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# A MULTIDIMENSIONAL EXTENSION OF A CARLSON TYPE INEQUALITY

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We give a multidimensional extension of a Carlson type inequality due to G.S. Yang and J.C. Fang, using a new theorem by the author.

Yang and Fang[4] proved the following generalization of Carlson's inequality (see[2]).

***Theorem* 1** — *Let* $f, g : [0, \infty) \to I\!R$ *be Lebesgue measurable functions and* $g$ *be continuously differentiable with* $g(0) = 0$, $\lim_{x \to \infty} g(x) = \infty$ *and*

$$0 < m = \inf_{x \in [0, \infty)} g'(x) < \infty.$$

*Suppose,* $p$ $\alpha$ *and* $r$ *are real numbers such that* $p > 2$ *and* $\alpha > 0$. *then*

$$\left( \int_0^\infty |f(x)|\, dx \right)^{2p} \le \left( \frac{\pi}{\alpha m} \right)^2 \left( \int_0^\infty g^{1-\alpha}(x) |f(x)|^{p(1+2r-rp)}\, dx \right)$$

$$\left( \int_0^\infty g^{1+\alpha}(x) |f(x)|^{p(1+2r-rp)}\, dx \right) \left( \int_0^\infty |f(x)|^{rp}\, dx \right)^{2(p-2)} \qquad \text{... (1)}$$

For other generalizations of Carlson's inequality, see[1&3] and the references given there.

Our aim in the present text is to prove the following multidimensional extension of (1), also allowing a more general setting of parameters and a much larger clas of functions $g$. It is shown in Remark 1 below that this includes Theorem 1 as a special case.

***Theorem* 2** — *Let* $n$ *be a positive integer. Suppose that* $p$, $q > 2$, $0 < \alpha < n$, *and* $r, s \in I\!R$. *Suppose, moreover, that for some positive constants* $m$ *and* $\gamma$, *the function* $g : I\!R^n \to (0, \infty)$ *satisfies*

$$g(x) \ge m |x|^\gamma. \qquad \text{... (2)}$$

*Then there is a constant* $B$, *independent of* $m$, $\alpha$ *and* $\gamma$, *such that*

$$\left( \int_{I\!R^n} |f(x)|\, dx \right)^{p+q} \le \frac{B}{\alpha^2 m^{2n/\gamma}}$$

$$\times \int_{\mathbb{R}^n} g(x)^{(n-\alpha)/\gamma} |f(x)|^{p(1+2r-rp)} dx \int_{\mathbb{R}^n} g(x)^{(n+\alpha)/\gamma} |f(x)|^{q(1+2s-sq)} dx$$

$$\times \left( \int_{\mathbb{R}^n} |f(x)|^{rp} dx \right)^{p-2} \left( \int_{\mathbb{R}^n} |f(x)|^{sq} dx \right)^{q-2}. \qquad \ldots (3)$$

*Remark* 1 : Let $n = 1$. Assume that $g$ satisfies the hypotheses of Theorem 1. Then

$$g(x) = \int_0^x g'(t)\, dt \geq mx.$$

Thus (2) is fulfilled with $\gamma = 1$. Put $p = q$ and $s = r$ in (3). If we restrict ourselves to $f$ supported on $[0, \infty)$ (cf. 1, p. 54], we get (1), except that we do not give an explicit value to the constant $B$.

We give the following simple example to illustrate the usefulness of the weaker assumptions on the function $g$ in Theorem 2 in comparison with those of Theorem 1.

*Example* 1 — Let $a$ and $b$ be positive real numbers, and define $g : [0, \infty) \to (0, \infty)$ by

$$g(x) = \begin{cases} a, & 0 \leq x < b, \\ 1 + a\sqrt{\frac{x}{b}}, & b \leq x < \infty. \end{cases}$$

Then $g$ is not differentiable (not even continuous at the point $x = b$) and $g(0) = a \neq 0$. Thus two of the conditions on $g$ in the hypotheses of Theorem 1 fail. Nevertheless, $g$ is an admissible function for Theorem 2, since (2) holds with $\gamma = \frac{1}{2}$ and $m = \frac{a}{\sqrt{b}}$.

