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AN ESTIMATION ON THE RATE OF CONVERGENCE FOR MODIFIED BETA OPERATORS

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In this paper, we study simultanous approximation for the linear combinations of modified Beta operators. We obtain a direct result in terms of higher order modulus of continuity. To prove the main result we use the technique of linear approximation method viz. Steklov mean.

Key Words: Modified Beta Operators; Linear Combinations; Modulus of Continuity

INTRODUCTION

Let f be a function defined on $[0, \infty)$ then the modified beta operators applied to f is defined by

$$B_n(f_r x) = \frac{n-1}{n} \sum_{\upsilon=0}^{\infty} b_{n,v}(x) \int_0^{\infty} P_{n,v}(t) f(t)\, dt, x \in [0, \infty) \qquad \text{... (1.1)}$$

Where,
$$b_{n,v}(x) = \frac{1}{B(\upsilon+1, n)} \frac{x^{\upsilon}}{(1+x)^{n+v+1}}$$

$$P_{n,u}(t) = \binom{n+\upsilon-1}{\upsilon} \frac{(t)^{v}}{(1+t)^{n+\upsilon}}$$

These operators (1.1) were introduced by Gupta and Ahmad[4] to approximate Lebesque function on the $[0, \infty)$

Let $C_{\gamma}[0, \infty) = \{f \in ([0, \infty) : |f(t)| \le Mt^{\gamma}$ for some $\gamma > 0$ and some constant $M > 0\}$

We define the norm $\|\cdot\|_{\gamma}$ on $C_{\gamma}[0, \infty)$ by $\|f\|_{\gamma} = \sup_{0<t<\infty} |f(t)| \pm t\gamma$

We note that order of approximation by these operators (1.1) is at best $O(n^{-1})$, howsoever smooth the function may be. Thus to improve the order of approximation, we consider the linear combination of operators (1.1) as described below.

For $d_0, d_1, d_2 \dots d_k$ arbitrary but fixed distinct positive integers, the linear combination $B_n(f, k, x)$ of $B_{djn}(f, x)$, $j = 0, 1, 2, \dots, n$ are defined by

$$B_n(f, k, x) = \sum_{j-u}^{k} C(j, k) B_{djn}(f, x)$$

Where $$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0, 0) = 1$$

Alternately the above linear combination may be defined as

$$B_n(f, k, x) = \begin{vmatrix} 1 & d_0^{-1} & \dots & d_0^{-k} \\ 1 & d_1^{-1} & \dots & d_1^{-k} \\ \dots & & & \\ 1 & d_k^{-1} & \dots & d_k^{-k} \end{vmatrix}^{-1} \begin{vmatrix} B_{d_0 n}(f, x) & d_0^{-1} & \dots & d_0^{-k} \\ B_{d_1 n}(f, x) & d_1^{-1} & \dots & d_1^{-k} \\ \dots & & & \\ B_{d_k n}(f, x) & d_k^{-1} & \dots & d_k^{-k} \end{vmatrix}$$

In the present paper we extend the results of [4] and obtain an estimate of error in terms of higher order modulus of continuity in simultaneous approximation for the linear combination of the operators (1.1).

AUXILIARY RESULTS

In this section, we shall mention some definition and certain lemmas to prove our main theorems.

Definition 1 — Let us assume that $0 < a < a_1 < b_1 < b < \infty$, for sufficiently small $\delta > 0$, the $(2K+2)$th order Steklov mean $g_{2K+2, i\delta}(t)$ corresponding to $g(t) \in C_\gamma[0, \infty)$ is defined by

$$g_{2k+2\, i\delta}(t) = \delta^{-(2k+2)} \int_{i\delta/2}^{\delta/2} \int_{i\delta/2}^{\delta/2} \text{———} \int_{\delta/2}^{\delta/2} [g(t) - \Delta_\eta^{2k+2} g(t)] \prod_{i=1}^{k+2} dt,$$

where $$\eta = \frac{1}{2k+2} \sum_{i=1}^{2k+2} t \qquad \text{and } t \in [a, b]$$

it is easily checked [2, 3, 5] that

(i) $g_{2k+2, \delta}$ has continuous derivatives upto order $(2k + 2)$ on $[a, b]$

