Indian J. pure appl. Math., **34**(6) : 963-972, June 2003

A NON-GAUSSIAN TWO-DIMENSIONAL DISPERSION MODEL WITH CONCENTRATION DEPENDENT WIND AND DIFFUSIVITY PROFILES

K. N. MEHTA#

Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi 110 016, *India*

AND

A. K. YADAV*

**Ansal Institute of Technology, Gurgaon 122 033, Haryana, India*

(*Received* 1 *June* 2001; *after revision* 20 *March* 2002; *accepted* 9 *December* 2002)

This paper presents an analytic solution to a two-dimensional steady-state non-Gaussian dispersion model obtained by following the variational imbedding approach. The solution has been obtained as the limit of a sequence of approximate polynomial solutions of the advection diffusion equation in which the downwind speed and the diffusivity are assumed to have the most general functional dependence on the local pollutant concentration. The source term for the onset of dispersion is taken to be the prescribed ground level concentration which in turn depends on the ratio of vertical height and depth of the momentum boundary layer. The dispersing pollutant is assumed to remain confined in the momentum boundary layer bounded by the impermeable ground surface and the upper zero-flux edge of the boundary layer. Particular cases arising from the suitable choices for the functional form of ground level concentration are considered to examine the non-Gaussian effects. Plume descriptors, constituting the first four moments of the non-Gaussian concentration profile, are also obtained to analyse the statistical behaviour of the plume.

Key Words : Variational Imbedding; Non-Gaussian Model; Plume Descriptors

1. INTRODUCTION

In this paper, we have analysed a non-Gaussian dispersion model of advective diffusive transport of ground level pollutants for the case when the downwind speed U and the diffusivity K have the most general functional dependence on the local pollutant concentration yielding the governing equation as a nonlinear partial differential equation. It is customary in non-Gaussian dspersion to consider power law profiles for wind and diffusivity fields. Here, we have digressed from the traditional approach and considered a more general form. The situation of U and K depending on local concentration may arise in the case of ground level sources with high emission rates. In other words, pollutants do not influence wind and diffusivity profiles significantly for very low concentration levels. Ultimately, U and K are expressed as polynomial functions of vertical height which can give conventional power law profiles for U and K as a particular case.

The dispersion is induced by prescribed ground level concentration (glc) just as in a recent study discussing the important of windward diffusion[1]. A unit step-function increase of surface concentration has been considered for the growth of internal boundary layer by Philip[1]. In the present work, the glc $c_g(x)$ is prescribed as a function of downwind distance x rather than a constant. This kind of situation may be encountered at a site where, for example, the nuclear wastes and activated biological or chemical weapons dumped over an area may cause the spread of radioactivity, epidemics

*Present address: 10-D, Vijay Mandal Enclave, Hauz Khas, New Delhi 110 016

or poisonous gases in the atmosphere. It could also pertain to closely placed ground level emission sources along a line whose emissions could be regulated through the parameter(s) appearing in functional forms of $Cg(x)$.

This study has originated from an earlier work[1] which dealt with the application of variational imbedding technique to two types of urban air pollution dispersion models describing; (*i*) unsteady diffusive transport, and (*ii*) steady, two dimensional advective diffusive transport. The primary motivation for reviewing the earlier work was to circumvent the ambiguity of arbitrarily assigning any positive integral value not less than three to the exponent n (degree) of the polynomial solutions. This objective has been achieved by first estimating the thickness of the momentum boundary layer for large values of n and finally evaluating the limit of the sequence of polynomial solutions as $n \to \infty$.

Variational imbedding technique has been used to obtain the analytic expression for theoretical prediction of non-Gaussian concentration profiles corresponding to realistic functional forms of the prescribed ground level concentration. This technique is equivalent to the inverse of the problem of calculus of variation, that is, to find the functional where stationary points are prescribed by the set of descriptive equations governing the nonlinear advective diffusive transport process. Variational imbedding has been practiced as an art, and the general problem of imbedding a given set of descriptive equations into a variational statement remains unsolved. It has been shown[3] that any operator equation can be imbedded into a variational statement if more unknown functions are used than those involved in the given set of descriptive equations. The unknown functions which are introduced for the purpose of variational imbedding of an operator equation are called as adjoint functions. The impetus for using adjoint functions first stemmed from the work of Morse and Feshback, Lewins and also from Edelen's work on non-local calculus of variation[4-6].

