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ALGEBRAIC ELEMENTS IN VALUED *-DIVISION RINGS

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The algebraic structure of a division ring D with involution is investigated. A valuation theoretical criterion is given to describe the algebraic elements of D.

Key Words : Valuations; Algebraic Elements; Division of Rings with Involution

1. INTRODUCTION

Rings with involution have been studied intensively, especially in some applications to Lie algebras, Jordan algebras, and rings of operators. More recently, the category of rings with involution has been taken under investigation (see[1]). In this paper, we investigate the algebraic structure of a *-division ring D, that is a division ring with involution * (an anti-automorphism of order 2). A main tool in this process is valuation theory. To define valuations on D we need our valuations to also be compatible with the involution. We define a *-valuation on D to be a valuation ω onto a linearly ordered group with the additional property that $\omega(x^*) = \omega(x)$ for all non-zero x in D.

To investigate algebraic elements in D, i.e., elements which are algebraic over the center of D, we need valuations with an additional restriction. We refer to these valuations as invariant *-valuations. Such invariant *-valuations exist. For example, for every ordering of a *-division ring we can associate a natural valuation which is invariant. Also, it is shown in[2], that the division ring generated by the universal envelope of a Lie algebra, has an invariant valuation.

In Section (3) of this paper, valuation theoretically criterions are given to describe the algebraic elements of a *-division ring. For example, a sufficient conditions is given to ensure that a symmetric element of D which is algebraic must be in the center of D. Also, sufficient conditions are given to give information about the ramification index and the residue degree of the given *-valuation. In the last section of this paper we give some examples of *-division rings with invariant *-valuations to illustrate the theory.

2. DEFINITIONS AND BASIC FACTS

Let D be a *-division ring with center Z and let $D^{\bullet}$ be the multiplicative group of non-zero elements of D. We begin with recalling the definition of a *-valuation on D. By a *-valuation we mean an onto mapping $\omega : D \to \Gamma$, where Γ is a linearly ordered additive group with positive infinitely adjoined, such that for all x, y in D,

(*i*) $\omega(x) = \infty$ if and only if $x = 0$,

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(*ii*) $\omega(x+y) \geq \min(\omega(x)+\omega(y))$,

(*iii*) $\omega(xy)=\omega(x)+\omega(y)$, and

(*iv*) $\omega(x^*)=\omega(x)$, for all $x \neq 0$.

It then follows that Γ is abelian, since $\omega(x)+\omega(y)=\omega(xy)=\omega(y^* x^*)=\omega(y)+\omega(x)$. Γ is called value group of ω and often we write Γ_ω instead of Γ if necessary. Each *-valuation ω of D defines a *-closed subring of D namely

$$R_\omega=\{x \in D \mid \omega(x) \geq 0\}.$$

Indeed, R_ω is a valuation subring of D (i.e., it contains x or x^{-1} and $x^* x^{-1}$ for all $x \in D^{\bullet}$). Also, R_ω is invariant under conjugation. Furthermore,

$$M_\omega=\{x \in D \mid \omega(x)>0\},$$

is the unique maximal *-closed ideal of R_ω, formed by the non-units in R_ω. So, $\overline{D}:=R_\omega/M_\omega$ is a division ring with involution induced by the involution of D. We shall write the induced involution on $\overline{D}$ as * also $((x+M_\omega^*)=x^*+M_\omega$ for $x \in R_\omega)$. R_ω is called the *-valuation ring of ω and $\overline{D}$ the residue *-division ring of ω. If there is no confusion, we will use R and M instead of R_ω and M_ω. Since M is invariant under conjugation when $1+M$ is also invariant under conjugation so that $1+M$ is a normal subgroup of $D^{\bullet}$. Hence, we have the quotient group $D^{\bullet}/1+M$ which enlarges the *-division ring $\overline{D}$, and carries the residue involution $(x(1+M))^*=x^*(1+M)$, for $x \in D^{\bullet}$.