*Remark* 2 : Although we can get away with weaker assumptions on the function $g$ than those imposed in Theorem 1, the condition $\lim_{x \to \infty} g(x) = \infty$ can not be relaxed too much. More precisely, we can not let the function $g$ be essentially bounded. To see this, assume that $0 \leq g(x) \leq G$ almost everywhere on $[0, \infty)$. For $R > 0$, let $f_R$ be the characteristic function of the interval $[0, R)$. Then for any $t$

$$\int_0^\infty |f_R(x)|^t dx = R,$$

and
$$\int_0^\infty g(x)^{(1 \mp \alpha)/\gamma} |f_R(x)|^t dx \leq RG^{(1 \mp \alpha)/\gamma},$$

and hence
$$\frac{\left( \int |f_R|\, dx \right)^{p+q} \left( \int |f_R|^{rp} dx \right)^{-p-2} \left( \int |f_R|^{sq} dx \right)^{-(q-2)}}{\int g^{(1-\alpha)/\gamma} |f_R|^{p(1+2r-rp)} dx \int g^{(1+\alpha)/\gamma} |f_R|^{q(1+2s-sq)} dx}$$

$$\geq \frac{R^{p+q}}{R^{p-2}\,R^{pq-2}\,RG^{(1-\alpha)/\gamma}\,RG^{(1+\alpha)/\gamma}} = \frac{R^2}{G^{2/\gamma}}\ .$$

This tends to infinity as $R \to \infty$, and thus the inequality can not hold for any finite constant. It is clear that this also shows failure of (3) in the general case.

To prove our main result, we apply Theorem 2 in Larsson[3], quoted below in a special case of the original, suitable for our present needs.

*Lemma* 1 — Let $(X, d\,\zeta)$ be a measure space on which weights $\beta \geq 0, \beta_0 > 0$ and $\beta_1 > 0$ are defined. Suppose that $p_0, p_1 \in (1, 2)$ and $\theta \in (0, 1)$. Suppose that there is a constant $C$ such that

$$\zeta\left(\left\{ z;\, 2^m \leq \frac{\beta_0\,(z)}{\beta_1\,(z)} < 2^{m+1} \right\}\right) \leq C, \quad m \in \mathbb{Z} \qquad \text{... (4)}$$

and that $$\frac{\beta}{\beta_0^{\theta}\,\beta_1^{1-\theta}} \in L^{\infty}\,(Z, d\,\zeta). \qquad \text{... (5)}$$

Then there is a constant $A$ such that

$$\| f\beta \|_{L^1\,(Z,\,d\,\zeta)} \leq A\, \| f\beta_0 \|^{\theta}_{L^{p_0}\,(Z,\,d\,\zeta)}\ \ \| f\beta_1 \|^{1-\theta}_{L^{p_1}\,(Z,\,d\,\zeta)}. \qquad \text{... (6)}$$

The constant $A$ can be chosen of the form

$$A = A_0\, C^{1-\theta/p_0 - (1-\theta)/p_1},$$

where $A_0$ does not depend on $C$.

We are now ready to give a proof of our main theorem.

PROOF OF THEOREM 2 : Let $Z = \mathbb{R}^n$ and define the measure $d\,\zeta$ on $Z$

$$d\,\zeta\,(x) = \frac{dx}{|\,x\,|^n},$$

where $dx$ denotes Lebesgue measure $\mathbb{R}^n$ Define, $Z, \beta\,(x) = |\,x\,|^n$, $\beta_0\,(x) = |\,x\,|^{n-\alpha/p}$ and $\beta_1\,(x) = |\,x\,|^{n+\alpha/q}$, and let $p_0 = p', p_1 = q'$ and

$$\theta = \frac{p}{p+q}.$$

Then $$\frac{\beta}{\beta_0^{\theta}\,\beta_1^{1-\theta}} \equiv 1 \in L^{\infty}\,(Z, d\,\zeta),$$

so (5) is satisfied. Moreover,

$$\frac{\beta_0\,(x)}{\beta_1\,(x)} = x^{-\alpha\left(\frac{1}{p}+\frac{1}{q}\right)}.$$