Analytical expressions for the thickness of the contaminated momentum boundary layer and for the non-Gaussian concentration profiles having continuous derivatives upto all orders have been obtained in this paper for the most general forms of wind and diffusivity fields. Cases arising from various choices for the prescribed inputs are also discussed in detail. In view of the importance and utility of plume descriptors as brought out in a recent study[7], the plume descriptors for the non-Gaussian profile have been obtained and their sensitivity to the changes in the input controls (downwind speed and diffusivity fields) has been also examined to describe the statistical behaviour of the plume.

2. Model Formulation and Solution

We take $z = 0$ as the ground level and x-axis is chosen in the direction of the downwind speed U in the plane of the ground. We consider two-dimensional steady-state advective diffusive transport of a pollutant being dispersed in the form of particulate suspension or gaseous state from the ground based sources into the space $x \geq 0, z \geq 0$ by diffusion in the z-direction and advection along the x-axis. The downwind speed U and the coefficient of eddy diffusivity K have been taken to vary with the local pollutant concentration in the most general form. The diffusing pollutant is assumed to remain confined in the momentum boundary layer bounded by the impermeable ground and the zero-flux top edge of the layer represented by $z = \delta(x)$ (see Fig. 1).

Let c_0 be the uniformly constant distribution of background concentration in the domain $0 < x < \infty, 0 < z < \infty$ before the onset of dispersion and let $c_g(x) > c_0$ be the prescribed line-source function associated with the pollutants situated at the ground level $z = 0$. The actual form of $c_g(x)$ can be chosen suitably based on the specific problem.

The model equations comprising the governing transport equation, the relevant boundary conditions and the smoothing conditions are[2] :

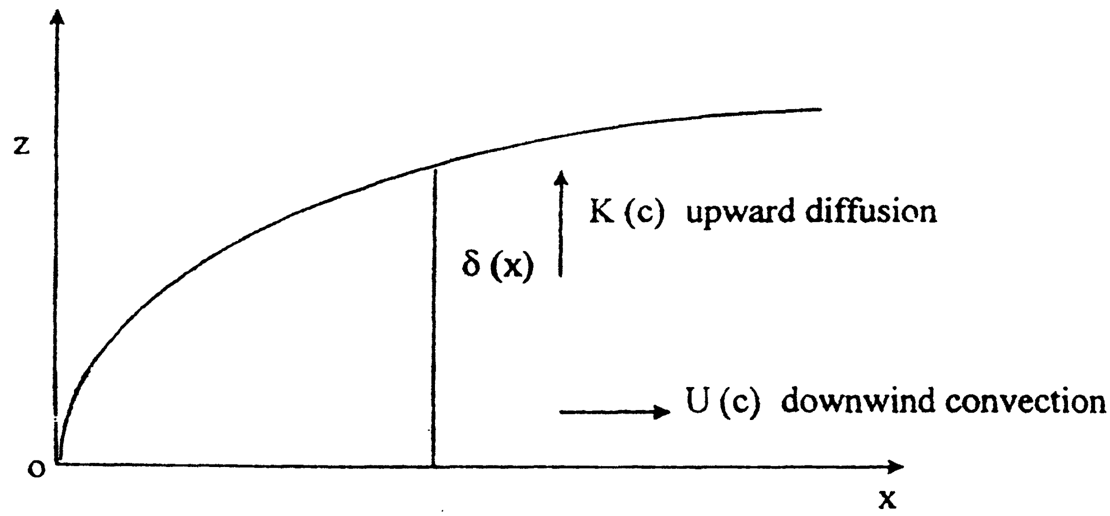

FIG. 1. Schematic diagram of dispersion in the momentum boundary layer

$$U(c)\frac{\partial c}{\partial x}=\frac{\partial}{\partial z}\left[K(c)\frac{\partial c}{\partial z}\right] \quad \text{... (1)}$$

$$c(x,0)=c_g(x),\ c_g(x)>c_0 \quad \text{... (2)}$$

$$c(x,z)=c_0,\ z\geq\delta(x) \quad \text{... (3)}$$

$$\frac{\partial c}{\partial z}=0,\ z=\delta(x) \quad \text{... (4)}$$

$$\frac{\partial^2 c}{\partial z^2}=0,\ z=\delta(x) \quad \text{... (5)}$$

More often in pollution studies, U and K are taken as power law functions of Z. However, here we have considered that for high pollutant concentration, U and K can be taken as polynomial functions of c which in turn are expressed as polynomials in powers of $Z/\delta(x)$, a more generalized approach.

