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MATRIX TRANSFORMATIONS OF SOME VECTOR-VALUED SEQUENCE SPACES

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Necessary and sufficient conditions have been established for an infinite matrix $A = (f_n^k)$ of continuous linear functionals on a Banach space X to transform the vector-valued sequence spaces of Maddox $l_\infty(X, p)$, $l(X, p)$, $c_0(X, p)$, and $c(X, p)$ into the scalar-valued sequence space $c(q)$, where $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers.

Key Words : Matrix Transformations; Maddox Vector-Valued Sequence Spaces

1. INTRODUCTION

The study of matrix transformations of scalar-valued sequence spaces is known since the turn of the century. In seventies, Maddox[12], Gupta[4] studied matrix transformations of continuous linear mappings on vector-valued sequence spaces. Das and Choudhury[1] gave conditions on the matrix $A = (f_n^k)$ of continuous linear mappings from a normed linear space X into a normed linear space Y under which A maps $c_0(X)$ into $c_0(Y)$, $l_1(X)$ into $l_\infty(Y)$, and $l_1(X)$ into $l_p(Y)$. Liu and Wu[22] gave the matrix characterizations from vector-valued sequence spaces $c_0(X, p)$, $l(X, p)$ and $l_\infty(X, p)$ into scalar-valued sequence spaces $c_0(q)$ and $l_\infty(q)$. Suantai[20] gave the matrix characterizations from the Naukano vector-valued sequence $l(X, p)$ into the vector-valued sequence spaces $c_0(Y, q)$, $c(Y)$, and $l_r(Y)$. In this paper, we continue the study of matrix transformations of continuous linear mappings on vector-valued sequences spaces.

The main purpose of this paper is to give the matrix characterizations from $c_0(X, p)$, $l(X, p)$, $l_\infty(X, p)$, and $l(X, p)$ into $c(q)$, where $c_0(X, p)$, $c(x, p)$, $l_\infty(X, p)$, and $l(X, p)$ are the vector-valued sequence spaces of Maddox as defined in Section 2. When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $l_\infty(p)$ and $l(p)$, respectively. Several papers deal with the problem of characterizing those matrices that map a scalar-valued sequence space of Maddox into another such spaces, see[6, 7, 11, 13, 15, 17, 18,19, 21]. Some of these results become particular cases of our theorems. Also some more interesting results are derived.

Section 2 deals with necessary preliminaries and some known results quoted as lemmas which are needed to characterize an infinite matrix $A = (f_n^k)$ such that A maps the vector-valued sequence spaces of Maddox into $c(q)$, and we also give some auxiliary results in Section 3. The main results of the paper is in Section 4.

2. PRELIMINARIES AND LEMMAS

Let $(X, \|\cdot\|)$ be a Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. Let N be the set of all natural numbers, we write $x=(x_k)$ with x_k in X for all $k \in N$. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences and the space of all finite sequences in X, respectively. When $X = K$, the scalar field of X, the corresponding spaces are written as w and ϕ respectively. An X-valued sequence space is a linear subspace of $W(X)$. The sequence spaces of Maddox are defined as

$$c_0(X,p)=\left\{x=(x_k): \lim_{k\to\infty} \|x_k\|^{p_k}=0\right\},$$

$$c(X,p)=\left\{x=(x_k): \lim_{k\to\infty} \|x_k-a\|^{p_k}=0 \text{ for some } a \in X\right\},$$

$$l_\infty(X,p)=\left\{x=(x_k): \sup_k \|x_k\|^{p_k}<\infty\right\},$$

$$l(X,p)=\left\{x=(x_k): \sum_{k=1}^{\infty} \|x_k\|^{p_k}<\infty\right\}.$$

When $X = K$, the scalar field of X, the corresponding spaces are written as $c_0(p), c(p), l_\infty(p)$ and $l(p)$ respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons[16] and Maddox[8, 9]. The space $l(p)$ was first defined by Nakano[14] and it is known as the Nakano sequence space. Also, we need to define the following sequence space :

$$M_0(X,p)=\left\{x=(x_k): \sum_{k=1}^{\infty} \|x_k\| n^{-1/p_k}<\infty \text{ for some } n \in N\right\}.$$

