x_i = 1$ and $x_k \le 1$ for all k.

We have

$$\{D(G)^{-1} A(G) D(G)\} X = \lambda_1(G) X. \qquad \ldots (7)$$

From the ith eq. of (7),

$$\lambda_1(G)\, x_i = \sum_j \left\{ \frac{d_j x_j}{d_i} : \upsilon_i \upsilon_j \in E \right\},$$

i.e.,
$$\lambda_1(G) = \sum_j \left\{ \frac{d_j x_j}{d_i} : \upsilon_i \upsilon_j \in E \right\}. \qquad \text{... (8)}$$

From the jth eq. of (7),

$$\lambda_1(G)\, x_j = \sum_k \left\{ \frac{d_k x_k}{d_j} : \upsilon_j \upsilon_k \in E \right\}. \qquad \text{... (9)}$$

Multiplying both sides of (9) by $\frac{d_j}{d_i}$ and taking summation over j on both sides such that $\upsilon_i \upsilon_j \in E$, we get

$$\lambda_1(G)^2 = \sum_j \left\{ \frac{1}{d_i} \sum_k \left\{ d_k x_k : \upsilon_j \upsilon_k \in E \right\} : \upsilon_i \upsilon_j \in E \right\}$$

$$\leq \sum_j \left\{ \frac{1}{d_i} \sum_k \left\{ d_k : \upsilon_j \upsilon_k \in E \right\} : \upsilon_i \upsilon_j \in E \right\}$$

$$= \frac{1}{d_i} \sum_j \left\{ d_j m_j : \upsilon_i \upsilon_j \in E \right\} \qquad \text{... (10)}$$

$$= \frac{TT_i}{d_i}, \text{ where } TT_i = \sum_j \left\{ d_j m_j : \upsilon_i \upsilon_j \in E \right\}.$$

Therefore
$$\lambda_1(G) \leq \max \left\{ \sqrt{\frac{TT_i}{d_i}} : 1 \leq i \leq n \right\}.$$

Corollary 2.3 — Let G be a connected graph and $\lambda_1(G)$ be the spectral radius of $A(G)$.

Then
$$\lambda_1(G) \leq \max \left\{ \sqrt{d_k m_k} : 1 \leq k \leq n \right\}. \qquad \text{... (11)}$$

PROOF : Let $d_k m_k = \max_j \left\{ d_j m_j : \upsilon_i \upsilon_j \in E \right\}, \upsilon_k \in V$. From eq. (10), we get

$$\lambda_1(G) \leq \sqrt{d_k m_k}.$$

Therefore,
$$\lambda_1(G) \leq \max \left\{ \sqrt{d_k m_k} : 1 \leq k \leq n \right\}.$$

Remark 2.4 : Now we show that (11) is better than (3). Let $\max \left\{ \sqrt{d_k m_k} : 1 \leq k \leq n \right\}$ gives the maximum at the ith vertex, we wish to prove that

$$d_i m_i \leq 2e - n + 1,$$

$$\text{i.e., } \sum_j \left\{ d_j : v_i v_j \in E \right\} \leq \sum_{i=1}^{n} d_i - (n-1),$$

$$\text{i.e. } \sum_j \left\{ d_j : v_i v_j \notin E \right\} - (n-1) \geq 0.$$

For connected graphs it is always true. Hence the value obtained by applying (11) is better than the value obtained by applying (3).

***Theorem* 2.5** — *Let G be a simple graph and $\lambda_1(G)$ be the spectral radius of A(G). Then*

$$\lambda_1(G) \leq \max_i \left\{ \min_k \left\{ \frac{k}{2} + \sqrt{T'_{ik} + \frac{k^2}{4}} : k \geq 0, D_k > 0 \right\} : 1 \leq i \leq n \right\}, \qquad \ldots (12)$$

where $T'_{ik} = T'_{i(k-1)} - D_k$, $k = 1, \ldots, (d_i - 1)$; $T'_{i0} = d_i m_i$ *and* D_k = *number of neighbors of* v_i *which are connected to more than or equal to k neighbours of* v_i *vertices.*

PROOF : Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of an eigenvalue $\lambda_1(G)$ of the adjacency matrix $A(G)$ of a graph G.

