Indian J. pure appl. Math., 34(6) : 935-939, June 2003

SUPER VERTEX - MAGIC LABELING

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(*Received* 26 *August* 2002; *accepted* 23 *December* 2002)

A vertex-magic labeling is an assignment of the integers from 1, 2, 3, ..., $v+\varepsilon$ to the vertices and edges of G so that at each vertex, the vertex label and the labels on the edges incident at that vertex add to a fixed constant. In this paper, we introduce a new concept super vertex-magic labeling of a graph and establish some families of graphs have super vertex-magic labeling.

Key Words : Vertex-Magic Labeling; Super Vertex-Magic Labeling

1. INTRODUCTION

In this paper, we consider only finite simple undirected graph. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we take $\varepsilon = |E|$ and $v = |V|$. The set of vertices adjacent to a vertex u of G is denoted by $N(u)$

In[1] the notion of a vertex-magic total labeling was introduced. This is an assignment of the integers from 1 to $v+\varepsilon$ to the vertices and edges of G so that at each vertex, the vertex label and the labels on the edges incident at that vertex, add to a fixed constant. More formally, a one-to-one map f from $V \cup E$ onto the integers $\{1, 2, 3, ..., v+\varepsilon\}$ is a vertex-magic total labeling if there is a constant k and so that for every vertex u,

$$f(u) + \Sigma\, f(u\,v) = k$$

where the sum runs over all vertices v adjacent to u.

In this paper we introduce the concept of super vertex-magic labeling of a graph. A vertex-magic labeling f is called super vertex-magic labeling if $f(E) = \{1, 2, 3, ..., \varepsilon\}$ and $f(V) = \{\varepsilon+1, \varepsilon+2, ..., \varepsilon+v\}$. A graph G is called super vertex-magic if there exists a super vertex-magic labeling of G.

2. MAIN RESULTS

Lemma 2.1 — If a nontrivial graph G is super vertex-magic then the magic number k is given by $k = \varepsilon + \dfrac{v+1}{2} + \dfrac{\varepsilon(\varepsilon+1)}{v}$.

PROOF : Let f be a super vertex-magic labeling of a graph G with the magic number k. Then $f(E) = \{1, 2, 3, ..., \varepsilon\}$ and $k = f(u) + \sum_{v \in N(u)} f(uv)$, for all $u \in V$.

Then,
$$vk = \sum_{u \in V} f(u) + \sum_{u \in V} \sum_{v \in N(u)} f(uv)$$
$$= \sum_{u \in V} f(u) + 2 \sum_{e \in E} f(e)$$
$$= (\varepsilon + 1) + (\varepsilon + 2) + \ldots + (\varepsilon + v) + \varepsilon(\varepsilon + 1)$$
$$= \varepsilon v + \frac{v(v+1)}{2} + \varepsilon(\varepsilon + 1)$$

Thus,
$$k = \varepsilon + \frac{(v+1)}{2} + \frac{\varepsilon(\varepsilon+1)}{v}.$$ ■

***Theorem* 2.2** — *A path P_n is super vertex-magic if and only if n is odd and $n \geq 3$.*

PROOF : Suppose there exists a super vertex-magic labeling f of P_n with the magic number k. Then by lemma 2.1

$$k = n - 1 + \frac{n+1}{2} + \frac{n(n-1)}{n} = \frac{5n-3}{2}$$

Since k is an integer, n must be odd.

Let n be an odd integer,

$$V(P_n) = \{v_1, v_2, ..., v_n\} \text{ and } E(P_n) = \left\{ e_i = v_i v_{i+1} / 1 \leq i \leq n-1 \right\}.$$

Define $f : V \cup E \to \{1, 2, ..., 2n - 1\}$ as follows :

$$f(v_1) = 2n - 1$$
$$f(v_i) = n - 2 + i \text{ for } 2 \leq i \leq n$$
$$f(e_i) = \frac{n-i}{2} \text{ if } i \text{ is odd}$$
$$= n - \frac{i}{2} \text{ if } i \text{ is even.}$$

It is easily seen that f is a super vertex-magic labeling with the magic number $\frac{5n-3}{2}$.