When $X = K$, the scalar field of X, the corresponding space is written as $M_0(p)$. This space was first introduced by Maddox[10]. Grosse-Erdmann[2] has investigated the structure of the spaces $c_0(p), c(p), l(p)$ and $l_\infty(p)$ and he also gave the matrix characterizations between scalar-valued sequence spaces of Maddox in[3]. Let E be an X-valued sequence space. For $x \in E$ and $k \in N$ we write that x_k stand for the kth term of x and for $x \in X$ and $k \in N$, let $e^{(k)}(x)$ be the sequence $(0, 0, 0, \ldots, 0, x, 0, \ldots)$ with x in the kth position and let $e(x)$ be the sequence $(x, x, x, \ldots)$, and we denote by e the sequence $(1, 1, 1 \ldots)$. An X-valued sequence space E is said to be *normal* if $(x_k) \in E$ and $(y_k) \in W(X)$ with $\|y_k\| \le \|x_k\|$ for all $k \in N$ implies that $(y_k) \in E$. For a fixed scalar sequence $u=(u_k)$ the sequence space E_u is defined as

$$E_u=\left\{x=(x_k) \in W(X): (u_k x_k) \in E\right\}.$$

The $\alpha-$, $\beta-$ and $\gamma-$ duals of a scalar-valued sequence space F are defined as

$$F^\zeta=\left\{z \in w: (x_k y_k) \in X_\zeta \text{ for every } y \in F\right\}$$

for $\zeta=\alpha,\beta,\gamma$ and $X_\alpha=l_1, X_\beta=cs$, and $X_\gamma=bs$, where l_1, cs and bs are defined as

$$l_1 = \left\{ x\,(x_k) \in w : \sum_{k=1}^{\infty} |x_k| < \infty \right\},$$

$$cs = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} x_k \text{ converges} \right\},$$

$$bs = \left\{ x = (x_k) \in w : \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}.$$

In the same manner, for an X-valued sequence space E, the $\alpha-, \beta-$ and $\gamma-$ duals of E are defined as

$$E^{\zeta} = \left\{ (f_k) \subset X' : (f_k(x_k)) \in X_{\zeta} \text{ for every } x = (x_k) \in E \right\}$$

for $\zeta = \alpha, \beta, \gamma$ where $X_\alpha = l_1, X_\beta = cs$ and $X_\gamma = bs$.

It is obvious from the definition that $E^{\alpha} \subseteq E^{\beta} \subseteq E^{\gamma}$ and it is easy to see that if E is normal, then $E^{\alpha} = E^{\beta} = E^{\gamma}$.

Let $A = (f_n^k)$ with f_k^n in X', the topological dual of X. Suppose that E is an X-valued sequence space and F a scalar-valued sequence space. Then A is said to map E into F, written by $A : E \to F$ if, for each $x = (x_k) \in E, A_n(x) = \sum_{k=1}^{\infty} f_k^n(x_k)$ converges for each $n \in N$ and the sequence $Ax = (A_n(x)) \in F$. We denote by (E, F) the class of all infinite matrices mapping E into F. If $u = (u_k)$ and $\upsilon = (\upsilon_k)$ are scalar sequences, let

$$_u(E, F)_{\upsilon} = \left\{ A = (f_k^n) : (u_n \upsilon_k f_k^n)_{n,k} \in (E, F) \right\}.$$

If $u_k \neq 0$ for all $k \in N$, we put $u^{-1} = (1/u_k)$. Suppose the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in N$ the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. A K-space that is Frechet (Banach) space is called an $FK-$ $(BK-)$ space.

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X, p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$, as a paranorm on $c_0(X, p)$, and it is known that $c_0(X, p)$ is an FK-space under the paranorm g defined as above. In $l(X, p)$, we consider it as a paranormed sequence space with the paranorm given by $\| (x_k) \| = \left(\sum_{k=1}^{\infty} \| x_k \|^{p_k} \right)^{1/M}$. It is known that $l(X, p)$ is an FK-space under the paranorm defined as above.

Now let us quote some known results as the following.

Lemma 2.1[10] — If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$, then

$$l(p)^{\beta} = \left\{ x \in w : \sum_{k=1}^{\infty} |x_k|^{t_k} M^{-t_k} < \infty \text{ for some } M \in N \right\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Lemma 2.2[16] — If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in N$, then $l(p)^{\beta} = l_{\infty}(p)$.

Lemma 2.3[6] — If $p = (p_k)$ is a bounded sequence of positive real numbers, then

$$l_{\infty}(p)^{\beta} = \left\{ x \in w : \sum_{k=1}^{\infty} |x_k| n^{1/p_k} < \infty \text{ for all } n \in N \right\}.$$

Lemma 2.4[10] — If $p = (p_k)$ is a bounded sequence of positive real numbers, then $c_0(p)^{\beta} = M_0(p)$.