We can assume that one of the eigencomponents (say x_i) is equal to 1 ad the other eigencomponents are less than or equal to 1, that is, $x_i = 1$ and $0 \leq x_k \leq 1$ for all k.

We have

$$A(G)X = \lambda_1(G)X. \qquad \ldots (13)$$

From the ith eq. of (13),

$$\lambda_1(G)x_i = \sum_j \left\{ x_j : v_i v_j \in E \right\},$$

$$\text{i.e., } \lambda_1(G) = \sum_j \left\{ x_j : v_i v_j \in E \right\}. \qquad \ldots (14)$$

From the *j*th eq. of (13),

$$\lambda_1(G)x_j = \sum_k \left\{ x_k : v_j v_k \in E \right\}. \qquad \ldots (15)$$

Taking summation over j on both sides of (15) such that $v_i v_j \in E$, we get

$$\lambda_1(G)^2 = \sum_j \left\{ \sum_k \left\{ x_k : v_j v_k \in E \right\} : v_i v_j \in E \right\},$$

$$\text{i.e., } \quad (\lambda_1(G) - k)\lambda_1(G) = \sum_j \left\{ \sum_k \left\{ x_k : v_j v_k \in E \right\} : v_i v_j \in E \right\}$$

$$- k \sum_j \left\{ x_j : v_i v_j \in E \right\},$$

i.e., $(\lambda_1(G) - k)\,\lambda_1(G) \le T'_{ik}$, ... (16)

where k is a non-negative integer and T'_{ik} is given by

$$T'_{i0} = d_i\, m_i$$

$$T'_{i1} = T'_{i0} - D_1$$

$$T'_{i2} = T'_{i1} - D_2$$

....

....

$$T'_{i,(d_i-1)} = T'_{i,(d_i-2)} - D_{(d_i-1)}$$

and D_k = number of neighbors of υ_i which are connected to more than or equal to k neighbours of υ_i vertices.

Therefore

$$\lambda_1(G) \le \frac{k}{2} + \sqrt{T'_{ik} + \frac{k^2}{4}}, \qquad \text{... (17)}$$

i.e., $\lambda_1(G) \le \min\limits_{k} \left\{ \frac{k}{2} + \sqrt{T'_{ik} + \frac{k^2}{4}} : k \ge 0, D_k > 0 \right\}$,

i.e., $\lambda_1(G) \le \max\limits_{i} \left\{ \min\limits_{k} \left\{ \frac{k}{2} + \sqrt{T'_{ik} + \frac{k^2}{4}} : k \ge 0, D_k > 0 \right\} : 1 \le i \le n \right\}$.

Corollary 2.6 — Let G be a simple graph $\lambda_1(G)$ be the spectral radius of $A(G)$. Then

$$\lambda_1(G) \le \max\left\{ \sqrt{d_i\, m_i} : 1 \le i \le n \right\}. \qquad \text{... (18)}$$

PROOF : Putting $k = 0$ in (17), we get

$$\lambda_1(G) \le \sqrt{d_i\, m_i}$$

Therefore $\lambda_1(G) \le \max\left\{ \sqrt{d_i\, m_i} : 1 \le i \le n \right\}$.

It is easily seen that the upper bound given by (12) be always less than or equal to (18).

Remark 2.7 : Now we will see that (18) is better than (4).

Let $t = \max\ \{\sqrt{d_i d_j} : \upsilon_i \upsilon_j \in E\}$, i.e, $d_j \le \frac{t^2}{d_i}\ \upsilon_i \upsilon_j \in E$.

Therefore, $\max\ \{\sqrt{d_i\, m_i} : 1 \le i \le n\} = \max\left\{ \sqrt{\Sigma_j \left\{ d_j : \upsilon_i \upsilon_j \in E \right\}} : 1 \le i \le n \right\} \le t$.

