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AN APPROACH FOR SOLVING A SYSTEM OF NONLINEAR EQUATIONS IN MINIMUM TIME

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In this paper we use measure theory to solve a wide range of a system of nonlinear equations in minimum time. First, by defining an error function, we transform the mentioned system to an optimal control problem. The new problem is modified into one consisting of the minimization of a linear functional over a set of Radom measures; then we obtain an optimal measure which is then approximated by a finite combination of atomic measures and the problem converted to an infinite-dimensional linear programming. We approximate the infinite linear programming to a finite-dimensional linear programming. Then by using the solution of the latter problem we obtain an approximate solution for the original problem. Furthermore, we obtain a path from the initial point of the system of the nonlinear equations to the approximate solution of the system in minimum time.

Key Words : Nonlinear Equations System; Measure Theory; Optimal Control; Optimal Time; Nonlinear Programming

1. INTRODUCTION

A well-known iterative method for solving a system of nonlinear equations is Newton's method, other famous methods are Halley's method, secant method and Broyden method (see [3], [9], [13], [19]). Now, we want to use a new method, called "Optimal Time Method", for solving a system of nonlinear equations in minimum time by using measure theory which is a method has recently been used for solving optimal control problems, replacing the classical problem by ones in measures spaces, for example see Wilson *et al.*[18], Rubio[14-15], Kamyad *et al.*[10-12], Farahi *et al.*[7] and Effati *et al.*[4-6]. This method is not an iterative method and it does not befall in zigzag case. So, we obtain an approximate optimal solution in a straightforward manner.

Let us consider a system of nonlinear equations in the form :

$$\left\{\begin{array}{l} f_1(x_1, x_2, ..., x_n) = 0 \\ f_2(x_1, x_2, ..., x_n) = 0 \\ \vdots \\ f_m(x_1 x_2, ..., x_n) = 0 \end{array}\right\} \quad ... (1)$$

where $f_i, i = 1, 2, ..., m$ are nonlinear functions in $C'(A^0)$, the space of real-valued continuously differentiable functions on A^0, where A^0 is the interior of the compact set A. We assume $f = (f_1, f_2, ..., f_m)$ and let

$$\| f(x) \|_2^2 = \sum_{i=1}^{m} f_i^2(x_1, x_2, ..., x_n), \quad ... (2)$$

which is the error function, and also let us assume $x = (x_1, x_2, ..., x_n)$ be a time varying vector that is

$$x(t) = (x_1(t), x_2(t), ..., x_n(t))$$

where the function $x(\cdot)$ is depend on time variable. We may write briefly (2) as :

$$\|f(x)\|_2^2 = \sum_{i=1}^{m} f_i^2(x(t)), \quad t \in J = [0, T] \qquad ... (3)$$

where T can be known or unknown. By differentiating of (3) with respect to t we have

$$\frac{d}{dt}\|f(x)\|_2^2 = \sum_{j=1}^{n}\sum_{i=1}^{m} 2f_i \frac{\partial f_i(x(t))}{\partial x_j(t)} \cdot x_j'(t). \qquad ... (4)$$

Let $x'(t) = (x_1'(t), x_2'(t), ..., x_n'(t))$, and define

$$g(x(t)) = \left(2\sum_{i=1}^{m} f_i \frac{\partial f_i}{\partial x_1}, ..., 2\sum_{i=1}^{m} f_i \frac{\partial f_i}{\partial x_n} \right).$$

Then (4) becomes :

$$\frac{d}{dt}\|f(x)\|_2^2 = g(x(t)) \cdot x'(t), \qquad ... (5)$$

where in the right hand side of (5) we used inner product. By integrating, we have

$$\|f(x(T))\|_2^2 - \|f(x(0))\|_2^2 = \int_0^T g(x(t)) \cdot x'(t)\, dt$$

where $x(0)$ is an initial point. Now we assume $x'(t) = u(t)$, and we call $u(\cdot)$ an artificial control function. Usually we would like to obtain an approximate solution of the system (1) by an acceptable error. Let $\varepsilon > 0$, be maximum acceptable error. If $x(T)$ is an acceptable approximate solution for system (1), so we want

$$\|f(x(T))\|_2^2 < \varepsilon,$$

where we assume $x(T)$ is an approximate solution of the system of nonlinear equations (1). It is obvious that if $x(T)$ be an exact solution for system (1), then we have $\|f(x(T))\|_2^2 = 0$. We may assume two cases for T:

Case 1 — T is known. We can transform the problem into an optimal control problem, as follows:

$$\text{Minimize} \int_0^T g(x(t)) \cdot u(t)\, dt + \|f(x(0))\|_2^2 \qquad ... (6)$$

Subject to $x'(t) = u(t), t \in [0, T]$ ... (7)

$x(0) = x_0, x(T)$ is approximate solution. ... (8)

By solving the problem (6)-(8) we obtain an acceptable approximate solution for system (1), where the value of objective function (6) is less than ε.