(ii) $\| g_{2k+2, \delta}^{(r)} \|_{C[a_1, b_1]} \leq M_1 \, \delta r_w (g, \delta, a, b), r = 1, 2 - (2k+2)$

(iii) $\| g - g_{2k+2, \delta} \|_{C[a_1, b_1]} \leq M_2 \, W_{2k+2} (g, \delta, a, b),$

(iv) $\| g_{2k+2, \delta} \|_{C[a_1, b_1]} \leq M_3 \| g \|_\gamma$

where M_i, $i = 1, 2, 3$ are certain unrelated constants independent of g and δ.

Definition 2 — The kth order modulus of continuity $\omega_k(f, \delta)$ for a function continuous on an interval $[a, b]$ is defined by

$$\omega_k(f, \delta) = \text{Sup}\left\{ \left| \Delta_h^k f(x) \right| : |h| \leq \delta, x, x+kh \in I \right\}$$

for $k = 1$, $w_k(f, \delta)$ is written simply as $w_i(\delta)$ or $w(f, \delta)$

Lemma 2.1 [4] — For $m \in N \cup \{0\}$ if

$$U_{n,m}(x) = \frac{1}{n} \sum_{v=0}^{\infty} b_{n,u}(x) \left(\frac{v}{n+1} - x \right)^m$$

then $$(n+1)\, U_{n,m+1}(x) = x(1+x) \left\{ U'_{n,m}(x) + m\, U_{n,m+1}(x) \right]$$

consequently

(*i*) $U_{n,m}(x)$ is a polynomial in x of degree $\leq m$.

(*ii*) $U_{n,m}(x) = O(n^{-[m+1/2]})$, where $[\beta]$ denotes the integral part of β.

Lemma 2.2 — Let the mth order moment be defined by

$$T_{n,m}(x) = \frac{n-1}{n} \sum_{v'=0}^{\infty} b_{n,v}(x) \int_0^{\infty} P_{n,v}(t)(t-x)^m\, dt$$

then $$T_{n,0}(x) = 1,\; T_{n,1}(x) = \frac{3x+1}{n-2}$$

and $$(n-m-2)\, T_{n,m+1}(x) = x(1+x) \left[T'_{n,m}(x) + 2m\, T_{n,,m\,1}(x) \right]$$

$$+ \left[(1+2x)(m+1) + x\right] T_{n,m}(x),\; n > m+2$$

Further, for all $x \in [0, \infty)$

$$T_{n,m}(x) = O(n^{-[m+1]^2})$$

PROOF : The proof of (2.1) can easily obtain by using the definition of $T_{n,m}(x)$.

For the proof of (2.2) we proceed as follows, first

$$x(1+x)\, T'_{n,m} = \frac{n-1}{n} \sum_{v=0}^{\infty} x(1+x)\, b'_{n,v}(x) \int_0^{\infty} P_{n,v}(t)(t-x)^m\, dt$$

$$-\; mx(1+x)\, T_{n,m+1}(x)$$

Using relations

$$x(1+x)\, b'_{n,v}(x) = [v - (n+1)]\, b_{n,v}(x)$$

and $$t(1+t)\, P'_{n,v}(t) = (v - nt)\, P_{n,v}(t)$$

we obtain

$$x(1+x)\left[T'_{n,m}(x)+m\,T_{n,m+1}(x)\right]=\frac{n-1}{n}\sum_{v=0}^{\infty}(v-(n+1)x]\,b_{n,v}(x)\int_0^{\infty}p_{n,v}(t)$$