Hence (18) is better than (4)

3. Lower Bound for Spectral Radius of Graphs

We now give some known lower bounds for the spectral radius $\lambda_1(G)$.

(1) (Collatz and Sinogowitz[3]). If G is a connected graph of order n, then

$$\lambda_1(G) \geq \lambda_1(P_n) = 2\cos(\pi/(n+1)). \qquad \text{... (19)}$$

The lower bound occurs only when G is the path P_n.

(2) (Hong[6]). If G is a connected unicyclic graph, then

$$\lambda_1(G) \geq \lambda_1(C_n) = 2, \qquad \text{... (20)}$$

where C_n denotes the cycle on n vertices. The lower bound occurs only when G is the cycle C_n.

Lemma 3.1 — Let a graph G have some pendant vertices. We separate the pendant vertices into groups such that all the pendant vertices in each group have a common neighbor. In each such group the eigencomponents of an eigenvector corresponding to the non-zero eigenvalues are equal.

PROOF : Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue $\lambda(G)$ of $A(G)$. Also let $\upsilon_1, \upsilon_2, \ldots, \upsilon_r$ vertices of $V = \{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$, $r < n$ are pendant vertices connected to the vertex υ_i.

We have $\quad A(G)X = \lambda(G)X$,

i.e., $\lambda(G)x_k = \sum_j \left\{x_j : \upsilon_k\upsilon_j \in E\right\}, k = 1, 2, \ldots, n.$

For $k = 1, 2, \ldots, r$

$$\lambda(G)x_k = x_i. \qquad \text{... (21)}$$

From (21), we conclude that x_k, $k = 1, 2, \ldots, r$ are equal, because $\lambda(G)$ is non-zero.

Lemma 3.2 — Let $d_1, d_2, \ldots, d_n$ are positive integers and $x_1, x_2, \ldots, x_n$ are positive real numbers. If d_i increases with x_i, then :

$$\frac{d_1x_1 + d_2x_2 + \ldots + d_nx_n}{x_1 + x_2 + \ldots + x_n} \geq \frac{d_1 + d_2 + \ldots + d_n}{n}. \qquad \text{... (22)}$$

For any two vertices u and υ connected by a path in a graph G, we define the distance between u and v, denoted by $d(u, \upsilon)$ to be the length of a shortest $u - \upsilon$ path. Let G be connected graph with vertex set V. For each $\upsilon \in V$, the eccentricity of υ, denoted by $e(\upsilon)$, defined by

$$e(\upsilon) = \max\left\{d(u, \upsilon) : u \in V, u \neq \upsilon\right\}.$$

Now we apply Lemmas 3.1 and 3.2 to obtain the following lower bound for $\lambda_1(G)$.

Lemma 3.3 —Let T be a tree of n vertices and suppose there exists a vcertex $\upsilon \in V$ such that $e(\upsilon) \leq 2$, then

$$\lambda_1(T) \geq \sqrt{d + m - 1}, \qquad \text{... (23)}$$

where d is the degree of $\upsilon \in V$ and m is the average of the degrees of the adjacent vertices of $\upsilon \in V$.

PROOF : Let $\upsilon_1, \upsilon_2, ..., \upsilon_d$ be the vertices adjacent to υ and corresponding degrees be $d_1, d_2, \ldots, d_d$.

Since $e(\upsilon) \leq 2$, therefore, $\upsilon_1, \upsilon_2, ..., \upsilon_d$ are connected to the $(d_1 - 1), (d_2 - 1), ..., ((d_d - 1)$ pendant vertices respectively.

Using the above Lemma 3.1 we can say that all the eigencomponents corresponding to the pendant vertices which are connected to υ_i are equal to x_i (say), $i = 1, 2, ..., d$.