In the contaminated layer $0<z<\delta(x)$, we assume a polynomial trial solution for the pollutant concentration $c(x, z)$, satisfying the conditions (3-5), in the form

$$c(x,z)=c_0+\theta(x)\left(1-\frac{z}{\delta(x)}\right)^n,\quad n\geq 3 \quad \text{... (6)}$$

where $\delta(x)$ denotes the thickness of the momentum boundary layer at location x and $\theta(x)=c_g(x)-c_0$ is a known function since $c_g(x)$ is prescribed. Here the exponent n can be assigned any positive integral value equal to or greater than 3. The main aim of this study was to obtain the form of solution (6) in the limit when $n\rightarrow\infty$. The eddy diffusivity K and the downwind speed U are assumed in the following form :

$$\left.\begin{array}{l} K(c) = K_0 \sum_{j=0}^{\infty} a_j \left(\dfrac{c - c_0}{\theta(x)} \right)^j, \\ U(c) = U_0 \sum_{j=0}^{\infty} b_j \left(\dfrac{c - c_0}{\theta(x)} \right)^j, \end{array}\right\} \quad \ldots (7)$$

Such an assumption excludes the possibility of a situation with $K = 0$ and $U = 0$. In the other words, it is to ensure that diffusivity does not vanish at the ground surface so that vertical diffusion is possible from $z = 0$ where sources are located. On using (6) eqs. (7) take the form

$$\left.\begin{array}{l} K(c) = K_0 \sum_{j=0}^{\infty} a_j \left(1 - \dfrac{z}{\delta(x)} \right)^{jn} \\ U(c) = U_0 \sum_{j=0}^{\infty} b_j \left(1 - \dfrac{z}{\delta(x)} \right)^{jn} \end{array}\right\} \quad \ldots (8)$$

Although (7) gives U and K as functions of local concentration, the choice of concentration distribution has changed it to dependence on vertical distance, a more general form than the conventional power law. The infinite series representation provided the most general forms for U and K which allow flexibility to the user to decide on the number of terms to achieve the desired accuracy. For example, one can have only one term for constant U and K, two terms for linear variation of U and K, and so on. In most of the practical situations, 2 or 3 terms would be sufficient to account for realistic variation of U and K with z. Following the method of variational imbedding (for details, see[2, 8]), one obtains

$$\delta^2(x) = \frac{K_0}{U_0} \frac{C}{A} [\theta(x)]^{-\frac{B}{A}} \int_0^x [\theta(\xi)]^{\frac{B}{A}} d\xi \quad \ldots (9)$$

in which the constants A, B, C depending on the coefficients a_j, b_j and the exponent n are given as

$$A = \sum_{j=0}^{\infty} \frac{nb_j}{(nj + 2n + 1)(nj + 2n)(nj + 2n - 1)} \quad \ldots (10a)$$

$$B = \sum_{j=0}^{\infty} \frac{b_j}{(nj + 2n + 1)(nj + 2n)} \quad \ldots (10b)$$

$$C = \sum_{j=0}^{\infty} \frac{n(nj + n - 1)a_j}{(nj + 2n - 1)(nj + 2n - 2)} \quad \ldots (10c)$$

For large n, the values of the ratios B/A and C/A involved in the expression for $\delta(x)$ can be approximated as

$$\frac{B}{A} \approx \left\{ \sum_{j=0}^{\infty} \frac{b_j}{(j+2)^2} \right\} \Bigg/ \left\{ \sum_{j=0}^{\infty} \frac{b_j}{(j+2)^3} \right\} = \alpha, \text{ say} \qquad \ldots (11a)$$

$$\frac{C}{A} \approx n^2 \left\{ \sum_{j=0}^{\infty} \frac{(j+1)\, a_j}{(j+2)^2} \right\} \Bigg/ \left\{ \sum_{j=0}^{\infty} \frac{b_j}{(j+2)^3} \right\} = n^2 B, \text{ say} \qquad \ldots (11b)$$

Since the value of α depends on the coefficients b_j which occur in $U(c)$ while β depends on both a_j and b_j, it is appropriate to regard α as wind parameter and β as wind-diffusivity parameter. Hence, for large n,

$$\delta^2(x) \approx n^2 \lambda(x)$$

where
$$\lambda(x) = \frac{\beta K_0}{U_0} F(x), \quad F(x) = [\theta(x)]^{-\alpha} \int_0^x [\theta(\xi)]^{\alpha} d\xi \qquad \ldots (12)$$

Then, the solution (6) for large n takes the form

$$c(x, z) = c_0 + \theta(x) \left(1 - \frac{z}{n\sqrt{\lambda(x)}} \right)^n \qquad \ldots (13)$$

which on taking the limit as $n \to \infty$, gives

$$c(x, z) = c_0 + \theta(x) \exp\left(\frac{-z}{\sqrt{\lambda(x)}} \right) \qquad \ldots (14)$$

Note that $F(x)$ and $\lambda(x)$ have dimensions of length and $(\text{length})^2$ respectively. The solution given by eq. (14) which is free from the exponent n may be regarded as the solution of the non-Gaussian dispersion model in the variational imbedding sense.