Let ω be a *-valuation on D. Let υ be the restriction of ω to the field Z. Clearly υ is a valuation of Z with value group $\Gamma_\upsilon=\{\omega(z) \mid z \in Z\}$, and valuation ring $Z \cap R_\omega$. Then Γ_υ is a subgroup of Γ_ω and the order of the quotient group $\Gamma_\omega/\Gamma_\upsilon$, if finite, is called the ramification index of ω (over Z) and is denoted by e (or e_ω). Let $\overline{Z}(Z \cap R)/M$. Then $\overline{Z} \subseteq Z(\overline{D})$, the center of the residue *-division ring $\overline{D}$. As a vector space over $\overline{Z}$, the division algebra $\overline{D}$ has a dimension, which is called the residue degree of ω over Z, and is denoted by f (or f_ω). By [3, Theorem 3], if D is finite dimensional over its center, then

$$[D : Z] = e.f.d$$

where $d=1$ for char $(\overline{D})=0$, and $d=p^k$ for char $(\overline{D})=p$ and k is a non-negative integer.

As we mentioned above, the valuation subring R of a *-valuation ω of D is invariant under conjugation, i.e., $aRa^{-1}=R$ for all non-zero a in D. This implies $aba^{-1} \in R$ for all non-zero a in D and all b in R. But generally we do not have $aba^{-1} \equiv b \pmod{M}$, i.e., $aba^{-1}-b \in M$.

Definition — A *-valuation ω of D is called invariant if $sbs^{-1} \equiv b \pmod M$ for all non-zero $s = s^*, b \in D$ and, s or b in R. ω is said to be strongly invariant if $sds^{-1} \equiv d \pmod{1 + M}$, i.e., $sds^{-1}d^{-1} \in 1 + M$, for all non-zero $s = s^*$, $d = d^*$ in D.

If * = identity, then D is commutative and every element is symmetric so that every valuation is strongly invariant. If D is an ordered *-division ring then there is a natural *-valuation associated to the ordering, which is called the order valuation. The valuation subring of the order valuation consisting of elements of D, which are bounded by some rational number with respect to the ordering. It is known (see[4]) that the symmetric elements of D commute modulo 1 + M, this means, that the order *-valuation is strongly invariant.

The following theorem gives another characterization of a strongly invariant *-valuation in terms of one arbitrary element in D.

Theorem 1 — *Let ω be a strongly invariant *-valuation on a *-division ring D with char $D \neq 2$. Then $sbs^{-1} \equiv b \pmod{1 + M}$ for all non-zero $s = s^*$, b in D.*

PROOF : Let $s = s^*, b \in D^\bullet$. Since b can be written as a sum of a symmetric element $\frac{1}{2}(b + b^*)$ and a skew symmetric element $\frac{1}{2}(b - b^*)$, we have

$$sbs^{-1} = s\left(\frac{1}{2}(b + b^*) + \frac{1}{2}(b - b^*)\right)s^{-1}$$

$$= \frac{1}{2}s(b + b^*)s^{-1} + \frac{1}{2}s(b - b^*)s^{-1}$$

$$\equiv \frac{1}{2}(b + b^*) + \frac{1}{2}s(b - b^*)s^{-1}.$$

If $sxs^{-1} \equiv x$ for skew symmetric elements x, then we would have

$$sbs^{-1} \equiv \frac{1}{2}(b + b^*) + \frac{1}{2}(b - b^*) = b,$$

as required. So, it suffices to prove the theorem for the case b a skew symmetric element.

Let $a = sbs^{-1}b^{-1}$ for some skew symmetric element b. We will show that $a \equiv 1$. Since $b^2 = -bb^*$ is symmetric, then

$$abab^{-1} = sb^2s^{-1}b^{-2}$$

$$\equiv b^2b^{-2} = 1,$$

which implies $b^{-1}a^{-1}b \equiv a$. From this,

$$sa^*s^{-1} = sb^{-1}s^{-1}b = b^{-1}a^{-1}b \equiv a,$$

so that $a^* \equiv s^{-1}as$ and $(a^*)^{-1} \equiv s^{-1}a^{-1}s$. We note that the element $(1 - a^{-1})\,sb$ is a symmetric element, because $(1 - a^{-1})\,sb = sb - bs = sb + (sb)^*$. Thus

$$s^{-1}((1-a^{-1})sb)s \equiv (1-a^{-1})sb$$

$$s^{-1}(1-a^{-1})s\cdot(bsb^{-1}s^{-1}) \equiv 1-a^{-1}$$

$$s^{-1}(1-a^{-1})s\cdot a^{-1} \equiv 1-a^{-1}$$

$$s^{-1}(1-a^{-1})s \equiv a-1$$

$$1-s^{-1}a^{-1}s \equiv a-1$$

$$1-(a^{*})^{-1} \equiv a-1$$

$$a^{*}-1 \equiv a^{*}(a-1),$$

i.e., $2a^{*} \equiv a^{*}a+1$. Since $a^{*}a+1$ is a symmetric element, this means that a^{*} is a symmetric element modulo $1+M$. Hence $a \equiv a^{*}$. Using this and $a^{*}-1 \equiv a^{*}(a-1)$, we get $(a-1) \equiv a(a-1)$, i.e., $a \equiv 1$.