$$(t-x)^m\,dt=\frac{n-1}{n}\sum_{v=0}^{\infty}b_{n,v}(x)\int_0^{\infty}[(v-nt)+n(t-x-x)\,P_{n,v}(t)\,(t-x)^m\,dt$$

$$=\frac{n-1}{n}\sum_{v=0}^{\infty}b_{n,v}(x)\int_0^{\infty}t(1+t)\,p'_{n,v}(t)\,(t-x)^m\,dt+n\,T_{n,m+1}(x)-x\,T_{n,m}(x)$$

$$=\frac{n-1}{n}\sum_{v=0}^{\infty}b_{n,v}(x)\int_0^{\infty}[(1+2x)+(t-x)^2\;+\;x(1+x)]\,P'_{n,v}\,t^2\,(t-x)\,dt$$

$$+\;n\,T_{n,m+1}(x)-x\,T_{n,m}(x)$$

$$=\;-(m+1)(1+2x)\,T_{n,m}(x)-(m+2)\,T_{n,m+1}(x)-mx(1+x)\,T_{n,m-1}(x)$$

$$+\;n\,T_{n,m+1}(x)-x\,T_{n,m}(x)$$

This leads to (2.2)

Lemma 2.3 — There exists the polynomial $q_{i,j,r}(x)$ independent of n and v such that

$$x^r(1+x)^r\frac{d^r}{dx^r}(x^v(1+x)^{-n-v})=\sum_{\substack{2i+j\le r\\ i,j>=0}}n'(v-n(x)^j\,q_{i,j,r}(x)\,x^v(1+x)^{-n-v}$$

Lemma 2.4 — Let f be r times differentiable on $[0,\infty)$ such $f^{(r-1)}=O(t^{\alpha})$ for some $\alpha>0$ as $t\to\infty$ then for r = 1, 2, 3 and $n>\alpha+r$, we have

$$B_n^{r}(f,x)=\frac{(n-r-1)!\,(n+r-1)!}{n!\,(n-2)!}\sum_{v=0}^{\infty}b_{n+r}v(x)\int_0^{\infty}P_{n-r}\,v+n^{(t)}f^{r}(t)\,dt$$

PROOF : We have

$$B_n^{(r)}(f,x)=\frac{n-1}{n}\sum_{v=0}^{\infty}b_{n,v}^{(r)}(x)\int_0^{\infty}P_{n,v}(t)\,f(t)\,dt$$

by using Leibnitz theorem

$$B_n^{(r)}(f,x)=\frac{n-1}{n}\sum_{i=0}^{r}\sum_{v=i}^{\infty}\binom{r}{i}\frac{(n+v+r-i)!}{(n-1)!\,(v-i)!}(-1)^{r-i}x^{k-i}(1+x)^{-n-v+r-i}$$

$$\int_0^\infty P_{n,v}(t) f(t)\,dt$$

$$= \frac{n-1}{n} \sum_{v=0}^{\infty} \frac{(n+v+r)!}{(n-1)!\,v!} \frac{x^v}{(1+x)^{n+v+r+1}} \int_0^\infty \sum_{k=0}^{n} (-1)^{r-i} \binom{r}{i} p_{n,v+i}(t) f(t)\,dt$$

$$= \frac{(n-1)(n+r-1)!}{n!} \sum_{v=0}^{\infty} b_{n+r,v}(x) \int_0^\infty \sum_{i=0}^{n} (-1)^{r-i} \binom{r}{i} P_{n,v+i}(t) f(t)\,dt$$

Again by using Leibnitz theorem, we get

$$P^{(r)}_{n-r,v+r}(t) = \frac{(n-1)!}{(n-r-1)!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} P_{n,v+i}(t)$$

Hence, $$B_n^r(f,x) = \frac{(n-r-1)!\,(n+r-1)!}{(n-2)!\,n!} \sum_{v=0}^{\infty} b_{n+r,v}(x) \int_0^\infty (-1)^r P^{(r)}_{n-v+r}(t) f(t)\,dt$$

integrating r times we get the required result.