From
$$F(x) = [\theta(x)]^{-\alpha} \int_0^x [\theta(\xi)]^{\alpha} d\xi$$

we note that $F(x)$ depends on $c_g(x)$, the ground level concentration and α, the wind parameter. The particular case $\alpha = \beta = 2$ corresponds to constant wind and diffusivity fields $U(c) = U_0$ and $K(c) = K_0$.

$$F_c(x) = F(x)\,|_{\alpha=2} = [\,\theta(x)]^{-2} \int_0^x [\theta(\xi)]^2 d\xi$$

giving
$$c_n(x, z) = \frac{c(x, z) - c_0}{c_0} = \frac{\theta(x)}{c_0} \exp\left(-\sqrt{\frac{U_0}{2K_0}} \; \frac{z}{\sqrt{F_c(x)}} \right)$$

The expression for the normalized concentration $c_n(x, z)$ for the non-Gaussian case is given by

$$c_n(x,z) = \frac{\theta(x)}{c_0}\exp\left(-\sqrt{\frac{U_0}{\beta K_0}}\ \frac{z}{\sqrt{F(x)}}\right), \beta \neq 2$$

In the simplest non-Gaussian case of linear dependence of U and K on c

$$U(c) = U_0 \frac{c - c_0}{\theta(x)}, \quad K(c) = K_0 \frac{c - c_0}{\theta(x)}$$

and we find that in this case $\alpha = 3$ and $\beta = 6$.

Special Cases

Different profiles for $c_g(x)$ such as constant, linear or exponentially varying with downwind distance x have been considered. Such choices for the specification of source strength would allow regulating the emissions, through the controlling parameter(s), for minimizing their adverse impact on the environment.

Case I — Uniform source function $c_g(x) = c_1 > c_0$. Then $\theta(x) = c_1 - c_0, F(x) = x$ from (12) and the corresponding solution for pollutant concentration $c(x, z)$ is given by

$$c(x,z) = c_0 + (c_1 - c_0)\exp\left(-\sqrt{\frac{U_0}{BK_0}}\ \frac{z}{\sqrt{x}}\right) \qquad \text{... (15)}$$

It may be noted that in the case of constant ground level concentration, $F(x)$, and hence $c(x, z)$ are independent of the wind parameter α. Further, if we consider a particular case of constant wind U_0 and diffusivity K_0, which corresponds to

$$a_0 = 1, b_0 = 1 \text{ and } a_j = b_j = 0 \text{ for } j = 2, 3, \text{ .... giving } \alpha = \beta = 2,$$

the above solutions becomes

$$c(x,z) = c_0 + (c_1 - c_0)\exp\left(-\frac{1}{\sqrt{2}}\sqrt{\frac{U_0}{K_0 x}}\ z\right) \qquad \text{... (16)}$$

Case II — Variable source function

(*a*) Linearly varying source function : $\theta(x) = c_0\left(1 + \gamma\frac{x}{L}\right)$

(*b*) Exponentially varying source function : $\theta(x) = c_0 \exp\left(\gamma\frac{x}{L}\right)$

Here γ is a real scalar characterizing the variation of the source strength and L is the characteristic length. Corresponding to these cases, the expressions for $F(x)$ found from eq. (12) come out to be

(a) $$F(x) = \frac{L}{\gamma(\alpha+1)}\left[\left(1 + \gamma\frac{x}{L}\right) - \frac{1}{\left(1+\gamma\frac{x}{L}\right)^{\alpha}}\right]$$

(b) $$F(x) = \frac{L}{\gamma\alpha}\left[1 - \exp\left(-\alpha\gamma\frac{X}{L}\right)\right]$$

These expressions for $F(x)$ and $\theta(x)$ when substituted in eq. (12) yield the pollutant concentration for the two cases.