Therefore, the eigencomponents of the vertices $\upsilon_1, \upsilon_2, ..., \upsilon_d$ corresponding to the greatest eigenvalue are $\lambda_1 x_1, \lambda_1 x_2, ..., \lambda_1 x_d$ respectively.

For vertex υ (let y be an eigencomponent corresponding to vertex υ),

$$\lambda_1 y = \lambda_1 x_1 + \lambda_1 x_2 + \ldots + \lambda_1 x_d, \text{ i.e., } y = \sum_{i=1}^{d} x_i \quad \ldots (24)$$

For vertex υ_i,

$$\lambda_1^2 x_i = y + (d_i - 1)\, x_i,\ i = 1, 2, ..., d;$$

i.e.,
$$y = (\lambda_1^2 - d_i + 1)\, x_i,\ i = 1, 2, ..., d. \quad \ldots (25)$$

Taking summation over $i = 1, 2, ..., d$ on both sides of (25) and using the relation (24), we get

$$d \sum_{i=1}^{d} x_i = \lambda_1^2 \sum_{i=1}^{d} x_i - \sum_{i=1}^{d} d_i x_i + \sum_{i=1}^{d} x_i \ ,$$

i.e.,
$$\lambda_1^2 = d + \frac{\sum_{i=1}^{d} d_i x_i}{\sum_{i=1}^{d} x_i} - 1. \quad \ldots (26)$$

From (25), d_i is proportional to x_i, $i = 1, 2, ..., d$. Therefore by Lemma 3.2, we get

$$\frac{d_1 x_1 + d_2 x_2 + \ldots\ldots + d_d x_d}{x_1 + x_2 + \ldots\ldots + x_d} \geq \frac{d_1 + d_2 + \ldots\ldots + d_d}{d} = m.$$

Hence from (26), $\lambda_1 \geq \sqrt{d + m - 1}$.

The following is a well known result of graph spectral theory[4].

Lemma 3.4 (The Interlacing Theorem) — Let G be an arbitrary graph on n vertices and let $\lambda_1(G) \geq \lambda_2(G) \geq \ldots\ldots \geq \lambda_n(G)$ be the eigenvalues of its $0 - 1$ adjacency matrix. Denote by H the subgraph obtained by deleting a vertex from G. Then for $i = 1, 2, ..., n - 1$,

$$\lambda_i(G) \geq \lambda_i(H) \geq \lambda_{i+1}(G).$$

***Theorem* 3.5** — *Let T be a tree of* n *vertices and $\lambda_1(T)$ be the spectral radius of $A(T)$. Then*

$$\lambda_1(T) \geq \sqrt{d+m-1}, \qquad \text{... (27)}$$

where d is the highest degree of vertex $\upsilon \in V$ and m is the average of the degrees of the adjacent vertices of $\upsilon \in V$.

PROOF : Using Lemmas 3.3 and 3.4 to obtain the lower bound for $\lambda_1(G)$.

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REFERENCES

1. A. Berman and X. D. Zhang, *Journal of Combinatorial Theory*, Series **B 83** (2001), 223-40.
2. R. A. Brualdi and A. J. Hoffman, *Linear Algebra Appl.*, **65** (1985), 133-46.
3. L. Collatz and U. Sinogowitz, *Abh. Math. Sem. Univ. Hamburg*, **21** (1957), 63-77.
4. D. Cvetkovic, M. Doob and H. Sachs, 1995. Spectra of graphs-Theory and Application, Academic Press, New York, 1980; III edition, Barth, Heidelberg.
5. F. R. Gantmacher, The Theory of Matrices, Volume Two, Chelsea Publishing Company, New York, N. Y., 1974.
6. Y. Hong, *J. East China Norm. Univ. Natur. Sci. Ed.* **1** (1986), 31-34.
7. Y. Hong, *Linear Algebra Appl.*, **108** (1988), 133-40.
8. Y. Hong, *Discrete Mathematics*, **123** (1993), 65-74.
9. R. P. Stanley, *Linear Algebra Appl.*, **67** (1987), 267-69.