Lemma 2.5[22] — Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X, p) \to c_0$ if and only if

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) $\lim_{m \to \infty} \sup_n \sum_{k=1}^{\infty} \| f_k^n \| m^{-1/p_k} = 0.$

Lemma 2.6[22] — Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : l_{\infty}(X, p) \to c_0$ if and only if

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) for each $M \in N$, $\sum_{j > k} \| f_j^n \| M^{1/p_j} \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Lemma 2.7[22] — Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ and $1/p_k + 1/t_k = 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : l(X, p) \to c_0$ if and only if

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) $\sum_{k=1}^{\infty} \|f_k^n\|^{t_k} m^{-t_k} \to 0$ as $m \to \infty$ uniformly on $n \in N$.

Lemma 2.8[22] — Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$ and let $A = (f_k^n)$ be an infinite matrix. Then $A : l(X, p) \to c_0$ if and only if

(1) $f_k^n \xrightarrow{w^*} 0$ as $n \to \infty$ for each $k \in N$ and

(2) $\sup_{n,k} \|f_k^n\|^{p_k} < \infty$.

3. Some Auxiliary Results

Suppose that E and F are sequence spaces and that we want to characterize the matrix space (E, F). If E and / or F can be derived from simpler sequence spaces in some fashion, then, in many cases, the problem reduces to the characterization of the corresponding simpler matrix spaces. We begin with giving various useful results in this direction.

Proposition 3.1 — Let E and E_n $(n \in N)$ be X-valued sequence spaces, and F and F_n $(n \in N)$ scalar-valued sequence spaces, and let u and υ be scalar sequences with $u_k \neq 0, \upsilon_k \neq 0$ for all $k \in N$. Then

(*i*) $\left(\bigcup_{n=1}^{\infty} E_n, F \right) = \bigcap_{n=1}^{\infty} (E_n, F)$,

(*ii*) $\left(E, \bigcap_{n=1}^{\infty} F_n \right) = \bigcap_{n=1}^{\infty} (E, F_n)$

(*iii*) $(E_1 + E_2, F) = (E_1, F) \bigcap (E_2, F)$,

(*iv*) $(E_u, F_\upsilon) = {}_\upsilon (E, F)_{u^{-1}}$.

PROOF : All of them are obtained directly from the definitions.

Proposition 3.2 — Let $p = (p_k)$ be a bounded sequences of positive real numbers. Then

(*i*) $c(X, p) = c_0(X, p) + \{e(x) : x \in X\}$

(*ii*) $M_0(X, p) = \bigcup_{n=1}^{\infty} l(X)_{(n^{-1/p_k})}$,

(*iii*) $l_\infty(X, p) = \bigcup_{n=1}^{\infty} l_\infty(X)_{(n^{-1/p_k})}$

PROOF : Assertions (*i*) and (*ii*) are immediately obtained from the definitions. To show (*iii*), let $x \in (X, p)$, then there is some $n \in N$ with $\| x_k \|^{p_k} \leq n$ for all $k \in N$. Hence $\| x_k \| n^{-1/p_k} \leq 1$ for all $k \in N$, so that $x \in l_\infty (X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \bigcup_{n=1}^{\infty} l_\infty (X)_{(n^{-1/p_k})}$, then there are some $n \in N$ and $M > 1$ such that $\| x_k \| n^{-1/p_k} \leq M$ for every $k \in N$. Then we have $\| x_k \|^{p_k} \leq n M^{p_k} \leq n M^{\alpha}$ for all $k \in N$, where $\alpha = \sup_k p_k$. Hence $x \in l_\infty (X, p)$. □

The next proposition give a relationship between the $\beta-$ dual of vector-valued and scalar-valued sequence spaces.

Proposition 3.3 — Let X be a Banach space and F a normal scalar-valued sequence space and define $F(X) = \{ (x_k) \in W(X) : (\| x_k \|) \in F \}$. Then for $(f_k) \subset X'$, the topological dual of X, $(f_k) \in F(X)^{\beta}$ if and only if $(\| f_k \|) \in F^{\beta}$.

PROOF : If $(\| f_k \|) \in F^{\beta}$, then for $x = (x_k) \in F(X)$ we have

$$\sum_{k=1}^{\infty} | f_k (x_k) | \leq \sum_{k=1}^{\infty} \| f_k \| \| x_k \| < \infty, \text{ so that } x \in F(X)^{\beta}.$$

Conversely, suppose that $(f_k) \in F(X)^{\beta}$ and $a = (a_k) \in F$. Since F is normal, $(| a_k |) \in F$. For each $k \in N$, we can choose $x_k \in X$ such that $\| x_k \| = 1$ and $| f_k (x_k) | \geq \frac{\| f_k \|}{2}$. Let $y = (a_k x_k)$, then $y \in F(X)$. Choose a sequence (t_k) of scalars such that $| t_k | \leq 1$ and $f_k (t_k a_k x_k) = | f_k (x_k) | \, | a_k |$ for all $k \in N$. Since F is normal $(t_k y_k) \in F(X)$, so we obtain that $\sum_{k=1}^{\infty} f_k (t_k y_k)$ converges. This implies $\sum_{k=1}^{\infty} \| f_k \| \, | a_k | \leq 2 \sum_{k=1}^{\infty} | f_k (x_k) | \, | a_k | < \infty$. It follows that $(\| f_k \|) \in F^{\beta}$.