Case 2 — T is unknown. We define an optimal time problem as follows, where $\| f(x)) \|_2^2$ is constant (see [8]),

$$\text{Minimize } T = \int_0^T 1\, dt \qquad \text{... (9)}$$

Subject to $x'(t) = u(t), t \in [0, T]$ ... (10)

$$\int_0^T g(x(t)) \cdot u(t)\, dt + \| \| f(x(0)) \|_2^2 < \varepsilon \qquad \text{... (11)}$$

$x(0) = x_0$, $x(T)$ is approximate solution. ... (12)

If $x^*(T) = (x_1^*(T), x_2^*(T), ..., x_n^*(T))$ be an optimal solution of (9)-(12) then $x^*(\cdot)$ is an approximate acceptable solution of system (1). In the next section we shall analyse bounded control problems where T is unknown.

2. Analysis of the Bounded Control Problems where T is Unknown

We assume that the set of all admissible pairs is non-empty and denote it by W. A pair $w = [x(\cdot), u(\cdot)]$ is said to be admissible if the following conditions hold :

(*i*) $x(t) \in A, t \in J$ and is absolutely continuous on J.

(*ii*) $u(t) \in U, t \in J, U$ is a closed and bounded set, and $u(\cdot)$ is Lebesgue measurable on J.

(*iii*) The boundary condition (12) is satisfied.

(*iv*) The pair w satisfies (10)-(11) a.e. on J^0 = int J.

Let $w = [x(\cdot), u(\cdot)]$ be an admissible pair, and B an open ball in $\mathbb{R}^{n+1}$ containing $J \times A$.

Let $\phi \in C'(B)$, the space of real-valued continuously differentiable function on B such that they and their first derivatives are bounded on B, and define function ϕ^u as follows :

$$\phi^u(t, x(t), u(t)) = \phi_x(t, x(t))\, u(t) + \phi_t(t, x(t)) \qquad \text{... (13)}$$

for all $(t, x(t), u(t)) \in \Omega = J \times A \times U$, note that both ϕ_x and u are two n-vectors, and the first term in the right-hand side of (13) is inner product of ϕ_x and u, and

$$\phi_x = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, ..., \frac{\partial \phi}{\partial x_n} \right)^t.$$

The function ϕ^u is in the space $C(\Omega)$ of all real-valued continuous functions on the compact set Ω. Since $w = [x(\cdot), u(\cdot)]$ is an admissible pair, we have

$$\int_0^T \phi^u(t, x(t), u(t))\, dt = \int_0^T (\phi_x(t, x(t))\, x'(t) + \phi_t(t, x(t)))\, dt$$

$$= \int_0^T \phi'(t, x(t))\, dt = \phi(T, x(T)) - \phi(0, x(0)) = \delta\phi, \quad \text{... (14)}$$

for all $\phi \in C'(B)$.

Let $D(J^0)$ be the space of all infinitely differentiable real-valued functions with compact support in J^0 (see [2] and [17]). Define

$$\psi_j(t, x(t), u(t)) = x_j(t)\, \psi'(t) + u_j(t)\, \psi(t), \quad \text{... (15)}$$

for $j = 1, ..., n$ and all $\psi \in D(J^0)$, where u_j is the jth component of the control function u. Then, if $w = [x(\cdot), u(\cdot)]$ be an admissible pair, we have, for $j = 1, 2, ..., n$ and $\psi \in D(J^0)$,

$$\int_0^T \psi_j(t, x(t), u(t))\, dt = \int_0^T x_j(t)\, \psi'(t)\, dt + \int_0^T u_j(t)\, \psi(t)\, dt$$

$$= x_j(t)\, \psi(t)\, |_J - \int_0^T (x_j'(t) - u_j(t)))\, \psi(t)\, dt = 0,$$

since the trajectory and control function are an admissible pair satisfying (10) a.e. on J^0, and, since the function ψ has compact support in J^0, $\psi(0) = \psi(T) = 0$.