***Theorem* 2.5** — *Let $f \in C_\gamma[0,\infty)$, if $f^{(2k+r+2)}$ exists at a point $x \in [0,\infty)$ then*

$$\lim_{n\to\infty} n^{k+1}\,[B_n^{(r)}(f,(d_0,d_1,d_2\,....\,d_k),x) - f^{(r)}(x)]$$

$$= \sum_{i=r}^{2K+r+2} Q(i,k,r,x) f^{(i)}(x),$$

where Q, (i, k, r, x) are certain polynomials in x.

The proof of above theorem follows along the lines[1] and [4].

Main Results

In this section we shall prove the following main results

***Theorem* 3.1** —*Let $f^{(r)} \in C_\gamma[0,\infty)$ and $0 < a < a_1 < b_1 < b < \infty$ then for* n *sufficiently large*

$$\| B_n^{(r)}(f,k,x) - f^{(r)} \|_{C\,|a_1,b_1|} \le Max \left\{ C_1\, W_{2k+2}(f^{(r)}, n^{-1/2}\, a, b),\, C_2\, n^{-(k+1)} \|f\|\, \gamma \right\}$$

Where $C_1 = C_1(k,r)$ and $C_2 = C_2(k,r,f)$.

Proof : First, we have by linearty property of the operators (1.2), we have

$$\| B_n^{(r)}(f,k) - f(n)_{C(a_1\,b_1)} \le \| B_n^{(r)} \| f - f_{2k+2},\, \delta]),\, k) \|_{C(a_1\,b_1)}$$

$$+ \| B_n^{(r)}(f_{2n+2,\,\delta}(d_0,d_1,..\,d_k),\cdot) - f^{(n)}_{2n+2\,\delta} \|_{C\,|a_1,b_1|})$$

$$+ \| f^{(r)} - f^{(r)}_{2k+2,\delta} \|_{C[a_1, b_1]}$$

$$= A_1 + A_2 + A_3, \text{ say}$$

By property (iii) of Steklov mean, we have

$$A_3 \le C_1 \, \omega_{2k+2} (f^{(r)} \delta, a, b)$$

Next by Theorem 2.5, we have

$$A_2 \le C_2 \, n^{-(k+1)} \sum_{j=r}^{2k+r+2} \| f^{(i)}_{2k+2,\delta} \|_{C[a,b]}$$

By interpolation property due to Goldberg and Meir[3] for each $j = r, r+1, \ldots 2k+r+2$, we have

$$\| f^{(i)}_{2k+2,\delta} \|_{C[a,b]} \le C_3 \left\{ \| f_{2k+2,\delta} \|_{C[a,b]} + \| f^{2k+r+2}_{2k+2,\delta} \|_{C[a,b]} \right\}$$

Therefore by properties (*ii*) and (*iv*) of Steklov mean, we have

$$A_2 \le C_4 \, n^{-(k+1)} \left\{ \| f \|_\gamma + \delta^{-2(k+2)} \, \omega_{2k+2} (f^{(r)}, \delta) \right\}$$

Finally we shall estimate A, choosing a^*, b^* satisfying the conditions,

$$0 < a < a^* < a_1 < b_1 < b^* < b < \infty$$

Also let $\psi(t)$ be a characteric function fo the interval $[a^*, b^*]$ then

$$A_1 \le \| B_n^{(r)} (\psi(t) (f(t) - f_{2k+2,\delta}(t)) \, k, \cdot) \|_{C_{[a_1, b_1]}}$$

$$+ \| B_n^{(r)} (1 - \psi((t)) (f(t) - f_{2k+2,\delta}(t)) \, k, \cdot) \|_{C[a_1 \, b_1]}$$

$$= A_4 + A_5, \text{ say}$$

We may not here that to estimate, A_4 and A_5, it is enough to consider their expressions without the linear combinations.