When x or y is a unit in the *-valuation ring R, then evidently $x \equiv y \pmod{1+M}$ $\Leftrightarrow x \equiv y \pmod{M}$. Then, by Theorem (1), every strongly invariant *-valuation is invariant. We note that the converse is not true, as we shall give an example of an invariant *-valuation which is not strongly invariant.

Let a be a non-zero element of D. Let $i_a : D \to D$ be the inner automorphism $i_a(x) = ax^{-1}a$. Since R is invariant, i_a induces an automorphism $\bar{i}_a : \bar{D} \to \bar{D}$, defined by $\bar{i}_a(\bar{x}) = \overline{axa^{-1}}$. We note that, by definition, if a *-valuation ω is invariant then $\bar{i}_s$ is the identity on $\bar{D}$ for each non-zero $s = s^{*}$ in D. By Theorem (1), ω is strongly invariant if and only if $\bar{i}_s$ is the identity on the quotient group $D^{\bullet}/1+M$ for every non-zero $s = s^{*}$ in D.

Lemma 2 — Let ω be an invariant *-valuation on a *-division ring D. Then $\bar{D}$ is a field, or $\bar{D}$ is 4-dimensional over its center.

PROOF : Let $s = s^{*} \in R$, then for all non-zero $b \in D, bsb^{-1} \equiv s \pmod{M}$. So, s maps into the center $Z(\bar{D})$ of $\bar{D}$, for all $s = s^{*}$ in R. By a theorem of Dieudonné[5] follows that either $\bar{D}$ is a field or $\bar{D}$ is 4-dimensional with the unique involution that has a fixed set precisely the subfield $Z(\bar{D})$.

Lemma 3 — Let ω be an invariant *-valuation on a *-division ring D, with char $(D) \neq 2$, then

$$bxb^{-1} \equiv x \text{ or } x^{*}$$

for all $x \in R, b \in D^{\bullet}$.

PROOF : Every element $x \in R$ can be written as $x = s + k$, where $s^{*} = s, k^{*} = -k \in R$. Also, $k^{2} = -kk^{*}$ is a symmetric element in R, then

$$bk^{2}b^{-1} \equiv k^{2}$$

$$(bkb^{-1})^2 \equiv k^2 = k^{*^2}$$

So, $\quad bkb^{-1} \equiv k$ or k^*, Now,

$$bxb^{-1} = bsb^{-1} + bkb^{-1}$$

$$\equiv s + k \text{ or } s + k^*$$

$$\equiv x \text{ or } x^*.$$

3. ALGEBRAICITY

We study in this section algebraic structure of a *-division ring D with invariant *-valuation. We note first that, if a non-zero element x is algebraic over Z, then there exist a natural number n, u a unit in R and $z \in Z$ such that $x^n = zu$. For, by a result of Wedderburn, if x is algebraic over Z with degree n, there are n conjugates $x_1, x_2, ..., x_n$ to x in D such that th eminimal polynomial p_x of x over Z can be factorized as

$$p_x = (t - x_1) \ldots (t - x_n),$$

where the factorization is in $D[t]$, the polynomial ring over D with centeral indeterminate t. Now, $x_i = y_i x y_i^{-1}$ for some $y_i \in D^\bullet$, so that $\omega(x_i) = \omega(y_i) + \omega(x) - \omega(y_i) = \omega(x)$. Let $z = x_1 x_2 \ldots x_n \in Z$, then

$$\omega(z) = \omega(x_1 \ldots x_n) = \omega(x_1) + \ldots + \omega(x_n) = n\,\omega(x) = \omega(x^n).$$

So, $x^n = zu$ for some unit $u \in R$ as required. If $x = x^*$, p_x has all its coefficients central symmetrics, in particular $z = z^*$. Hence, the unit element u is symmetric.