Let $C(\Omega)$ be of all real valued continuous functions on Ω, and with the choice of functions which depend only on the time variable, and independent of x and u. We are led thus to consider $C_1(\Omega)$ as a subspace of the space $C(\Omega)$, of the continuous functions dependent only on the time variable t. Thus we have,

$$\int_0^T f(t, x(t), u(t))\, dt = a_f, f \in C_1(\Omega),$$

where a_f is the integral of $f(\cdot, x, u)$ over J, independent of x and u. Now consider :

(1) The mapping

$$\Lambda_w : F \to \int_J F(t, x(t), u(t))\, dt,\ F \in C(\Omega),$$

defines a positive linear functional on $C(\Omega)$.

(2) By the Riesz representation theorem (see [16]), there exists a unique positive Radon measure μ on Ω such that

$$\Lambda_w(F) = \int_J F(t, x(t), u(t))\, dt = \int_\Omega F\, d\mu \equiv \mu(F),\ F \in C(\Omega).$$

Thus, the minimization of the functional (9) is equivalent to the minimization of

$$E[w] = \Lambda_w = (1) \qquad \text{... (16)}$$

Subject to

$$\left\{\begin{array}{l} \Lambda_w(\phi^u) = \delta\phi, \quad \phi \in C'(B) \\ \Lambda_w(\psi_j) = 0, j = 1, 2, \ldots, n, \quad \psi \in D(J^0) \\ \Lambda_w(f) = a_f, \ f \in C_1(\Omega) \\ \Lambda_w(g \cdot u) + \| f(x) \|_2^2 < \varepsilon \end{array}\right\} \qquad \text{... (17)}$$

Now, suppose that the space of all positive Radon measures on Ω will be denoted by $M^+(\Omega)$. By the Riesz representation, the positive linear functionals above will be replaced by their representing measures, thus we seek a measure in $M^+(\Omega)$, to be normally denoted by μ^* which minimizes the functional E. Thus, the minimization of the functional E in (16) over W is equivalent to the minimization of

$$E[\mu] = \int_\Omega 1 \, d\mu \equiv \mu(1) \in I\!R \qquad \text{... (18)}$$

over the set of all positive measures μ corresponding to admissible pairs w, which satisfy

$$\begin{array}{l} \mu(\phi^u) = \delta\phi, \phi \in C'(B) \\ \mu(\psi_j) = 0, j = 1, 2, \ldots, n, \psi \in D(J^0) \\ \mu(f) = a_f, f \in C_1(\Omega) \\ \mu(g \cdot u) + \| f(x(0)) \|_2^2 < \varepsilon. \end{array} \qquad \text{... (19)}$$

We shall consider the minimization of (18) over the set Q of all positive Radon measures on Ω satisfying (19). Now if we 'topologize' the space $M^+(\Omega)$ by the weak*-topology, it can be seen from (see [14]) that Q is compact. The functional $E : Q \to I\!R$, defined by

$$E[\mu] = \int_\Omega 1 \, d\mu \equiv \mu(1) \in I\!R, \mu \in Q$$

is a linear continuous functional on a compact set Q; so attains its minimum on Q (see [14]), thus, the measure-theoretical problem, which consists of finding the minimum of the functional (18) over the subset Q of $M^+(\Omega)$, possesses a minimizing solution μ^*, say in Q.

3. First Approximation

For the estimation by the nearly optimal piecewise constant control, consider the minimization of the functional (18) not over the set Q but over a subset of $M^+(\Omega)$ which is defined by requiring only a finite number of the constraints in (19) to be satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces, and then selecting a finite number of them. In the first step, we obtain an approximation of the optimal measure μ^* by a finite combination of atomic measures, that is, from the Theorem (see [14], appendix), μ^* has the form $\mu^* = \sum_{i=1}^{N} \alpha_i^* \delta(z_i^*)$ where $\alpha_i^* \geq 0$ and $x_i^* \in \Omega$ for $i = 1, 2, ..., N$ (here $\delta(z)$ is a unitary atomic measure, characterized by $\delta(z)(F) = F(z)$ where $F \in C(\Omega)$ and $z \in \Omega$). Then, we construct a piecewise- constant control function corresponding to the finite-dimensional problem. Therefore in the infinite-dimensional linear programming problem (18) with restriction defined by (19), we shall consider only a finite number M_1 of functions ϕ of the type $\phi_1 = x_1, \phi_2 = x_2, \phi_3 = x_3, ..., \phi_n = x_n, \phi_{n+1} = x_1^2, \phi_{n+2} = x_2^2 \ldots$.