By Lemma 2.4, we have

$$B_n^{(r)} \; \psi(t) (f(t) - f_{2k+2,\delta}(t), x)] = \frac{(n-r-1)!\,(n+r-1)!}{n!\,(n-2)!}$$

$$\sum_{v=0}^{\infty} b_{n+r,v}(x) \int_0^\infty P_{n-r,v+r}(t) \cdot \psi(t) (f^r - f^{(r)}_{2k+2,\delta}(t)) \, dt$$

Hence

$$\| B_n^{(r)} (\psi(t) (f(t) - f_{2k+2,\delta}(t), \cdot) \|_{C[a,b]} \le C_5 \| f^{(r)} - f^{(r)}_{2k+2,\delta} \|_{C[a^*, b^*]}$$

Now for $x \in [a_1, b_1]$ and $t \in [0, \infty) \mid [a^*, b^*] \mid$, we choose a δ_1 sstisfying $\mid t - x \mid \geq \delta_1$ Therefore by Lemma 2.3 and Schwarz inequality, we have

$$I = B_n^{(r)}\,((1 - \psi(t))\,(f(t) - f_{2k+2,\,\delta}(t))\,x) \mid \leq \frac{(n-1)}{n}$$

$$\sum_{\substack{2i \mid j \leq r \\ i, j \geq 0}} n' \frac{\mid q_{i,j,r}(x) \mid}{x^r (1 + x)^r} \sum_{v=0}^{\infty} b_{n,v}(x).$$

$$|v - nx|^j \int_0^{\infty} P_{n,v}(t)\,(1 - \psi(t)) \mid f(t) - f_{2k+2,\,\delta}(t) \mid dt$$

$$\leq C_6 \| f \|_\gamma \left(\frac{n-1}{n} \right) \sum_{\substack{2i + j \leq r \\ i, j \geq 0}} n^i b_{n,v}(x) \mid v - nx \mid^j \int_{\mid t - x \mid \geq \delta_i} P_{n,v}(t)\,dt \mid, (t - x) \geq \delta_1$$

$$\leq C_6\, \delta_1^{-2s} \| f \|_\gamma \frac{n-1}{n} \sum_{\substack{2i + j \leq r \\ i, j \geq 0}} n^i \sum_{v=0}^{\infty} b_{n,v}(x) \mid v - nx |^j$$

$$\left(\int_0^{\infty} P_{n,v}(t)\,dt \right)^{/2} \cdot \left(\int_0^{\infty} p_{n,v}(t)\,(t - x)^{4s}\,dt \right)^{1/2}$$

$$\leq C_6\, \delta_1^{-2s} \| f \|_\gamma \sum_{\substack{2i + j \leq r \\ i, j \geq 0}} n^i \left\{ \sum_{v=0}^{\infty} b_{n,v}(x)\,(v - nx)^i \right\}^{1/2}$$

$$\left\{ \frac{n-1}{n} \sum_{v=0}^{\infty} b_{n,v}(x) \left(\int_0^{\infty} p_{n,v}(t)\,(t - x)^{4s}\,dt \right) \right\}^{1/2}$$

Hence by Lemma 2.1 and 2.2

$$1 \leq C_7 \| f \|_\gamma O\,((n^{\left(i + \frac{j}{2} - s \right)} \leq C_7\, n^{-\gamma} \| f \|_\gamma$$

where $q = (s - n/2)$. Now choose $\delta > 0$ such $q \geq (k+1)$

Then $\qquad I \leq C_7\, n^{-(k+1)} \| f \|_\gamma$

Therefore by property (iii) of Steklov mean, we get

$$A_1 \leq C_8 \| f^{(r)} - f_{2k+2,\,\delta}^{(r)} \|_{C[a^*, b^*]} + C_7\, n^{-(k+1)} \| f \|_\gamma$$

$$\leq C_9\, \omega_{2k+2}\,(f(r), \delta, a, b) + C_7\, n^{-(k+1)} \| f \|_\gamma$$

Hence with $\delta = n^{-1/2}$, the theorem follows.

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