Proposition 4 — Let ω be an invariant *-valuation on a *-division ring D. Let $s = s^*$ be an algebraic unit in R, then $s \equiv z$ for some unit $z = z^* \in R \cap Z$.

PROOF : If s is algebraic over Z with degree n, then there are n conjugates $x_1, x_2, \ldots, x_n$ to s in D such that the minimal polynomial p_s of s over Z can be written as

$$p_s = (t - x_1) \ldots (t - x_n).$$

Hence, $x_1 + x_2 + \ldots = x_n = z$, where $-z$ is the coefficient of t^{n-1} in p_s. Since s is a symmetric unit in R, then the minimal polynomial of s has all its coefficients central symmetrics. Now, $s \equiv x_i$ for every $i = 1, \ldots, n$. Then, $ns \equiv x_1 + \ldots + x_n = z$, so that $s \equiv \frac{1}{n} z$, and $\frac{1}{n} z$ is as required.

Lemma 5 — If a is algebraic over $Z \cap R$, the center of the *-valuation ring R, p_a is the minimal polynomial of a over $Z \cap R$, and g_a is the minimal polynomial of a over Z, then g_a divides p_a over $Z \cap R$.

PROOF : We first claim that a is a unit in R. Since a is algebraic over $Z \cap R$, it follows that a is a polynomial expression of a^{-1} over $Z \cap R \subseteq R$. Similarly, a^{-1} is a polynomial expression of a over $Z \cap R$. But R is a *-valuation ring, i.e. a or a^{-1} belongs to R, so it follows that both a and a^{-1} belong to R. By a result of Wedderburn, if a is algebraic over X with degree r, then there are r conjugates $a_1, a_2, \ldots, a_r$ to a in D such that $g_a(t) = (t - a_1) \ldots (t - a_r)$. Since a is a unit in R and R is closed under conjugation, it follows $a_1, a_2 \ldots, a_r \in R$, so that $g_a(t) \in (Z \cap R)[t]$. Since g_a is the minimal polynomial of a over Z, we have $p_a = hg_a$ for some $h \in Z[t]$. Because a is a unit of R, one can show that $h \in (Z \cap R)[t]$.

Theorem 6 — *Let ω be an invariant *-valuation on a *-division ring D with char* $(\overline{D}) = 0$.

(1) If a is algebraic over $Z \cap R$ then a is a unit in R of degree 1 or 2 over Z.

(2) If $a = a^*$ is algebraic over $Z \cap R$, then $a \in Z$.

(3) If $a = a^*$ is algebraic over Z, and ω is strongly invariant, then $a \in Z$.

PROOF : (1) Assume a of degree $n > 2$ over Z. By Wedderburn's result, the minimal polynomial g_a of a over Z can be written in the form

$$g_a = (t - x_1 a x_1^{-1}) \ldots (t - x_n a x_n^{-1})$$

Since a is a unit in R, and ω is invariant, we have $x_i a x_i^{-1} \equiv a \pmod{M}$ or $x_i a x_i^{-1} \equiv a^*$ (mod M) (by Lemma (3)). As ω is invariant, the symmetric elements of $(\overline{D})$ are central and so $\overline{a}$ and $\overline{a}^*$ commute. Let $\overline{Z}(\overline{a}, \overline{a}^*)$ be the subfield of $\overline{D}$ generated by $\overline{Z}$ and $\overline{a} = a + M, \overline{a}^* = a^* + M$. The polynomial g_a maps into the polynomial $\overline{g}_a \in \overline{D}[t]$ of degree n, where

$$\overline{g}_a = (t - \overline{a}_1) \ldots (t - \overline{a}_n),$$

and $\overline{a}_i = \overline{a}$ or $\overline{a}_i = \overline{a}^*$, $i = 1, \ldots, n$. Since $n > 2$, $\overline{g}_a$ has a multiple root in $\overline{Z}(\overline{a}, \overline{a}^*)$. If p_a is the minimal polynomial of a over $Z \cap R$, then by Lemma (5), g_a divides p_a and hence $\overline{g}_a$ is a factor of $\overline{p}_a$. By the above the minimal polynomial $\overline{p}_a$ of $\overline{a}$ has a multiple root. But in char $\overline{D}$ = 0, it is well known that the minimal polynomial has no multiple roots, which contradicts the assumption $n > 2$.