Also, only a finite number of functions $\chi_h, h = 1, 2, ..., M_2$ that before defined in (15), when the functions ψ are considered by

$$\sin\left(\frac{2\pi rt}{T}\right) \text{ or } 1 - \cos\left(\frac{2\pi rt}{T}\right), \; r = 1, 2, \ldots .$$

Also, only a finite number L of functions f of the type

$$f_s(t) = \begin{cases} 1 & \text{if } t \in J_s \\ 0 & \text{otherwise} \end{cases}$$

will be considered, when $J_s = ((s-1)d, sd)$, and $d = \frac{T}{L}$, $s = 1, \ldots L$.

The set $\Omega = J \times A \times U$ is covered by a partition, where the partition is defined by taking all points in Ω as $z_j = (t_j, x_j, u_j)$. Of course, we only need to construct the control function $u(\cdot)$, since the trajectory is then simply the corresponding solution of the differential equation (10), with condition (12), which can be estimated numerically. The infinite-dimensional linear programming problem (18) with restriction defined by (19) can be approximated by the following problem, which z_j for $j = 1, ..., N$ belong to dense subset of Ω.

Minimize

$$\sum_{j=1}^{N} \alpha_j \qquad \ldots (20)$$

Subject to

$$\left\{\begin{array}{l} \sum_{j=1}^{N} \alpha_j \phi_i^u(z_j) = \delta\phi_i,\ i = 1, \dots, M_1 \\ \sum_{j=1}^{N} \alpha_j \chi_h(z_j) = 0,\ h = 1, \dots, M_2 \\ \sum_{j=1}^{N} \alpha_j f_s(t_j) = a_f,\quad s = 1, \dots, L \\ \sum_{j=1}^{N} \alpha_j (g \cdot u)(z_j) + \| f(x(0)) \|_2^2 < \varepsilon \\ \alpha_j \geq 0,\ j = 1, \dots, N, T > 0. \end{array}\right\} \quad \dots (21)$$

Note that the elements $z_j, j = 1, 2, .., N$ are fixed, the only unknowns are the numbers

$\alpha_j,\ j = 1, 2, \dots, N$ and T.

The procedure to construct a piecewise constant control function approximating the action of the optimal measure is based on the analysis [14].

In the following section we intend to transform the nonlinear programming (20)-(21) to a linear programming.

4. Second Approximation

In this section we solve approximately the nonlinear programming problem (20)-(21), and intend to develop a method for solving it. Since the functions ψ's are as follows :

$$\psi(t) = \sin\left(\frac{2\pi rt}{T}\right), r = 1, 2, \dots ,$$

or

$$\psi(t)\ 1 - \cos\left(\frac{2\pi rt}{T}\right), r = 1, 2, \dots$$

and T is unknown, and the functions $\chi_h, h = 1, 2, \dots, M_2$ contain the variable T, thus we have M_2 nonlinear constraints in (21). Therefore (20)-(21) is a nonlinear programming problem which we can solve it with a suitable software. But here for solving this nonlinear programming problem we transform it approximately into a linear programming problem by using the functions ψ's as follows:

$$\psi(t) = \begin{cases} \sin\left(\frac{2\pi rt}{T_1}\right) & \text{if } t \in [0, T_1] \\ 0 & \text{if } t \in (T_1, T] \end{cases} \quad \dots (22)$$

or
$$\psi(t)=\begin{cases}1-\cos\left(\dfrac{2\pi rt}{T_1}\right) & \text{if } t\in[0,T_1]\\ 0 & \text{if } t\in(T_1,T]\end{cases} \qquad \ldots (23)$$

where we choose T_1 as a known real number and an initial time for T, such that $T_1 < T$. We obtain T_1 by two steps as follows :

Step 1 — Choose $T = T_0$ where T_0 is a known real number. Now solve the problem (20)-(21). If the solution isn't an admissible approximate solution, put $T_0 := T_1$, else go to step (2).

Step 2 — Put $T_0 := \frac{2}{3} T_0$ and go to step (1).

Now we explain how to obtain T_1, let $T_0 = 1$ (or an arbitrary positive number) be an admissible time. We choose T_1 as rT_0 where $0 < r < 1$, is any arbitrary real number, (here for example we choose $r = \frac{2}{3}$). Now solve the problem (20)-(21). If we obtain an unadmissible approximated solution, then we obtain T_1, otherwise, put $T_0 := T_1$, and continue step (2). By this process we decrease it until we obtain an unadmissible time for T_1. Now, we divide the interval J = $[0, T]$ into two subintervals $[0, T_1]$ and $(T_1, T]$ such that $[0, T] = [0, T_1] \cup (T_1, T]$. Thus the problem (20)-(21) transform to a linear programming problem that we can solve it by revised simplex method (see[1]).