Concentration Distribution

The behaviour of the concentration distribution in the light of these special cases discussed above has been studied through certain plots of normalized concentration versus dwnwind distance. In Fig. 2 there are two sets of curves : solid ones correspond to constant wind and diffusivity fields whereas dashed ones refer to linear dependence of U and K on concentration. For concentration dependent U and K distributions, the pollutant concentration at all downwind locations is greater than its value corresponding to constant U and K. The nature of variation of $c_g(x)$ with x is also reflected in the concentration profiles at all altitudes (Fig. 3). For a given downwind location, the concentration profiles to bring out the response of rate of variation of linear glc.

3. Plume Descriptors

Plume descriptors, namely centroid, variance, skewness and kurtosis, obtained from the first four moments of the concentration distribution are defined as[7]

$$\text{Centroid } \bar{z}(x) = \int_0^\infty z\,\tilde{c}(x,z)\,dz \Big/ \int_0^\infty \tilde{c}(x,z)\,dz$$

$$\text{Variance } \sigma_z^2(x) = \int_0^\infty (z-\bar{z})^2\,\tilde{c}(x,z)\,dz \Big/ \int_0^\infty \tilde{c}(x,z)\,dz$$

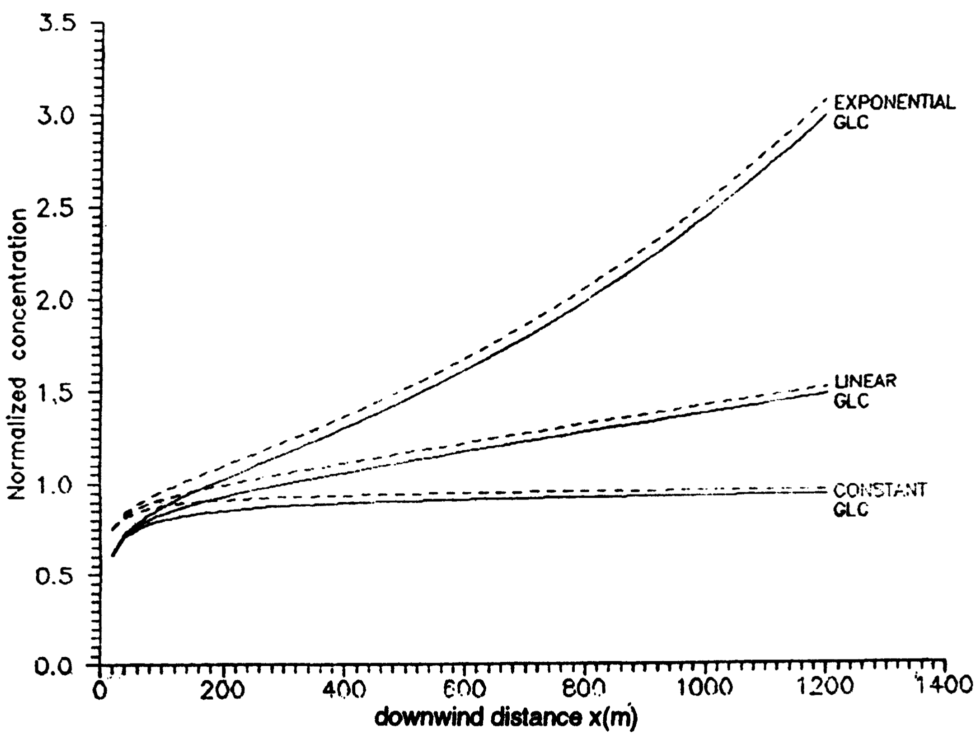

FIG. 2 Variation of concentration with downwind distance for constant, linearly and exponentially varying source strength, $\gamma = .1$, $z = 5$m, $L = 100$ m. (a) $\alpha=\beta=2$ (solid curves); constant U and K, (b) $\alpha = 3$, $\beta = .6$ (dashed curves); linear dependence of U and K on c.