(2) Now, if we take $a = a^*$, then the above arguments can be repeated to show that $\overline{g}_a = (t - \overline{a})^n$. So, the minimal polynomial $\overline{p}_a$ of $\overline{a}$ over $\overline{Z}$ is $\overline{p}_a = t - \overline{a}$, and hence $\overline{a} \in \overline{Z}$, i.e., $a \in Z$.

(3) If we take $a = a^*$ is algebraic over Z, then a need not be a unit in R, so we need ω to be strongly invariant to do the above arguments.

If D is an ordered *-division ring, $\overline{D}$ has an induced ordering and so char $\overline{D} = 0$. Hene, by Theorem 6, all algebraic symmetric elements of D are central.

Denote by A the subgroup of $D^{\bullet}$ generated by the set

$$\{ada^{-1}d^{-1} \mid a, d = d^* \in D^{\bullet} \text{ and, } d \text{ algebraic over } Z\}.$$

Let ω be a *-valuation of D. Then, denote by $\overline{A}$ the subgroup of the multiplicative group $\overline{D}^{\bullet}$ of $\overline{D}$ generated by the set $\{\overline{x} \mid x \in A\}$. Since the value group of ω is abelian, then $\omega(ad) = \omega(da)$ so that $\omega(ada^{-1}d^{-1}) = 0$ for all $a, d \in D^{\bullet}$. Thus ω maps the subgroup A into 0.

Proposition 7 — Let ω be an invariant *-valuation of D. Then each element $\overline{x}$, where x is a generator of A, has finite order.

PROOF : We show that each $\overline{ada^{-1}d^{-1}}$ has finite order in $\overline{D}$ where $d = d^*$ is algebraic over Z. Then there exist a natural number n, u a unit in R and $z \in Z$ such that $d^n = zu$. Put $c = ada^{-1}d^{-1}$ and then

$$ad^n a^{-1} = (ada^{-1})^n = (cd)^n = c\,(dcd^{-1})\,(d^2cd^{-2})\ldots(d^n cd^{-n+1})\,d^n.$$

Replacing d^n by zu and canceling z we obtain

$$aua^{-1}c\,(dcd^{-1})\,(d^2cd^{-2})\ldots(d^{n-1}cd^{-n+1})\,u.$$

Since $d, d^2, \ldots, d^{n-1}$ are all symmetric elements and $c \in R$ (as $\omega(c) = 0$), then when we pass to $\overline{D}$ we get $\overline{u} = \overline{c}^n\,\overline{u}$ (as ω is invariant). Then $\overline{c}^n = \overline{1}$ as required.

We note that if $\overline{D}$ is commutative, then by Proposition (7), each element of $\overline{A}$ has finite order. important case of an invariant *-valuation is the one for which the subgroup $\overline{A}$ is trivial. For example, if ω is the order *-valuation corresponding to a strong ordering of a *-division ring then the subgroup $\overline{A}$ is trivial. For, every element $ada^{-1}d^{-1}$ is positive for every non-zero a, $d = d^*$ in D, and so $\overline{ada^{-1}d^{-1}}$ is positive in the induced ordering of $\overline{D}$ (see [4]). If in addition d is algebraic then $\overline{ada^{-1}d^{-1}}$ is a positive element of finite order in $\overline{D}$ (Proposition 7)), i.e., $\overline{ada^{-1}d^{-1}} = \overline{1}$. Then $\overline{A}$ is trivial. Algebraic elements in division rings with invariant valuation for which $\overline{A}$ is trivial, has been studied in[6]. For the case of a *-division ring we have the following theorem.

Theorem 8 — *Let D be a *-division ring and let ω be an invariant *-valuation of D such that $\overline{A}$ is trivial. If char $(\overline{D}) = p > 0$, then $[Z(s) : Z]$ is a p-power for each non-zero element $s = s^*$ of D algebraic over Z.*

PROOF : Let $s = s^*$ be an algebraic element of D. As mentioned above, $[Z(s) : Z] = e \cdot f \cdot p^k$. If Char $\overline{Z} = 0$ then $e = f = 1$ has to be shown and if Char $\overline{Z} = p > 0$ then e and f have to be a p-power. One can adopt the remaining proof of Theorem 3.2 in [6], to prove that.