Finally, in the next section we shall analyse bounded control problems where T is known.

5. Analysis of the Problem where T is Known

We note that T may be known or unknown. If T is known we can transform the problem into an optimal control problem like as (6)-(8). Now suppose that the mapping

$$\Lambda_w : F \to \int_J F(t, x(t), u(t))\, dt, \quad F \in C(\Omega),$$

defines a positive linear functional on $C(\Omega)$, then by the Riesz representation theorem (see [16]), there exists a unique positive Radon measure μ on Ω such that

$$\Lambda_w(F) = \int_J F(t, x(t), u(t))\, dt = \int_\Omega F\, d\mu \equiv \mu F, \quad F \in C(\Omega).$$

Since by the above assumption, the positive linear functionals will be replaced by their representing measures, thus we seek a measure in $M^+(\Omega)$, to be normally denoted by μ^* which minimizes the functional E.

Thus, the problem (6)-(8) is equivalent to the minimization of

$$E[w] = \Lambda_w(g \cdot u) + \| f(x(0)) \|_2^2 \qquad \ldots (24)$$

Subject to

$$\left\{\begin{array}{l} \Lambda_w(\phi^u) = \delta\phi,\ \phi \in C'(B) \\ \Lambda_w(\psi_j) = 0,\ j = 1, 2, \ldots, n,\ \psi \in D(J^0) \\ \Lambda_w(f) = a_f,\ f \in C_1(\Omega) \end{array}\right\} \quad \ldots (25)$$

Therefore, this problem is equivalent to the minimization of

$$E[\mu] = \int_\Omega (g \cdot u)\, d\mu + \| f(x(0)) \|_2^2 \equiv \mu(g \cdot u) + \| f(x(0)) \|_2^2 \quad \ldots (26)$$

Subject to

$$\left\{\begin{array}{l} \mu(\phi^u) = \delta\phi,\ \phi \in C'(B) \\ \mu(\psi_j) = 0, j = 1, 2, \ldots, n,\ \psi \in D(J^0) \\ \mu(f) = a_f,\ f \in C_1(\Omega) \end{array}\right\} \quad \ldots (27)$$

Here, we have an infinite-dimensional linear programming problem like as (18)-(19) which can be apprximated by the following problem, where z_j for $j = 1, \ldots, N$ belong to dense subset of Ω.

$$\text{Minimize} \quad \sum_{j=1}^{N} \alpha_j (g \cdot u)(z_j) + \| f(x(0)) \|_2^2 \quad \ldots (28)$$

Subject to

$$\left\{\begin{array}{l} \sum_{j=1}^{N} \alpha_j \phi_i^u(z_j) = \Delta\phi_i,\ i = 1, \ldots, M_1 \\ \sum_{j=1}^{N} \alpha_j \chi_h(z_j) = 0,\ h = 1, \ldots, M_2 \\ \sum_{j=1}^{N} \alpha_j f_s(t_j) = a_f,\ s = 1, \ldots, L \\ \alpha_j > 0, j = 1, \ldots, N, T > 0. \end{array}\right\} \quad \ldots (29)$$

Note that the elements $z_j, j = 1, 2, \ldots, N$ are fixed, and T is known and only unknowns are the numbers α_j, $j = 1, 2, \ldots, N$. Since, this problem is a linear programming problem that we can solve it by revised simplex method (see[1]).

6. Numerical Examples

Some numerical examples are considered below to illustrate our method. The software which we use for solving our problems by Newton's method and the linear programming associated to our method is "MATLAB 5.3".

We note that in our method, "Optimal Time Method", by choosing suitable set of controls U and suitable choosing of meshes it is possible to reach to an admissible solution of the nonlinear system of equations from any initial point. But in the other methods, say Newton's method, the choosing of initial point is critical, that is, it may have no solution or it is possible to reach an unadmissible solution of the nonlinear system of equations. Furthermore, there are some systems of nonlinear equations which is not possible to get or achieve to an approximate solution by Newton's method but by Optimal Time method we can obtain an approximate solution for each of them.