(See Figure 1)

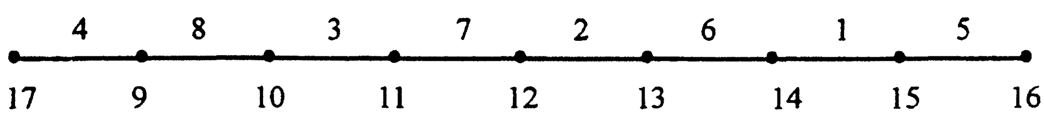

FIG. 1

***Theorem* 2.3** — *A cycle C_n is super vertex-magic if and only if n is odd.*

PROOF : Suppose there exists a super vertex-magic labeling f of C_n with the magic number k. Then by lemma 2.1

$$k = n + \frac{n+1}{2} + \frac{n(n+1)}{n} = \frac{5n+3}{2}.$$

Since k is an integer, n must be odd.

Let n be an odd integer,

$$V(C_n) = \left\{ v_1, v_2, ..., v_n \right\} \text{ and } E(C_n) = \left\{ v_n v_1 \right\} \cup \left\{ v_i v_{i+1} / 1 \leq i \leq n-1 \right\}$$

Define $f : V \cup E \rightarrow \{1, 2, 3, ..., 2n\}$ as follows :

$$f(v_i) = 2n + 1 - i \text{ for } 1 \leq i \leq n$$

$$f(v_i v_{i+1}) = \frac{i+1}{2} \text{ if } i \text{ is odd}$$

$$= \frac{n+1+i}{2} \text{ if } i \text{ is even}$$

$$f(v_n v_1) = \frac{n+1}{2}$$

It is easily seen that f is a super vertex-magic labeling with the magic number $\frac{5n+3}{2}$. (See Figure 2).

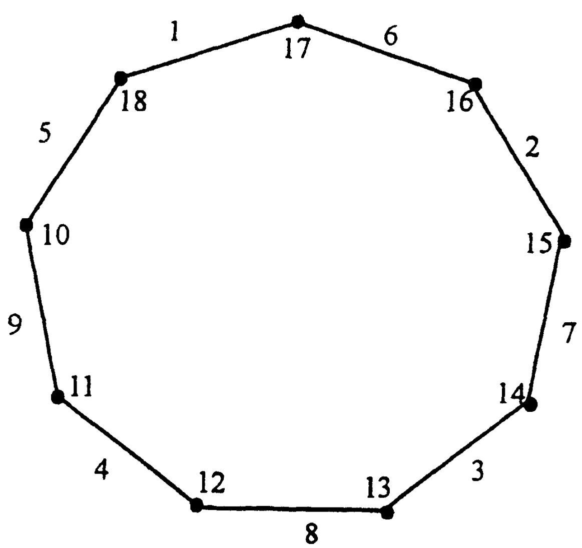

FIG. 2

***Theorem* 2.4** — *Let G be a graph and g is a bijection from E onto {1, 2, 3, ..., ε}. Then g can be extended to a super vertex-magic labeling of G if and only if*

$\left\{ w(u) = \sum_{v \in N(u)} g(uv)/u \in V \right\}$ *consists of* $|V|$ *sequential integers.*

PROOF : Assume that $\{w(u) . u \in V\}$ consists of $|V|$ sequential integers. Let $t = \min \{w(u)/u \in V\}$. Define $f: V \cup E \rightarrow \{1, 2, 3, ..., v+\varepsilon\}$ as $f(xy) = g(xy)$ for $xy \in E$ and $f(x) = t + v + \varepsilon - w(x)$. Then $f(E) = \{1, 2, 3, ..., \varepsilon\}$ and $f(V) = \{\varepsilon + v, v + \varepsilon - 1, ..., \varepsilon + 1\}$. Hence f is a super vertex-magic labeling with $k = t + v + \varepsilon$.

Suppose g can be extended to a super vertex-magic labeling f of G with a constant k. Now let $t = \min \{w(u)/u \in V\}$. Since for every $u \in V, f(u) + w(u) = k$, we have $w(u) = k - f(u)$. Thus, $\{w(u)/u \in V\} = \{k - \varepsilon - v, k - \varepsilon - v + 1, ..., k - \varepsilon - 1\} = \{t, t+1, ..., t + v - 1\}$. ■

***Theorem* 2.5** — *A star graph S_n is super vertex-magic if and only if $n = 2$.*

PROOF : Let $V(S_n) = \{c, u_1, u_2, ..., u_n\}$ and $E(S_n) = \{cu_i/1 \le i \le n\}$. Let f be super vertex-magic labeling of S_n. Then by Theorem 2.4, $\{w(u)/u \in V\} = \{1, 2, ..., n + 1\}$. Again $w(c) = \frac{n(n+1)}{2}$ and $w(u_i) = i$ for $1 \le i \le n$. Hence $n + 1 = \frac{n(n+1)}{2}$ Thus $n = 2$.