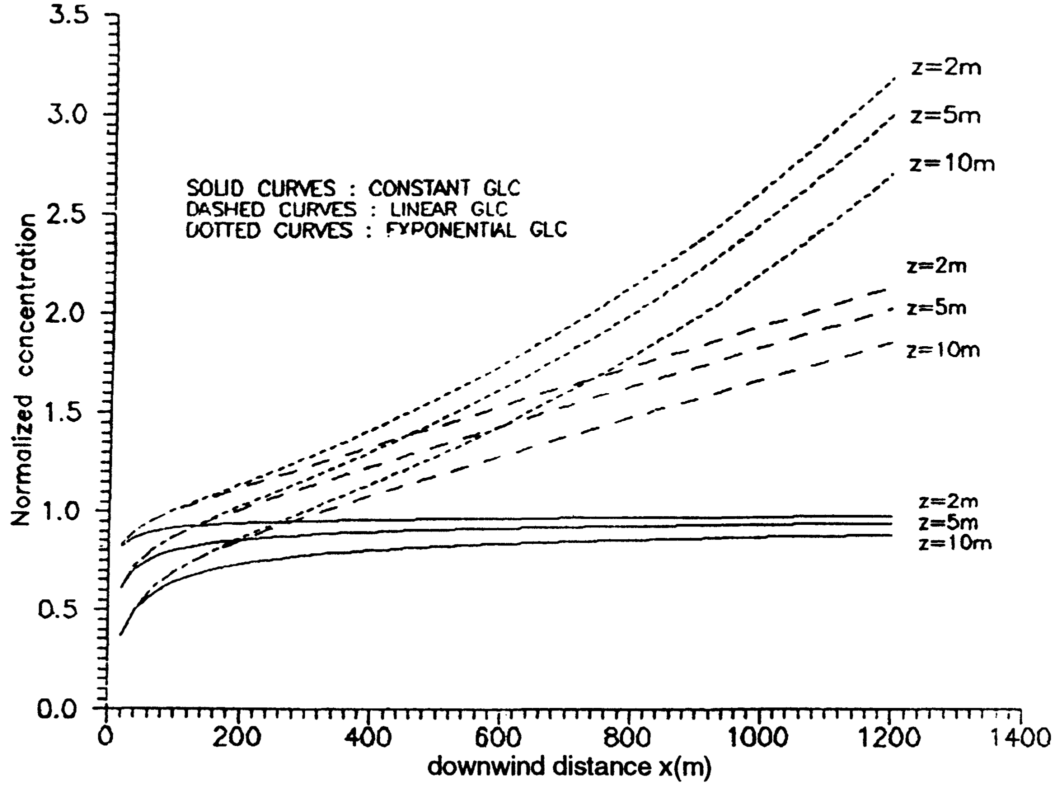

FIG. 3. Normalized concentration versus downwind distance at different altitudes z = 2, 5, 10m. Solid, dashed and dotted curves refer to constant, linear and exponential variation of glc. $\alpha=\beta=2$, $\gamma=.1$, L = 100m.