Example 6.1 — Consider the following nonlinear system

$$\left\{ \begin{array}{l} f_1(x_1, x_2) = \exp(x_1) + x_1 x_2 - 1 = 0 \\ f_2(x_1, x_2) = \sin(x_1 x_2) + x_1 + x_2 - 1 = 0 \end{array} \right\} \quad \ldots (30)$$

with initial point (0.1, 0.5). By applying our method to problem (30), we transform it to the following optimal control problem :

Minimize

$$T = \int_0^T 1 dt$$

Subject to

$$\begin{aligned} &x_1'(t) = u_1(t) \\ &x_2'(t) = u_2(t) \\ &\int_0^T g(x(t)) \cdot u(t)\, dt + 0.1466 < \varepsilon \end{aligned} \quad \ldots (31)$$

where $x_1(0) = 0.1, x_2(0) = 0.5$, and $x(T) = (x_1(T), x_2(T))$ is the solution of the nonlinear system (32). Let $t \in J$, and $J = J_1 \cup J_2$ and $x(t) = [x_1(t), x_2(t)] \in A = A_1 \times A_2$, and $u = (u_1, u_2) \in U = U_1 \times U_2$, where $J = [0, T]$, $J_1 = [0, T_1], J_2 = (T_1, T], T_1 = 0.4$, $A_1 = [0, 1], A_2 = [0, 1]$, $U_1 = [-0.3, -0.2]$ and $U_2 = [0, 2, 2]$. Let the set $J = J_1 \cup J_2$ is divided into 6 subintervals such that $J_1 = [0, T_1]$, is divided into 5 subintervals and $J_2 = (T_1, T]$ is 6th subinterval, the sets A_1 and A_2 are divided respectively into 5, and 6 subintervals, and the sets U_1 and U_2 are divided respectively into 5, and 6 subintervals, so that $\Omega = J \times A \times U$ is divided into 5400 meshes. We assume $M_1 = 2, M_2 = 8, L = 6$ and $\varepsilon = 10^{-3}$. Then we solve the linear programming associated to our problem. Finally, by using the solution of the LP, when the optimal time T = 0.4052, the approximate solution of the nonlinear system (30) will be obtain as $x(T) = x(0.4052) = (0.0096, 0.9976)$. Also we obtain the approximate solution of the nonlinear system (30), by Newton's method as you see the summarized results in Table 6.1.

TABLE 6.1 : Approximate solution when T = 0.4052

Method	Initial Point	Solution
Newton	$x(0)$ = (0.1, 0.5)	$x^{(5)}$ = (0.0000, 1.0000)
Optimal Time	$x(0)$ = (0.1, 0.5)	$x(T)$ = (0.0096, 0.9976)

The graphs of the piecewise constant optimal control functions and the corresponding trajectory functions are shown in Figures 1-5.