By using Proposition 3.3, the following results are obtained immediately from Lemmas 2.1-2.4 respectively.

Proposition 3.4 — If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in N$ then

$$l(X, p)^{\beta} = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} \| f_k \|^{t_k} M^{-t_k} < \infty \text{ for some } M \in N \right\}$$

where $1/p_k + 1/t_k = 1$ for all $k \in N$.

Proposition 3.5 — If $p = (p_k)$ is a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in N$, then $l(X, p)^{\beta} = l_\infty (X', p)$.

Proposition 3.6 — If $p=(p_k)$ is a bounded sequence of positive real numbers, then

$$l_\infty(X,p)^\beta=\left\{(f_k)\subset X':\ \sum_{k=1}^{\infty}\|f_k\|\,n^{1/p_k}<\infty \text{ for all } n\in N\right\}.$$

Proposition 3.7 — If $p=(p_k)$ is a bounded sequence of ppsitive real numbers, then $c(X,p)^\beta=M_0(X',p)$.

4. Main Results

We begin with the following useful result.

Theorem 4.1 — *Let $q=(q_k)$ be a bounded sequence of positive real numbers and let E be a normal X-valued sequence space which is an FK-space containing $\Phi(X)$. Then*

$$(E,c(q))=(E,c_0(q))\oplus(E,\langle e\rangle).$$

To prove this theorem, we need the following two lemmas.

Lemma 4.1 — Let E be an X-valued sequence space which is an FK-space containing $\Phi(X)$. Then for each $k\in N$, the mapping $T_k: X\to E$, defined by $T_k x=e^k(x)$, is continuous.

PROOF : For each $k\in N$, we have that $V=\{e^k(x): x\in X\}$ is a closed subspace of E, so it is an FK-space. Since E is a K-space, the coordinate mapping $p_k: V\to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, hence $p_k^{-1}: X\to V$ is continuous. It follows that T_k is continuous because $T_k=p_k^{-1}$. □

Lemma 4.2 — If E and F are scalar-valued sequence spaces such that E is normal containing ϕ, F is an FK-space and there is a subsequence (n_k) with $x_{n_k}\to 0$ as $k\to\infty$ for all $x=(x_n)\in F$, then $(E,F\oplus\langle e\rangle)=(E,F)\oplus(E,\langle e\rangle)$.

PROOF : See [3, Proposition 3.1 (vi)] □

PROOF OF THEOREM 4.1 : Since $c(q)=c_0(q)\oplus\langle e\rangle$, it is clear that $(E,c_0(q))+(E,\langle e\rangle)\subseteq(E,c_0(q)\oplus\langle e\rangle)=(E,c(q))$. Moreover, if $A\in(E,c_0(q))\cap(E,\langle e\rangle)$, then $A\in(E,c_0(q)\cap\langle e\rangle)$, so that $A\in(E,0)$, which implies that $A=0$ because E contain $\Phi(X)$. Hence $(E,c_0(q))+(E,\langle e\rangle)$ is a direct sum. Now, we will show that $(E,c(q))\subseteq(E,c_0(q))\oplus(E,\langle e\rangle)$. Let $A=(f_k^n)\in(E,c(q))=(E,c_0(q)\oplus\langle e\rangle)$. For $x\in X$ and $k\in N$., we have $\left(f_n^n(x)\right)_{n=1}^{\infty}=Ae^k(x)\in c_0(q)\oplus\langle e\rangle$, so that there exist unique $\left(b_k^n(x)\right)_{n=1}^{\infty}\in c_0(q)$ and $\left(c_k^n(x)\right)_{n=1}^{\infty}\in\langle e\rangle$ with

$$\left(f_n^n(x)\right)_{n=1}^{\infty}=\left(b_k^n(x)\right)_{n=1}^{\infty}=\left(c_k^n(x)\right)_{n=1}^{\infty}. \quad \dots (4.1)$$

For each $n, k \in N$, let g_k^n and h_k^n be the functionals on X defined by

$$g_k^n(x) = b_k^n(x) \text{ and } h_k^n(x) = c_k^n(x) \text{ for all } x \in X.$$