Let $$\tau = \alpha\left(\frac{1}{p} + \frac{1}{q}\right) > 0.$$

Thus $\beta_0(x)/\beta_1(x) \in [2^m, 2^{m+1})$ if and only if

$$2^{-(m+1)/\tau} < |x| \le 2^{-m/\tau}.$$

Hence, using polar coordinates, we get

$$\zeta\left(\left\{\frac{\beta_0}{\beta_1} \in [2^m, 2^{m+1})\right\}\right) = \omega_n \int_{2^{-(m+1)/\tau}}^{2^{-m/\tau}} \frac{dr}{r} = \frac{\omega_n \log 2}{\tau},$$

where $\omega_n$ denotes the surface area of the unit sphere in $\mathbb{R}^n$, so (4) holds with

$$C = \frac{\omega_n \log 2}{\tau} = \frac{C_0}{\alpha}.$$

Thus (6) implies

$$\int_{\mathbb{R}^n} |f(x)|\, dx = \int_Z |f(x)\, \beta(x)|\, d\,\zeta(x)$$

$$\le A\left(\int_Z |f(x)\,\beta_0(x)|^{p_0}\, d\,\zeta(x)\right)^{\theta/p_0}$$

$$\times\left(\int_Z |f(x)\,\beta_1(x)|^{p_1}\, d\,\zeta(x)\right)^{(1-\theta)/p_1}$$

$$= A\left(\int_{\mathbb{R}^n} |x|^{(n-\alpha)/(p-1)}\,|f(x)|^{p'}\, dx\right)^{(p-1)/(p+q)}$$

$$\times A\left(\int_{\mathbb{R}^n} |x|^{(n+\alpha)/(q-1)}\,|f(x)|^{q'}\, dx\right)^{(q-1)/(p+q)}$$

or

$$\left(\int_{\mathbb{R}^n} |f(x)|\, dx\right)^{p+q} \le A^{p+q}$$

$$\times\left(\int_{\mathbb{R}^n} |x|^{(n-\alpha)/(p-1)}\,|f(x)|^{p'}\, dx\right)^{p-1}$$

$$\times\left(\int_{\mathbb{R}^n} |x|^{(n+\alpha)/(q-1)}\,|f(x)|^{q'}\, dx\right)^{q-1}.$$

We write

$$|x|^{(n-\alpha)/(p-1)}\,|f(x)|^{p'} = \left(|x|^{(n-\alpha)/(p-1)}\,|f(x)|^{p'r(p-2)}\right)$$

and $$|x|^{(n+\alpha)/(q-1)}\,|f(x)|^{q'} = \left( |x|^{(n+\alpha)/(q-1)}\,|f(x)|^{q's(q-2)} \right)$$

and apply Hölder's inequality with the pairwise conjugate exponents $p-1$ and $(p-1)/(p-2)$ in the first integral, and the exponents $q-1$ and $(q-1)/(q-2)$ in the second integral, respectively. This gives

$$\left( \int_{\mathbb{R}^n} |f(x)|\,dx \right)^{p+q} \leq A^{p+q}$$
$$\times \int_{\mathbb{R}^n} |x^{n-\alpha}|f(x)|^{p(1+2r-rp)}\,dx \int_{\mathbb{R}^n} x^{n+\alpha}|f(x)|^{p(1+2s-sq)}\,dx$$
$$\times \left( \int_{\mathbb{R}^n} |f(x)|^{rp}\,dx \right)^{p-2} \left( \int_{\mathbb{R}^n} |f(x)|^{sq}\,dx \right)^{q-2}$$

Since

$$1 - \frac{\theta}{p_0} - \frac{1-\theta}{p_1} = \frac{2}{p+q},$$

we can choose

$$A = A_0\,C^{2/(p+q)} = A_0 \left( \frac{C_0}{\alpha} \right)^{2/(p+q)},$$

so that

$$A^{p+q} = \frac{B}{\alpha^2},$$

where $B$ does not depend on $\alpha$. By (2),

$$|x| \leq \left( \frac{g(x)}{m} \right)^{1/\gamma},$$

so estimating $|x|^{n-\alpha}$ and $|x|^{n+\alpha}$ in the integrals on the right of (7) yields (3), and the proof is complete. □

*Remark* 3 : We do not give a specific value to the constant $B$ in our results, while the corresponding constant in (1) is given as $\pi^2$. However, by examining the proof of (1) in [4], we find that equality can be attained in (1) only if $r = 1/p$, in which case the inequality reduces to the significantly simpler

$$\left( \int_0^\infty |f(x)|\,dx \right)^4 \leq \left( \frac{\pi}{\alpha m} \right)^2 \int_0^\infty g^{1-\alpha}(x)\,|f(x)|^2\,dx \int_0^\infty g^{1+\alpha}(x)\,|f(x)|^2\,dx.$$

It is therefore reasonable to believe that only in this special case, the constant $\pi^2$ can be best possible. It is also reasonable that the sharp constant in (1) depends on $r$, and that that of (3) depends on $r$ and on $s$. Thus, both of these two inequalities should be possible to sharpen.

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