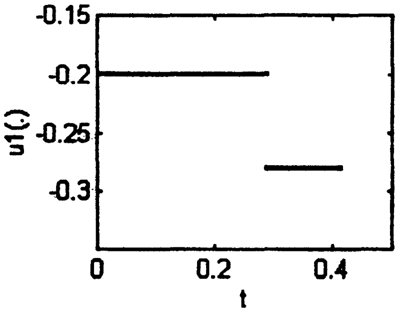

FIG. 1. Piecewise constant control

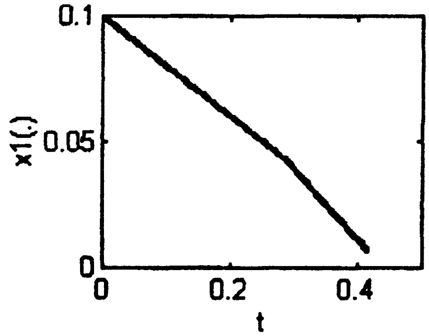

FIG. 2. Approximate solution

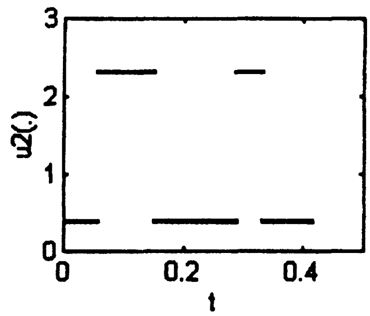

FIG. 3. Piecewise constant control

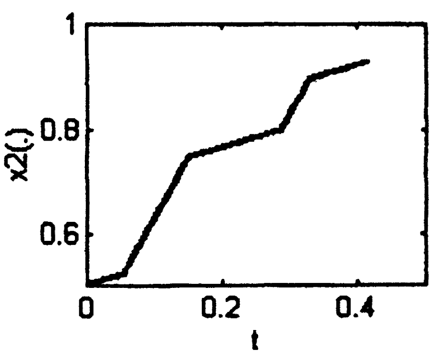

FIG. 4. Approximate solution

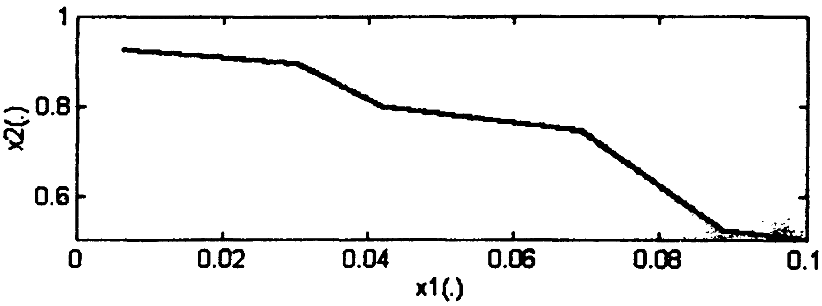

FIG. 5. Trajectory of solution from $x(0)$ = (0.1, 0.5) to $x(T)$ = (0.0096, 0.9976)

Example 6.2 — Consider the following nonlinear system which arose in the study of the stress at a point within a plate under a uniformly distributed load in mechanical engineering. Angles x_1 and x_2 define the position of point with respect to the load (see [19]).

$$\left\{ \begin{array}{l} f_1(x_1, (x_2) = \cos(2x_1) - \cos(2x_2) - 0.4 = 0 \\ f_2(x_1, x_2) = 2(x_2 - x_1) + \sin(2x_2) - \sin(2x_1) - 1.2 = 0 \end{array} \right\} \quad \ldots (32)$$

with initial point (0.1, 0.5), we obtain $x(T) = (x_1(T), x_2(T))$ and the optimal time T such that $f(x(T)) = (f_1(x(T)), f_2(x(T)) = (0, 0)$. First we define the function

$$\| f(x) \|_2^2 = f_1^2(x) + f_2^2(x),$$

and we obtain

$$\nabla \| f(x) \|_2^2 = g(x)$$

$$= \left(2f_1(x)\frac{\partial f_1(x)}{\partial x_1} + 2f_2(x)\frac{\partial f_2(x)}{\partial x_1}, 2f_1(x)\frac{\partial f_1(x)}{\partial x_2} + 2f_2(x)\frac{\partial f_2(x)}{\partial x_2} \right),$$

thus the problem reduce to the following problem :

$$\text{Minimize } T = \int_0^T 1\, dt$$

Subject to

$$x_1'(t) = u_1(t)$$

$$x_2'(t) = u_2(t) \quad \ldots (33)$$

$$\int_0^T g(x(t)) \cdot u(t)\, dt + 0.0605 < \varepsilon,$$

where $x_1(0) = 0.1$, $x_2(0) = 0.5$, $f(x_1(0), x_2(0) = 0.0605$ and $x(T) = (x_1(T), x_2(T))$ is the solution of the nonlinear system (33).

Let $t \in J = [0, T]$, and $J = J_1 \cup J_2$ and $x(t) = (x_1(t), x_2(t)) \in A = A_1 \times A_2$ and $u = (u_1, u_2) \in U = U_1 \times U_2$ where $J = [0, T]$, $J_1 = [0, T_1]$, $J_2 = (T_1, T]$, $T_1 = 0.3$, $A_1 = [0, 1]$, $A_2 = [0, 1]$; $U_1 = [0, 0.289]$ and $U_2 = [-0.039, 0]$. Let the set $J = J_1 \cup J_2$ is divided into 6 subintervals such that here the subinterval $J_1 = [0, T_1]$, is divided into 5 subintervals and $J_2 = (T_1, T]$ is 6th subinterval, the sets A_1 and A_2 are divided respectively into 5, and 6 subintervals, and the sets U_1 and U_2 are divided into 5400 meshes. Now if $M_1 = 2, M_2 = 8$, $L = 6$ and $\varepsilon = 10^{-4}$, then we have a linear programming associated to our problem. Finally, by using the solution of the LP, when the optimal time $T = 0.3165$, the approximate solution of the nonlinear system (32) will be obtain as $x(T) = (0.1565, 0.4927)$. Now, we obtain the approximate solution of the nonlinear system (32), by using Newton's method. Let the initial point be $x(0) = (0.1, 0.5)$, the approximate solution of the system (32), by using Newton's method is better than the approximate solution by our method (see Table 6.2), but in some cases, the approximate solution of the system (32) by our method is better than the approximate solution by Newton's method for example, when the initial point is (0, 0) and (0.2, 0.2) and (0.25, 0.25). The results are summarized in following Tables.