One can also adopt the proof of Theorem (3.3) in [6], to give the following theorem, which gives some information about the algebraic symmetric elements of D even if $\bar{A}$ is not trivial.

Theorem 9 — *Let D be a *-division ring and let* ω *be an invariant *-valuation of D and* $s = s^*$ *an algebraic element of D with* $n = [Z(s) : Z]$. *If p is a prime divisor of n then char* $\bar{Z} = p$ *or* $\bar{A}$ *has an element of order p.*

Let D be a *-division ring with a *-valuation ω such that $\bar{D} \cong \boldsymbol{R}$, the field of all real numbers. Since $\boldsymbol{R}$ has a trivial automorphism group, then ω is invariant. Now by Theorem (9), $\bar{A}$ has an element of order 2, i.e., $\bar{D}$ has a primitive square root of unity. Thus $[Z(s) : Z]$ is a 2-power for each algebraic symmetric element s of D. This shows that in general not all algebraic symmetric elements are central.

It is known from Lemma (2), that $\bar{D}$ is commutative or $\bar{D}$ is 4-dimensional over its center. If further D is algebraic over its center we have

Theorem 10 — *Let D be a *-division ring which is algebraic cover its center, and* ω *be a *-valuation on D such that* $bsb^{-1} \equiv s \pmod{M}$ *for all non-zero* $s' = s^* \in R, b \in D$ *(in particular if* ω *is invariant).*

(1) If $\bar{D}$ *is commutative, then the residue degree f of* ω *over Z is 1 or 2.*

(2) If $\bar{D}$ *is noncommutative, then the residue degree f of* ω *over Z is 4.*

PROOF : (1) In this case, $\bar{Z} = (Z \cap R)/M$ is a *-subfield of the field $\bar{D}$. Also, by Proposition (4), $\bar{Z}$ contains the symmetrics of $\bar{D}$. Thus, $f = [\bar{D} : \bar{Z}] = 1$ or 2.

(2) Here, $\bar{D}$ is a *-division ring which is 4-dimensional over its center, i.e., $[\bar{D} : Z(\bar{D})] = 4$. Since, $f [\bar{D} : Z(\bar{D})]\, [Z(\bar{D}) : \bar{Z}]$, then it remains to show that $[Z(\bar{D}) : \bar{Z}] = 1$. Assume that $Z(\bar{D})$ strictly contains $\bar{Z}$. Since all symmetric units are central in $\bar{D}$ (by assumption), then there exist a skew symmetric unit $\bar{b} = b + M$ which is central in $\bar{D}$ and $\bar{b} \notin \bar{Z}$. Then there exist a central skew symmetric unit b in D. Hence all units are central which contradicts that $\bar{D}$ is noncommutative.

Theorem 11 — *Let D be a *-division ring which is algebraic over its center, and* ω *be an invariant *-valuation on D. Then the ramification index e of* ω *over Z is a power of 2.*

PROOF : Let Γ_{ω} be the value group of ω and $\Gamma_{\upsilon} = \{ \omega(z) \mid z \in Z \}$, where υ is the restriction of ω to Z. Let $\gamma \in \Gamma_{\omega}$ and $x \in D^{\bullet}$ such that $\omega(x) = \gamma$. Then $2\gamma = 2\,\omega(x) = \omega(xx^*)$. Since xx^* is a symmetric algebraic element then by Proposition (4), $x\overset{*}{x} \equiv z$ for some $z \in R \cap Z$. Thus $2\gamma = \omega(xx^*) = \omega(z) \in \Gamma_{v}$, as desired.

Corollary 12 — Let D be a finite dimensional *-division ring, and ω be an invariant *-valuation on D such that char $(\bar{D})= 0$. Then $[D : Z]$ is a perfect square, it is then a power of 4.

4. EXAMPLES

In this section we study some examples of *-division rings with invariant *-valuations. More details on the construction of these division rings can be found in [7].