Clearly, g_k^n and h_k^n are linear, and by (4.1)

$$f_k^n = g_k^n + h_k^n \text{ for all } n, k \in N. \quad \ldots (4.2)$$

Note that $c_0(q) \oplus \langle e \rangle$ is an FK-space in its direct sum topology. By Zeller's theorem, $A : E \to c_0(q) \oplus \langle e \rangle$ is continuous. For each $k \in N$, let $T_k : X \to E$ be defined by $T_k(x) = e^k(x)$. By Lemma 4.1, we have that T_k is continuous for all $k \in N$,. Since the projection P_1 of $c_0(q) \oplus \langle e \rangle$ onto $c_0(q)$ and the projection P_2 if $c_0(q) \oplus \langle e \rangle$ onto $\langle e \rangle$ are continuous and $g_k^n = p_n \circ P_1 \circ A \circ T_k$ and $h_k^n = p_n \circ P_2 \circ A\, T_k$ for all $n, k \in N$, we obtain that g_k^n and h_k^n are continuous, so $g_k^n, h_k^n \in X'$ for all $n, k \in N$. Let $B = (g_k^n)$ and $C = (h_k^n)$. By (4.1) and (4.2), we have $A = B + C$, $B = (g_k^n) \in (\Phi(X), c_0(q))$ and $C = (h_k^n) \in (\Phi(X), \langle e \rangle)$. We will show that $B \in (E, c_0(q))$ and $C \in (E, \langle e \rangle)$. To do this, let $x = (x_k) \in E$. Then for $\alpha = (\alpha_k) \in l_\infty$, we have $\| \alpha_k x_k \|| = | \alpha_k | \| x_k \| \leq \| M x_k \|$, where $M = \sup_k | \alpha_k |$. Then the normality of E implies that $(\alpha_k x_k) \in E$. Hence $(f_k^n(x_k))_{n,k} \in (l_\infty, c_0(q) \oplus \langle e \rangle)$, moreover, we have $(g_k^n(x_k))_{n,k} \in (\Phi, c_0(q))$, $(h_k^n(x_k))_{n,k} \in (\Phi, \langle e \rangle)$, and $(f_k^n(x_k))_{n,k} = (g_k^n(x_k))_{n,k} + (h_k^n(x_k))_{n,k}$. Since l_∞ is normal containing ϕ and $c_0(q) \subseteq c_0$, it follows from Lemma 4.2 that $(g_k^n(x_k)_{n,k}) \in (l_\infty, c_0(q))$ and $(h_k^n(x_k)_{n,k}) \in (l_\infty, \langle e \rangle)$. This implies that $Bx \in c_0(q)$ and $Cx \in \langle e \rangle$, so we have $B \in (E, c_0(q))$ and $C \in (E, \langle e \rangle)$, hence $A \in (E, c_0(q)) \oplus (E, \langle e \rangle)$. This completes the proof. □

***Theorem* 4.2** — *Let $q = (q_k)$ be a bounded sequence of posiive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : l_\infty(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \| f_k \| < \infty$,

(2) $m^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$ *for every* $k, m \in N$ *and*

(3) for each $m \in N$, $\sum_{j>k} m^{1/q_n} \| f_j^n - f_j \| \to 0$ *as* $k \to \infty$ *uniformly on* $n \in N$.

PROOF : Necessity. Let $A \in (l_\infty(X), c(q))$. It follows from Theorem 4.1 that $A = B + C$, where $B \in (l_\infty(X), c_0(q))$ and $C \in (l_\infty(X), \langle e \rangle)$. Then there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k} \in (l_\infty(X), c_0(q))$, which implies that $(f_k) \in l_\infty(X)^\beta$, so (1) is obtained by Proposition 3.6. Since $c_0(q) = \bigcap_{m=1}^{\infty} c_{0_{(m^{1/q_k})}}$ (by [2, Theorem 0 (i)]), we have by Proposition 3.1 (*ii*) and (*iv*) that for each $m \in N$, $(m^{1/q_n}(f_k^n - f_k)_{n,k}) : l_\infty(X) \to c_0$. Hence, (2) and (3) are obtained by Lemma 2.6.

Sufficiency — Suppose that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that conditions (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = -(f_k)_{n,k}$. It is obvious that $A = B + C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1 (*ii*) and (*iv*) that $B \in (l_\infty(X), c_0(q))$. By Proposition 3.6, condition (1) implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in l_\infty(X)$, which implies that $C \in (l_\infty(X), \langle e \rangle)$. Hence, we obtain by Theorem 4.1 that $A \in (l_\infty(X), c(q))$. This completes the proof. □

Theorem 4.3 — *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : l_\infty(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2) and (3) are satisfied, where*

(1) for each $m \in N$, $\sum_{k=1}^{\infty} \|f_k\| m^{1/p_k} < \infty$,

(2) $r^{1/q_n}(f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $k, r \in N$ and

(3) for each $m, r \in N$, $r^{1/q_n} \sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| \to 0$ as $k \to \infty$ uniformly on $n \in N$.