TABLE 6.2 :
Approximate solution for example 2 when $T = 0.3165$

Method	Initial Point	Solution
Newton	$x(0) = (0.1, 0.5)$	$x^{(5)} = (0.1565, 0.4933)$
Optimal Time	$x(0) = (0.1, 0.5)$	$x(T) = (0.1565, 0.4927)$

TABLE 6.3 :
Approximate solution for example 2 when $T = 0.3232$.

Method	Initial Point	Solution
Newton	$x(0) = (0, 0)$	No solution
Optimal Time	$x(0) = (0, 0)$	$x(T) = (0.1617, 0.5273)$

TABLE 6.4 :
Approximate solution for example 2 when $T = 0.3119$

Method	Initial Point	Solution
Newton	$x(0) = (0.25, 0.25)$	No solution
Optimal Time	$x(0) = (0.25, 0.25)$	$x(T) = (0.1570, 0.5105)$

TABLE 6.5 :
Approximate solution for example 2 when $T = 0.3119$

Method	Initial Point	Solution
Newton	$x(0) = (0.2, 0.2)$	No solution
Optimal Time	$x(0) = (0.2, 0.2)$	$x(T) = (0.1504, 0.4898)$

One can show that Newton's method has no solution for system (32) when we choose initial points say, (0.3, 0.3), (0.5, 0.5) and (1, 1), but by Optimal Time method we obtain approximate solution.

In this case, the graphs of the piecewise constant control functions and the trajectory functions are shown in Figures 6-10.

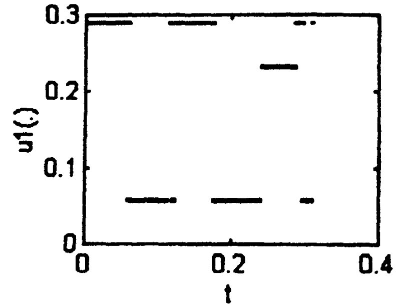

FIG. 6. Piecewise constant control

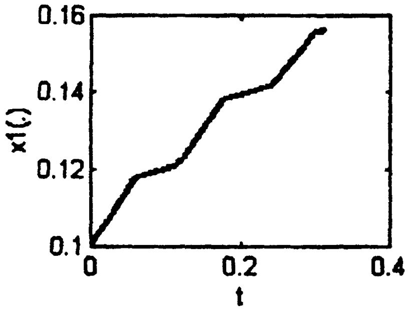

FIG. 7. Approximate solution

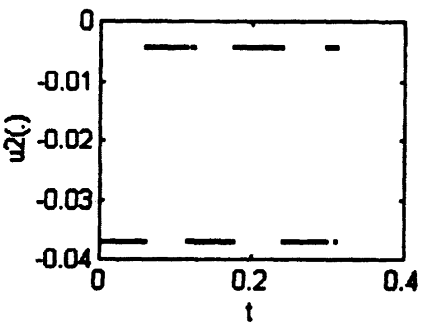

FIG. 8. Piecewise constant control

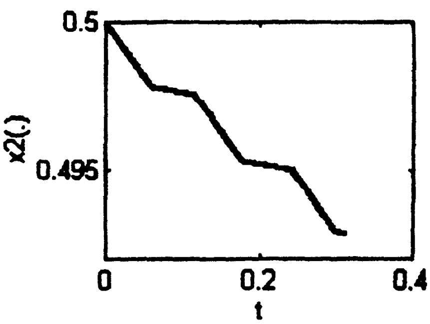

FIG. 9. Approximate solution

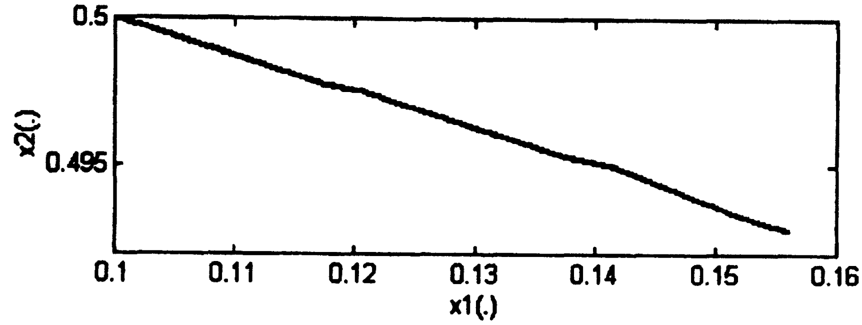

FIG. 10. Trajectory of solution from $x(0) = (.1, .5)$ to $x(T) = (.1565, .4927)$

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