Indian J. pure appl. Math., 34(6) : 991-994, June 2003

SOME RESULTS ON Γ-RINGS

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(*Received* 1 *May* 2001; *after revision* 5 *August* 2002; *accepted* 2 *January* 2003)

In this note some results on Γ-rings are generalized which extend a few results obtained in a work done by Satyanarayana, Pradeepkumar and Srinivasa Rao. Some more results on idempotents and commutativity of Γ-rings are also included.

Key Words : Γ-rings; Left Ideal; Prime Ideals and Idempotents

1. INTRODUCTION

A generalization of the concept of Γ-rings has been done in Barnes[1]. Earlier the concept of Γ-ring was introduced by Nobusawa[3]. The aim of our present note is to obtain few results on ideals of Γ-ring which extend some results of Satyanarayana, Pradeep Kumar and Srinivasa Rao[5]. Moreover in later sections we give some extensions of generalized form of Prime ideals in the Γ-ring structure. The definition given below was introduced by Barnes.

Definition 1.1 — Let $(M, +)$ and $(\Gamma, +)$ be additive Abelian groups. If there exists a mapping $M \times \Gamma \times M \to M$ (the image of (a, α, b) is denoted by $a \alpha b$ where $a, b \in M$ and $\alpha \in \Gamma$) satisfying the following conditions :

(*i*) $(x+y)\, \alpha z = x\, \alpha z + y\, \alpha z$

(*ii*) $x\, \alpha\, (y+z) = x\, \alpha y = x\, \alpha z$

(*iii*) $x\, (\alpha + \beta)\, z = x\, \alpha z + x\, \beta z$

(*iv*) $x\, \alpha\, (y\, \beta z) = (x\, \alpha y)\, \beta z$

for all x, y, z in M and α, β in Γ, then M is called a Γ-ring.

Ideals in Γ-rings

Throughout the matter below M denotes a Γ-ring.

An additive subgroup A of M is said to be a (i) right ideal

If $a\, \alpha m \in M$ for all $a \in A, \alpha \in \Gamma$ and $m \in M$.

(*ii*) left ideal if $m\, \alpha a \in A$ for all $a \in A, \alpha \in \Gamma$ and $m \in M$

(*iii*) ideal if it is both left and right ideal

The smallest ideal containing a is denoted by $\langle a \rangle$. The smallest ideal containing a subset X in M is denoted by $\langle X \rangle$.

As in ref.[5] we give some definitions of γ-ideals in M

Definition 1.2 — Fix $\gamma \in \Gamma$. A subgroup I in M is said to be (*i*) a γ-right ideal If $I_\gamma M \subseteq I$ (*ii*) a γ-left ideal if $M_\gamma I \subseteq I$ (*iii*) a γ-ideal if it is both a γ-right ideal and a γ-left ideal.

Remark : It is clear that, I is an ideal $\Leftrightarrow$ I is γ-ideal for all

$\gamma \in \Gamma$.

We given alternative definition of a IFP ideal P as follows ref.[5]

Definition 1.3 — A (left/right) ideal P is said to be of (*a*) type one-IFP if for $a, b \in M, \gamma \in \Gamma, a\gamma b \in P \Rightarrow a\Gamma M\gamma b \subseteq P$.

(*b*) is type-two IFP.

if for $a, b \in M, \gamma \in \Gamma, a\gamma b \in P \Rightarrow a\gamma M\Gamma b \subseteq P$

Let P in the following be a left ideal of Γ-ring M.

Write $(P : \Gamma J) = \{x \in M : x\Gamma J \subseteq P\}$ and $(P : rJ) = \{x \in M : x r J \subseteq P\}$ where J is a left ideal of M.

Note. It is clear that $(P : \Gamma J) = \bigcap_{r \in \Gamma} (P : rJ)$

Theorem 1.1 — *The following statements are true*

(i) $(P : \Gamma J)$ is a left ideal of M

(ii) $(P : \Gamma J)$ is a γ-ideal of M

(iii) If P has type one or type two IFP property then

$(P : \Gamma J)$ is an ideal in M.

PROOF OF THEOREM 1.1

(*i*) Follows from Thoremm 1.6 (*i*) of [5] and above note

(*ii*) Similar to that fo Theorem 1.6 (*ii*) of[5]

(*iii*) Suppose P is a IFP ideal fo type one. Then the proof is same as in Theorem 1.6 (*v*) Suppose P is a IFP ideal of type two. Then the proof to Theorem 1.6 (*v*).

Section 2

Prime ideals in Γ-rings

We follow the convention of g-system as in[5]

Definition 2.1 — (*i*) A subset X of M is said to be a left generalized g^1-system if either $X = \phi$ or for any a, b in X there exists $\alpha \in \Gamma, b^1 \in g(b)$ such that a $\Gamma M\alpha b^1 \subseteq X$.