Moreover, (3) is equivalent to (3'), where

(3') for each $m \in N$, $\lim_{k \to \infty} \sup_n \left(\sum_{j>k} m^{1/p_j} \|f_j^n - f_j\| \right)^{q_n} = 0.$

PROOF : Necessity. Suppose that $A : l_\infty(X, p) \to c(q)$. By Theorem 4.1, $A = B + C$, where $B \in (l_\infty(X, p), c_0(q))$ and$C \in (l_\infty(X, p), \langle e \rangle)$. Then there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k) \in (l_\infty(X, p), c_0(q))$. Since $C = (f_k)_{n,k} : l_\infty(X, p) \to \langle e \rangle$, it implies by Proposition 3.6 that (1) holds. Since $c_0(q) = \bigcap_{m=1}^{\infty} c_{0_{(m^{1/q_k})}}$, we have by Proposition 3.1 (ii) that for each $r \in N$, $(r^{1/q_n}(f_k^n - f_k))_{n,k} : l_\infty(X,p) \to c_0$. Hence, (2) and (3) holds by an application of Lemma 2.6.

Sufficiency — Suppose that there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that condition (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. It is obvious that $A = B + C$. By condition (2) and (3), we obtain by Lemma 2.6 and Proposition 3.1 *(ii)* and *(iv)* that $B \in (l_\infty(X,p), c_0(q))$. By Proposition 3.6, condition (1) implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in l_\infty(X,p)$, which implies that $C \in (l_\infty(X,p), \langle e \rangle)$. Hence, we obtain by Theorem 4.1 that $A \in (l_\infty(X,p), c(q))$.

Now we shall show that (3) and (3') are equivalent. Suppose (3) holds and let $\varepsilon > 0$. Choose $r \in N$ such that $1/r < \varepsilon$. By (3), there exists $k_0 \in N$ such that

$$r^{1/q_n} \sum_{j>k} m^{1/p_j} \| f_j^n - f_j \| < 1 \text{ for all } k \geq k_0 \text{ and all } n \in N,$$

which implies that

$$\sup_n \left(\sum_{j>k} m^{1/p_j} \| f_j^n - f_j \| \right)^{q_n} \leq 1/r < \varepsilon \text{ for } k \geq k_0,$$

hence, (3') holds.

Conversely, assume that (3') holds. Let $m, r \in N$ and $0 < \varepsilon < 1$. Then there exists $k_0 \in N$ such that

$$\sup_n \left(\sum_{j>k} m^{1/p_j} \| f_j^n - f_j \| \right)^{q_n} < \varepsilon^H / r \text{ for all } k \geq k_0$$

where $H = \sup_n q_n$. This implies that

$$r^{1/q_n} \sum_{j>k} m^{1/p_j} \| f_j^n - f_j \| < \varepsilon^{H/q_n} < \varepsilon \text{ for all } k \geq k_0 \text{ and all } n \in N$$

hence, (3) holds. □

***Theorem* 4.4** — *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c_0(X,p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2) and (3) are satisfied, where*

(1) $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for $M \in N$,

(2) $m^{\frac{1}{q_n}} (f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for every $m, k \in N$ and

(3) for each $m \in N$, $\sup_n \left(m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \right) \to 0$ as $r \to \infty$.

Moreover, (3) is equivalent to (3') where

$$\text{(3')} \lim_{r \to \infty} \sup_n \left(\sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{1-p_k} \right)^{q_n} = 0.$$

PROOF : Necessity — Suppose that $A : c_0(X, p) \to c(q)$. By Theorem 4.1, we have $A = B + C$, where $B \in (c_0(X, p), c_0(q))$ and $C \in (c_0(X, p), \langle e \rangle)$. It follows that there is a sequence $(f_k) \subset X'$ such that $C = (f_k)_{n,k}$ and $B = (f_k^n - f_k)_{n,k}$. Since $c_0(q) = \bigcap_{r=1}^{\infty} c_{0_{(r^{1/q_k})}}$, it follows from Proposition 3.1 (ii) and (iv) that for each $m \in N$, $(m^{1/q_n}(f_k^n - f_k))_{n,k} \in (c_0(X, p), c_0)$, hence, conditions (2) and (3) hold by using the result from Lemma 2.5. Since $C = (f_k)_{n,k} \in (c_0(X, p), \langle e \rangle)$, we have that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X, p)$, so that $(f_k) \in c_0(X, p)^{\beta}$, hence, by Proposition 3.7, we obtain that there exists $M \in N$ such that $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ Hence, (1) is obtained.