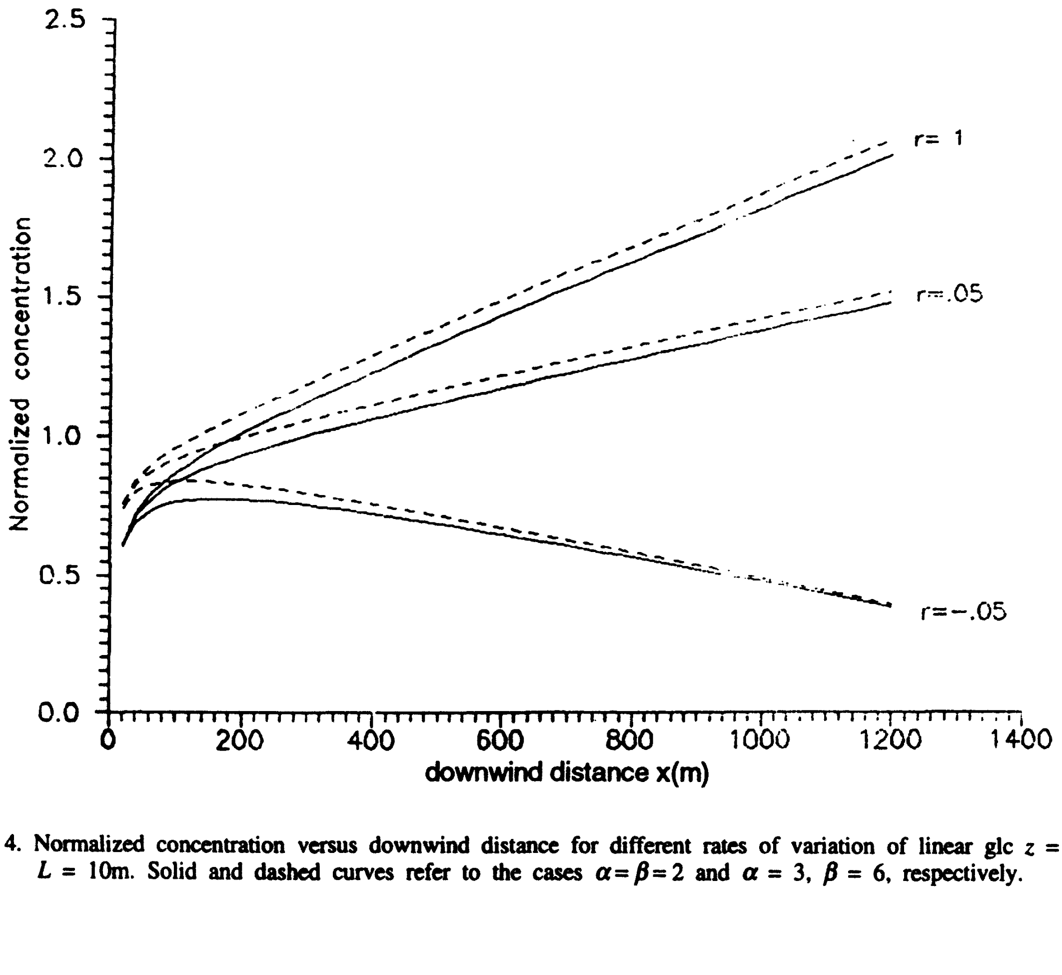

FIG. 4. Normalized concentration versus downwind distance for different rates of variation of linear glc z = 5m, L = 10m. Solid and dashed curves refer to the cases $\alpha=\beta=2$ and α = 3, β = 6, respectively.

The skewness and kurtosis are defined by the ratio

$$\int_0^{\infty} (z-\bar{z})^m \tilde{c}(x,z)\,dz \Big/ \sigma_z^m \int_0^{\infty} \tilde{c}(x,z)\,dz$$

with m = 3 and 4 respectively. Here $\tilde{c}(x,z) = c(x,z) - c_0$

Using the above formulae, the plume descriptors from the non-Gaussian solution (14) come out to be

$$\bar{z}(x) = \sqrt{\lambda(x)},\ \sigma_z^2(x) = \lambda(x),\ Sk(x) = 2,\ Ku(x) = 9$$

While skewness and kurtosis are constants, the centroid and the standard deviation depend on the location x, and both are given by the same expression, namely

$$\bar{z}(x) = \sigma_z(x) = \sqrt{\lambda(x)} = \sqrt{\frac{\beta K_0 F(x)}{U_0}},\quad F(x) = [\theta(x)]^{-\alpha} \int_0^{x} [\theta(\xi)]^{\alpha} d\xi \qquad \dots (17)$$