(*ii*) Fix $\gamma \in \Gamma$. A subset X of M is said to be a left generalized g^1-γ-system if either $X = \phi$ or for any a, b in X there exists b^1 in $g(b)$ such that a $\Gamma M\gamma b^1 \subseteq X$. In the following we take a left ideal P satisfying following condition :

$$a\gamma b \in P \Rightarrow a\Gamma M\gamma b \subseteq P.$$

Theorem 2.1 — *Let P be a ideal (resp. left ideal). Then the following conditions are equivalent :*

(i) P is left $g^i-\gamma$-prime ideal (resp left $g-\gamma$-prime left ideal)

(ii) a, b are in M such that $\langle a \rangle_i \Gamma M_\gamma b \subseteq P$ implies that $a \in P$ or $b \in P$.

(iii) A is an ideal of M and B is an ideal of M such that $A\,\Gamma M\,\gamma(B+g(0)) \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

(iv) $M \quad P$ is a left generalized g^i-γ-system.

PROOF : Proof of the equivalence of several parts in the above theorem follows by the methods in[5].

Idempotents in Γ-rings.

An element e in a Γ-ring is said to be an idempotent, if there exists $\gamma \in \Gamma$ such that $e\,\gamma e = e$. In this case we also say that e is a γ-idempotet.

We give some theorem on ideals involving idempotents.

***Theorem* 2.2** — *Suppose e in M is an idempotent.*

The following statements are true

(i) If L is a left ideal of M then $L \cap e\,\gamma M = e\,\gamma L$.

(ii) If A is an ideal of M then

$$A \cap e\,\gamma M\,\gamma e = e\,\gamma A\,\gamma e.$$

PROOF : (*i*) If $e\,\gamma r \in L \cap e\,\gamma M$ then

$e\,\gamma r = e\,\gamma(e\,\gamma r) \in e\,\gamma L$, which proves the one way inclusion. The other inclusion is clear.

(*ii*) This part can be proved as in part (*i*).

The following result can be termed as generalized "Peirce Decomposition".

***Theorem* 2.3** — *If e is an idempotent of M then*

$$M = e\,\gamma M\,\gamma e \oplus (1-e)\,\gamma M\,\gamma e \oplus e\,\gamma M\,\gamma(1-e) \oplus (1-e)\,\gamma M\,\gamma(1-e).$$

PROOF : For r in M take $r_1 = e_\gamma r\,\gamma e$, $r_2 = (1-e)_\gamma r\,\gamma e$,

$$r_3 = e\,\gamma r\,\gamma(1-e) \text{ and } r_4 = (1-e)\,\gamma r\,\gamma(1-e)$$

Then it can be verified that $(r_1 + r_2 + r_3 + r_4) = r$.

A lengthy but direct verification shows that the representation is unique.

Section 3 — Commutativity conditions on gamma rings :

In this section we give certain conditions on which the commutativity of gamma rings are dependent.

***Theorem* 3.1** — *A gamma ring is commutative if for every element a in R we have $a\,\gamma a = a$ or $\gamma \in \Gamma$ (fix γ in Γ)*

PROOF : Suppose $(a+b)\,\gamma(a\ \ b) = (a+b)$, then after simplification we get,

$$a\,\gamma b = -\,b\,\gamma a.$$

Which implies $a\,\gamma(a\,\gamma b) = a\,\gamma(-\,b\,\gamma a)$.

Therefore, $a\,\gamma b = -\,a\,\gamma(b\,\gamma a)$. ... (*)

Similarly $a\,\gamma(b\,\gamma a) = -b\,\gamma a.$ ... (**)

From (*) and (**) we get that $a\,\gamma b = b\,\gamma a$.

***Theorem* 3.2** — *For idempotent e which satisfies* $e\,\gamma e + e$ *(*γ *in* Γ *fixed) we have* $e\,\gamma x = x\,\gamma\,e$ *for all x in gamma ring R if there is no nil potent element in R. (an element n in R is nipotent if* $n\,\gamma n = 0$).

PROOF : Let $e\,\gamma e = e$. Now evaluate the expression $(e\,\gamma x\,\gamma e - e\,\gamma x)\,\gamma(e\,\gamma x\,\gamma e - e\,\gamma x)$ after simplification it can be seen that the expression above is zero. Which implies because of non existence of nonzero nilpotent element, $e\,\gamma x\,\gamma e = e\,\gamma x$. Similarly one can prove that,$e\,\gamma x\,\gamma e = x\,\gamma e$. From above the proof follows.

Example of idempotent

If gamma is taken as the ordinary ring R then the binary operation $R \times R \times R \to R$ makes R a gamma ring where the space set gamma is replaced by R. If in R e is an idempotent then e also becomes the idempotent in gamma ring R, because $e\,e\,e = e\,e = e$.

ACKNOWLEDGEMENT

The author is grateful to the referee for his valuable comments on the previous draft of the paper.

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