Sufficiency — Assume that there is a sequence $(f_k) \subset X'$ such that conditions (1), (2) and (3) hold. Let $B = (f_k^n - f_k)_{n,k}$ and $C = (f_k)_{n,k}$. Then $A = B + C$. By conditions (2) and (3), we obtain from Proposition 3.1 (*ii*) and (*iv*) and Lemma 2.5 that $B \in (c_0(X, p), c_0(q))$.The condition (1) implies by Proposition 3.7 that $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X, p)$, so that $C \in (c_0(X, p), \langle e \rangle)$. Hence, by Theorem 4.1, we obtain that $A \in (c_0(X, p), c(q))$.

Now, we shall show that conditions (3) and (3') are equivalent. To do this, suppose that (3) holds and let $\varepsilon > 0$. Choose $m \in N$, $1/m < \varepsilon$. From (3), there is $r_0 \in N$ such that

$$\sup_n m^{1/q_n} \sum_{k=1}^{\infty} \|f_k^n - f_k\| r^{-1/p_k} \le 1 \text{ for all } r \ge r_0.$$

This implies that $\sup_n \left(\sum_{k=1}^{\infty} \| f_k^n - f_k \| r^{-1/p_k} \right)^{q_n} \leq 1/m < \varepsilon$ for all $r \geq r_0$. Hence, (3′) holds.

Conversely, suppose that (3') holds. Let $m \in N$ and $0 < \varepsilon < 1$. Then there exists $r_0 \in N$ such that $\sup_n \left(\sum_{k=1}^{\infty} \| f_k^n - f_k \| r^{-1/p_k} \right)^{q_n} < \varepsilon^H/m$ for all $r \geq r_0$, where $H = \sup_n q_n$. Hence, we have

$$m^{1/q_n} \sum_{k=1}^{\infty} \| f_k^n - f_k \| r^{-1/p_k} < \varepsilon^{H/q_n} \leq \varepsilon \text{ for all } r \geq r_0 \text{ and } n \in N,$$

so that (3) holds. This completes the proof. □

Theorem 4.5 — *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers and $A = (f_k^n)$ an infinite matrix. Then $A : c(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that (1), (2), (3) and (4) are satisfied, where*

(1) $\sum_{k=1}^{\infty} \| f_k \| M^{-1/p_k} < \infty$ *for some* $M \in N$,

(2) for each $m, k \in N$, $m^{1/q_n} (f_k^n - f_k) \xrightarrow{w^*} 0$ *as* $n \to \infty$,

(3) for each $m \in N$, $\sup_n m^{1/q_n} \sum_{k=1}^{\infty} \| f_k^n - f_k \| r^{-1/p_k} \to 0$ *as* $r \to \infty$ *and*

(4) $\left(\sum_{k=1}^{\infty} f_k^n(x) \right)_{n=1}^{\infty} \in c(q)$ for all $x \in X$.

Moreover, (3') is equivalent to (3') where (3')

$$\lim_{r \to \infty} \sup_n \left(\sum_{k=1}^{\infty} \| f_k^n - f_k \| y^{-p/k} \right)^{q_n} = 0.$$

PROOF : Since $c(X, p) = c_0(X, p) + \{ e(x) : x \in X \}$ (Proposition 3.2 (*i*)), it follows from Proposition 3.1 (*iii*) that $A \in (c(X, p), c(q))$ if and only if $A \in (c_0(X, p), c(q))$ and $A \in (\{ e(x) : x \in X \}, c(q))$. By Theorem 4.4, we have $A \in (c(X, p), c(q))$ if and only if conditions (1)-(3) hold and it is clear that $A \in (\{ e(x) : x \in X \}, c(q))$ if and only if (4) holds. We have by Theorem 4.4 that (3) and (3') are equivalent. Hence, the theorem is proved. □

Wu and Liu (Lemma 2.7) have given a characterization of an infinite matrix A such that $A \in (l(X, p), c_0)$ when $p_k > 1$ for all $k \in N$. By applications of Proposition 3.1 (*ii*) and (*iv*), Proposition

3.4, and Theorem 4.1, and using the fact that $c_0(q) = \bigcap_{m=1}^{\infty} c_{0_{(m^{1/q_k})}}$, we obtain the following result.

