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A NOTE ON TWO RECENT PAPERS ON APPROXIMATION OF FIXED POINTS

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Recently, Sharma and Sahu (*Indian J. pure Appl. Math.* **31** (2000), 185-96) claimed to have improved a theorem of Schu (*J. Math. Anal. Appl.* **158** (1991), 407-13) from Hilbert spaces to Banach spaces satisfying Opial's condition. These spaces include l_p spaces, $1 < p < \infty$, but exclude L_p $(1 < p < \infty, p \neq 2)$. In a subsequent paper, the two authors, in collaboration with Bounias claimed to have extended this result to Banach spaces with property $(U, \lambda, m+1, m)$, $\lambda \in I\!R$, $m \in I\!N$. These spaces include the L_p spaces, $p \geq 2$. It is shown in this note that these claims are false. The proofs of all the results in these two papers of Sharma *et al.* are valid in Hilbert spaces. The validity of the theorems in L_p (or l_p), $p > 2$, has not been proved.

Key Words : Accretive Operators; Asymptotically Pseudocontractive Maps

Let K be nonempty subset of a real Banach space E. A mapping $T : K \to K$ is said to be *asymptotically hemicontractive*[1] if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim k_n = 1$ such that for each $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y) \rangle \leq k_n \| x - y \|^2 \quad \forall n \geq 1,$$

where $J : X \to 2^{E^*}$ denotes the normalized duality map on E. A mapping $T : K \to K$ is called *asymptotically hemicontractive* if $F(T) := \{x \in K : Tx = x\} \neq 0$ and there exists a sequence $\{k_n\}$ of positive numbers with $\lim_{n \to \infty} k_n = 1$ such that for each $x \in K$ and $x^* \in F(T)$, there exists $j(x - x^*) \in J(x - x^*)$ such that

$$\langle T^n x - x^*, j(x - x^*) \rangle \leq k_n \| x - x^* \|^2 \quad \forall n \geq 1,$$

The class of asymptotically pseudocontractive maps contains the class of asymptotically nonexpansive maps introduced by Göebel and Kirk[2], i.e. the class of mapping $T : K \to K$ such that for each $x, y \in K$ there exists a sequence $\{k_n\}$ of positive numbers $k_n \geq 1$ with $\lim_{k_n} = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|^2.$$

Fixed point theory for these classes of nonlinear maps are intimately connected with the mapping theory of the important classes of *accretive* operators. For this connection, the reader may consult[3-6].

In[1], Schu proved the following theorem

Theorem JS[1] — *Let H be a Hilbert space. $0 \neq K \subseteq H$ closed, convex and bounded; $T: K \to K$ be completely continuous uniformly L-Lipschitzian for some $L > 0$ and asymptotically pseudocontractive with sequence $\{k_n\} \subseteq [1, \infty)$; $q_n = 2k_n - 1$ for all integers $n \geq 1$; $\Sigma\,(q_n^2 - 1) < \infty$; $\{\alpha_n\}$, $\{\beta_n\} \subseteq [0, 1]$; $\varepsilon \leq \alpha_n \leq \beta_n \leq b$ for all integers $n \geq 1$, some $\varepsilon > 0$ and some $b \in (0, L^{-2}\left\{(1 + L^2)^{\frac{1}{2}} - 1\right\}$; $x_1 \in K$. For all integers $n \geq 1$, define the sequence $\{x_n\}$ by $x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n z_n$ and $z_n = (1 - \beta_n)\, x_n + \beta_n T^n x_n$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Sharma, Sahu and Bounias weakened the condition of complete continuity in theorem JS and claimed to have proved the following weak convergence theorem.

Theorem SSB **([7], Theorem 4.1)** — *Let $(E, \|\cdot\|)$ be a Banach space with property $(U, \lambda, m+1, m)$, $\lambda \in \mathcal{R}$, $m \in I\!N$ and a uniformly Gâteaux differentiable norm; $J^{-1}: E^* \to E$ weakly sequentially continuous at zero, $0 \neq K \subseteq E$ be closed, convex and bounded; $T: K \to K$ be uniformly* L-Lipschitzian for some $L > 0$ and asymptotically hemicontractive with sequence $\{k_n\}$ and $L < N(E)^{\frac{1}{2}}$. Suppose in addition, T satisfies the condition :

$$\| x - T^n y \|^2 \leq \langle x - T^n y, j(x - y) \rangle, \ \forall\, x, y \in K, n \in N \cup \{0\}.$$

Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $0 < a \leq \alpha_n \leq \alpha < 1$ and $0 < b \leq \beta_n \leq \beta < 1 \ \forall\, n \in \mathcal{N}$;

(ii) $\sum_{n=1}^{\infty} (v_n c - m) < \infty$, where $v_n := (m+1)\, k_n - m$ and $c = \left(\dfrac{\lambda}{2^m - 1}\right)$;

(iii) $(1 - 2\beta^m c - \beta^{m+1} L^{m+1} c)\, c + 1 - \beta^m c - c^2 > 0$

and $1 - \alpha^m c - (1 - mb)\, c^2 > 0$. For arbitrary $x_1 \in K$ define the sequence $\{x_n\}$ by $x_{n+1} := (1 - \alpha_n)\, x_n + \alpha_n T^n z_n$ and $z_n := (1 - \beta_n)\, x_n + \beta_n T^n x_n$ for all integers $n \geq 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.

Remark : It is well known (Sharma and Sahu have also explicitly stated this, see e.g. [8], p. 187) that l_p and L_p spaces, $2 \leq p < \infty$, have property $(U, p-1, 2, 1)$. This implies that for these spaces, $\lambda = p - 1, m = 1$, We examine condition (ii) in Theorem SSB, i.e. $\sum_{n=1}^{\infty} (v_n c - m) < \infty$, where

$v_n := (m+1)\,k_n - m$ and $c := \left(\dfrac{\lambda}{2^m - 1}\right)$. Clearly $v_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty} (v_n\, c - m) < \infty$ implies $\lim_{n \to \infty} (v_n\, c - m) = 0$. So, $c = m$. This implies for L_p (or l_p) spaces, $2 \le p < \infty$, that $c = m = 1$. From $c = \dfrac{\lambda}{2^m - 1}$ we obtain that $c = \lambda = 1$. But $\lambda = p - 1$,and so $p - 1 = 1$, i.e., $p = 2$. This reduces Theorem SSB to the L_2 (or l_2), Hilbert space, situation.

This condition *(ii)* is a hypothesis in *all* the theorems in the two papers under review and is central in the argument of the proofs. Thus the proofs of all the results in the two papers are true for Hilbert spaces. The validity of the proofs in L_p (or l_p) spaces, $1 < p < \infty, p \neq 2$ has not been proved as claimed by the authors. Furthermore, in this Hilbert space case, the situation is even worse. If one takes $x = y$ and $n = 1$ in the displayed condition of Theorem SSB, then it is immediate that the condition implies $Tx = x$ for all $x \in K$. This trivializes the result even in Hilbet spaces.

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