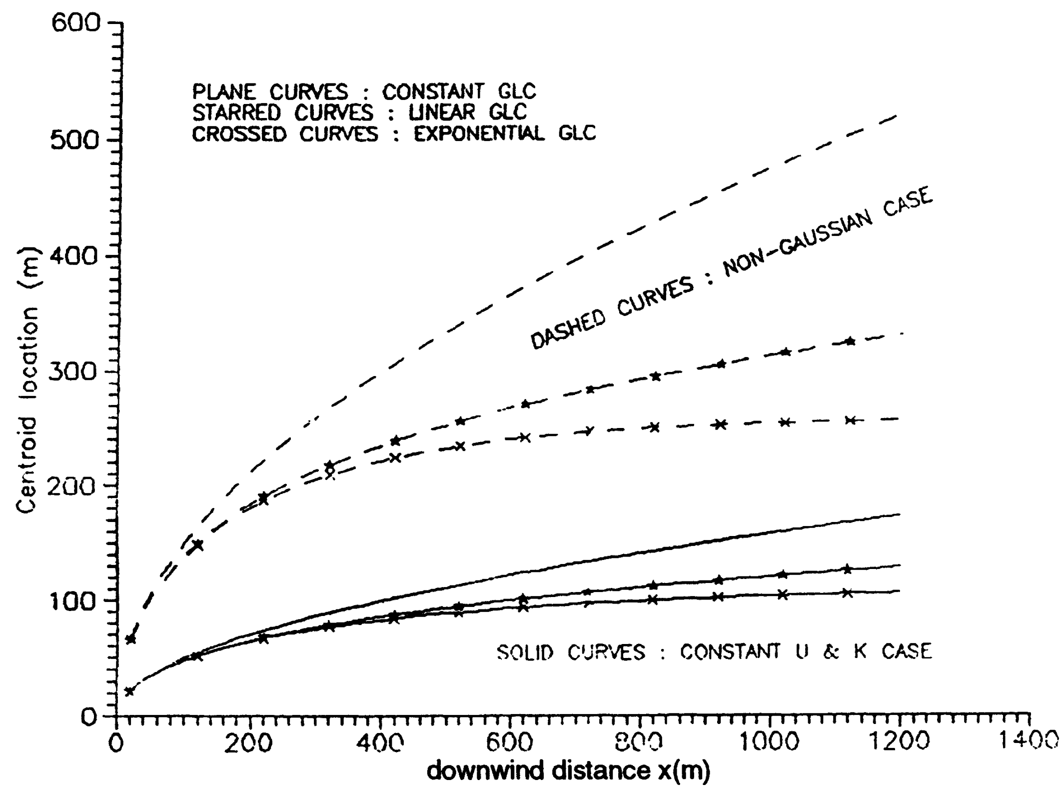

FIG. 5. Position of centroid for different downwind locations when γ = .1 and L = 100m. Solid and dashed curves refer to the cases $\alpha = \beta = 2$ and α = 3, β = 6, respectively.

In the special case of uniform source function $\theta(x) = c_1 - c_0$, $F(x) = x$, and hence, both standard deviation and centroid location vary as square root of the downwind distance x. Even in the case of a non-reflected Gaussian plume and a Gaussian deposition plume, centroid and standard deviation vary as square root of the downwind distance[7, 9].

Note that these two descriptrs are influenced by the wind and diffusivity fields by the appearance of parameters α and β in their common expression $\sqrt{\lambda(x)}$.

The effect of concentration dependent wind and diffusivity fields on the location of centroid is shown in Fig. 5. It may be seen that the dependence of U and K on c (the non-Gaussian case) induces upward movement of the centroid due to enhanced vertical dispersion. Furthermore, the centroid is located at the highest location in the case of constant glc and it moves downward with the increase in the source strength at the ground surface (linearly and exponentially increasing glc).

It may be recalled here that the values of skewness and kurtosis for a non-reflected gaussian plume are 0 and 3 respectively[7].

4. CONCLUSIONS

This study demonstrates the application of variational imbedding technique for analysing non-Gaussian vertical dispersion of pollutants from a ground based line-source. Analytical solution is presented for the advection diffusion equation when the downwind speed and the diffusivity have a general dependence on the local pollutant concentration expressed as a function of vertical height. Wind and diffusivity profiles get affected by high concentration from the emission source and hence considered functions of local concentration. Traditionally, in such a situation hydrodynamic equations coupled with the diffusion equation are considered. However, to avoid the complexities involved in solving the coupled system, an alternative approach is attempted by considering U and K as functions of local concentration in the advection-diffusion equation. Due to the form of the solution assumed in this study the dependence of U and K on local concentration ultimately reduces to dependence on the ratio of vertical height and depth of the momentum boundary layer (eqs. 7 and 8). The solution thus obtained overcomes the arbitrariness in the degree of the approximate polynomial solutions given in the earlier work[2]. It may be further pointed out that the approach adopted here enables one to deduce the concentration profiles for constant wind and diffusivity fields as a particular case of the general solution presented in this paper. The statistical behaviour of the plume movement is examined by obtaining the plume descriptors from the non-Gaussian concentration distribution. For the simplest case, the centroid and the standard deviation have been found to vary as square root of the downwind distance which is also the case in Gaussian models whereas the coefficient of skewness and kurtosis have the values 2 and 9 respectively.

REFERENCES

1. J. R. Philip, *J. of Applied Meteoro*, **36** (1997), 974-77.
2. K. N. Mehta and R. Balasubramaniam, *Atmos. Environ.*, **11** (1977), 109-12.
3. V. P. Bhatkar, *Ph. D. Thesis*, 1972 Indian Institute of Technology, New Delhi
4. P. M. Morse and H. Feshback, *Methods of Theoretical Physics*, Vol. 1, McGraw-Hill, New York, 1953.
5. J. Lewins, Importance: *The Adjoint Functions*, Pergamon Press, Oxford, 1965.
6. D. G. B. Edelen, *Non-local Variations and Local Invariance of Fields*. American Elsevier, New York, 1969.
7. M. J. Brown, S. P. Arya and W. H. Synder, *Atmos. Environ.*, **31** (1997), 183-89.
8. K. N. Mehta and R. Balasubramaniam, *Atmos. Environ.*, **12** (1978), 1343-47.
9. A. K. Yadav and K. N. Mehta, *Il Nuovo Cimento*, **23c** (2000), 251-62.