Theorem 4.6 — *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in N$ and $1/p_k + 1/t_k = 1$ for all $k \in N$, and let $A = (f^n_k)$ be an infinite matrix. Then $A : l(X, p) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that*

(1) $\sum_{k=1}^{\infty} \| f_k \|^{t_k} M^{-t_k} < \infty$ for some $M \in N$,

(2) $m^{1/q_n} (f^n_k - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $m, k \in N$ and

(3) for each $m \in N$, $\sum_{k=1}^{\infty} m^{t_k/q_n} \| f^n_k - f_k \|^{t_k} r^{-t_k} \to 0$ for all $m \in N$.

By using Lemma 2.8, Proposition 3.1 (*ii*) and (*iv*), Proposition 3.5 and Theorem 4.1, we also obtain the following result:

***Theorem* 4.7** — *Let $P = (p_k)$ and $g = (g_k)$ be bounded sequences of positive real numbrs with $p_k \leq 1$ for all $k \in N$ such that*

(*i*) $\sup_k \| f_k \|^{p_k}_n \subset \infty$

(*ii*) $m^{1/q_n} (f_k - f_k) \to 0$ as $n \to \infty$ for all $m, k \in N$ and

(*iii*) $\sup_{n,k} m^{p_k | q_n} \| f^n_k - f_k \|^{p_k} < \infty$ for all $n \in N$

When $p_k = 1$ for all $k \in N$, we obtain the following.

Corollary 4.8 — Let $q = (q_k)$ be a bounded sequence of positive real numbers and let $A = (f^n_k)$ be an infinite matrix. Then $A : l_1(X) \to c(q)$ if and only if there is a sequence (f_k) with $f_k \in X'$ for all $k \in N$ such that

(1) $\sup_k \| f_k \| < \infty$,

(2) $m^{1/q_n} (f^n_k - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all m, $k \in N$ and

(3) $\sup_{n,k} m^{1/q_n} \| f^2_k - f_k \| < \infty$ for every $m \in N$.

Theorem 4.9 — *Let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (f^n_k)$ an infinite matrix. Then $A : M_0(X, p) \to c(q)$ if and only if there is a sequence (f_k) of bounded linear functionals on X* such that

(1) $\sup_k m^{1/p_k} \| f_k \| < \infty$ for all $m \in N$,

(2) for each $m, r \in N$, $r^{1/q_n} m^{1/p_k} (f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $k \in N$ and

(3) for each $m, r \in N$, $\sup_{n,k} r^{1/q_n} m^{1/p_k} \| f_k^n - f_k \| < \infty$.

PROOF : It follows from Theorem 4.1 that $A \in (M_0(X, p), c_0(q) \oplus \langle e \rangle)$ if and only if there is a sequence (f_k) of bounded linear functionals on X such that $A = B + (f_k)_{n,k}$ where $B : M_0(X, p) \to c_0(q)$ and $(f_k)_{n,k} : M_0(X, p) \to \langle e \rangle$. Since $B = (f_k^n - f_k)_{n,k}$ and $M_0(X, p) = \bigcup_{m=1}^{\infty} l_1 (X)_{(m^{-1/p_k})}$ (by Proposition 3.2 (*ii*)), we have by Proposition 3.1 (*i*) and (*iv*) that $B : M_0(X, p) \to c_0(q)$ if and only if $(m^{1/p_k} (f_k^n - f_k))_{n,k} : l_1(X) \to c_0(q)$ for all $m \in N$. Since $c_0(q) = \bigcap_{r=1}^{\infty} c_{0_{(r^{1/q_k})}}$, by Proposition 3.1 (*ii*) and (*iv*), we have $(m^{1/p_k} (f_k^n - f_k))_{n,k} : l_1(X) \to c_0(q)$ if and only if $(r^{1/q_n} m^{1/p_k} (f_k^n - f_k))_{n,k} : l_1(X) \to c_0$ for all $r \in N$. By Lemma 2.8, we have

$(r^{1/q_n} m^{1/p_k} (f_k^n - f_k))_{n,k} : l_1(X) \to c_0$ if and only if

(*a*) $r^{1/q_n} m^{1/p_k} (f_k^n - f_k) \xrightarrow{w^*} 0$ as $n \to \infty$ for all $k \in N$ and

(*b*) $\sup_{n,k} r^{1/q_n} m^{1/p_k} \| f_k^n - f_k \| < \infty$.

By Proposition 3.1 (*i*) and (*iv*),we have $(f_k)_{n,k} : M_0(X, p) \to \langle e \rangle$ if and only if $(m^{1/p_k} f_k)_{n,k} : l_1(X) \to \langle e \rangle$ for all $m \in N$. By Proposition 3.5, we obtain that $(m^{1/p_k} f_k)_{n,k} : l_1(X) \to \langle e \rangle$ if and only if $\sup_k m^{1/p_k} \| f_k \| < \infty$. Hence, the theorem is proved. □

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