When $n = 2, S_n = P_z$ which is super vertex-magic. ■

3. SUPER VERTEX-MAGIC LABELING ON A DISCONNECTED GRAPH

In this section we give a super vertex-magic labeling for the disconnected graph $m\,C_n$, that is, the disjoint union of m cycles of length n, where m and n are odd.

***Theorem* 2.6** — *$m\,C_n$, is super vertex-magic if and only if both m and n are odd.*

PROOF : Suppose there exists a super vertex-magic labeling of $m\,C_n$, with the magic number k. Then by lemma 2.1

$$k = mn + \frac{mn+1}{2} + \frac{mn(mn+1)}{mn} = \frac{5mn+3}{2}$$

Thus, k is an integer only when both m and n are odd.

Let m and n are odd integers. Assume that the graph $m\,C_n$, has vertex set $V = V_1 \cup V_2 \cup ... \cup V_m$. Where $V_i = \{v_i^1, v_i^2 ..., v_i^n\}$, and the edge $E = E_1 \cup E_2 \cup ... \cup E_m$ where $E_i = \{e_i^1, e_i^2 ..., e_i^n\}$, and $e_i^j = v_i^j v_i^{j+1}$ for $1 \le i \le m, 1 \le j \le n-1, e_i^n = v_i^n v_i^1$

Define $f: V \cup E \rightarrow \{1, 2, 3, ..., 2mn\}$ as follows :

For $1 < i \le \frac{(m-1)}{2}$

$$f(v_i^j) = (2n - j)m + 1 - 2i \text{ for } 1 \le j \le n-2$$

$$= mn + i \text{ for } j = n - 1$$

$$= \frac{1}{2}(4n-1)\,m + \frac{1}{2} + i \text{ for } j = n$$

$$f(e_i^j) + \frac{1}{2}(j-1)\,m + i \text{ for } j = 1, 3, \ldots, n-2$$

$$= \frac{1}{2}(n+j)\,m + \frac{1}{2} + i \text{ for } j = 2, 4, \ldots, n-1$$

$$= \frac{1}{2}(n+1)\,m + 1 - 2i \text{ for } j = n$$

For $\frac{m+1}{2} \le i \le m$

$$f(v_i^j) = (2n+1-j)\,m + 1 - 2i \text{ for } 1 \le j \le n-2$$

$$= mn + i \text{ for } j = n-1$$

$$= \frac{1}{2}(4n-3)\,m + \frac{1}{2} + i \text{ for } j = n$$

$$f(e_i^j) = \frac{1}{2}(j-1)m + i \text{ for } j = 1, 3, \ldots, n-2$$

$$= \frac{1}{2}(n+j-2)\,m + \frac{1}{2} + i \text{ for } j = 2, 4, \ldots, n-1$$

$$= \frac{1}{2}(n+3)\,m + 1 - 2i \text{ for } j = n.$$

It is easily verified that f is a super vertex-magic labeling of mC_n with $k = \frac{5mn+3}{2}$ (See Figure 3).

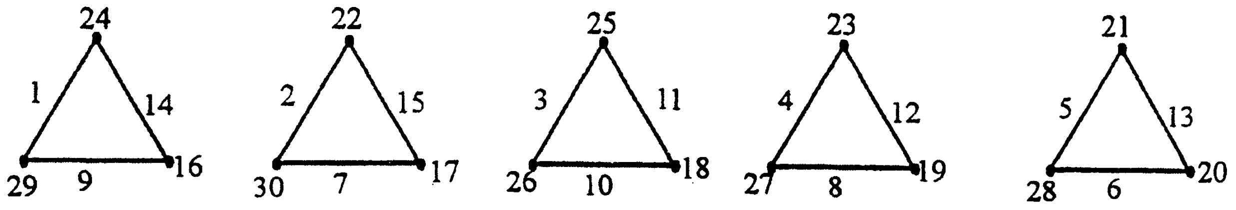

FIG. 3

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