Example 1 — Let $D = C(x)$, the division ring of Laurent series over the complex numbers C, where $xa = a^* x$ for $a \in C$ and a^* is the complex conjugate of a. Extend * to D by

$(\Sigma a_k x^k)^* = \Sigma(-1)^k a_k^{*'} x^k$, where $a_k^{*'} = a_k^*$ if k is even and $a_k^{*'} = a_k$ if k is odd. One can check that $*$ is an involution on D, $x^* = -x$ and the set of all symmetric elements of D is $\left\{ \Sigma a_{2k} x^{2k} \mid a_{2k} \in \boldsymbol{R} \right\} = Z$, the center of D. Define a valuation on D by $\omega(\Sigma a_k x^k) = m \in \Gamma = \boldsymbol{Z}$, the ring of integers, where a_m is the first non-zero coefficient. This is a *-valuation, and since all symmetric elements are central, ω is strongly invariant. Therefore D is standard quaternion *-division ring over Z generated by x and i, with strongly invariant *-valuation.

Example 2 — Let $D = C(x, y)$, the division ring of Laurent series over the complex numbers $\boldsymbol{C}$ in two variables with $xy = -yx$. Define $*$ on D so that it will be conjugation on $\boldsymbol{C}$ with $x^* = x$, $y^* = y$ and $(\Sigma\, a_{rs} x^r y^s)^* = \Sigma(-1)^{rs} a_{rs}^* x^r y^s$. One can check that $*$ is an involution on D. The center of D is $Z = \boldsymbol{C}(x^2, y^2)$. So D is a quaternion *-division ring over its center. The set of symmetric elements is

$$\{\Sigma a_{rs} x^r y^s \mid a_{rs} \in \boldsymbol{R} \text{ if } rs \text{ is even, and } a_{rs} \in i\boldsymbol{R} \text{ if } rs \text{ is odd}\}.$$

Define a valuation on D, $\omega : D^\bullet \to \Gamma = Z \times Z$ (ordered lexicographically) by

$$\omega(a_{mn} x^m y^n + \ldots) = (m, n).$$

One can check that ω is a *-valuation with $\overline{D} = \boldsymbol{C}$. Since x, y are symmetric elements with $xy = -yx$, i.e., $x\, y\, x^{-1} = -y$, then ω is not strongly invariant. Here the value group $\{\omega(z) \mid z \in Z\} = 2\boldsymbol{Z} \times 2\boldsymbol{Z}$ and so the ramification index e of ω is 4 and the residue degree f of ω is 1. Let $D_0 = \boldsymbol{R}(x, y)$, a sub-*-division ring of D. The set of symmetric elements of D_0 is $\{\Sigma a_{rs} x^r y^s \mid rs \text{ is even}\}$ and $Z(D_0) = \boldsymbol{R}(x^2, y^2)$. Let υ be the restriction of ω to D_0. Then υ is *-valuation on D_0 with value group $\Gamma_\upsilon = \Gamma = \boldsymbol{Z} \times \boldsymbol{Z}$ and $\overline{D} = \boldsymbol{R}$. Thus υ is invariant, but not strongly invariant. We note that in D not all algebraic symmetric elements are central.

Example 3 — Let D be any finite dimension *-division ring with invariant *-valuation ω. Let D^n be the division ring of Laurent series over D with $2n$ indeterminates x_i, y_i, $i = 1, 2, \ldots, n$, such that $x_i y_i = -y_i x_i$, $x_i y_j = y_j x_i$, $x_i x_j = x_j x_i$ and $y_i y_j = y_j y_i$ for $i \neq j$, and all x_i, y_i, commute with the elements of D. One can check that D^n carries an extended involution at which $x_i^* = x_i$, $y_i^* = y_i$. The center Z^n of D^n is $Z(D)(x_1^2, y_1^2, \ldots, x_n^2, y_n^2)$. We extend ω to an invariant *-aluation ω^n on D^n in a standard way, $\omega^n : D^n \to \Gamma_{\omega^n} = \Gamma_\omega \times Z \times \ldots \times \boldsymbol{Z}$, as in example (2). The value group of the restriction of ω^n to the center Z^n is $\Gamma_\omega \times 2\boldsymbol{Z} \times \ldots \times 2\boldsymbol{Z}$.

Since D is finite dimensional over its center, we have $e_{\omega^n} = 4^n e_\omega$ and $f_{\omega^n} = f_\omega$. Hence, $[D^n : Z^n] = 4^n [D : Z]$. If we take $D^\infty = \bigcup D^n$, then D^∞ is a *-division ring with invariant *-valuation

ω^{∞} such that $f_{\omega^{\infty}} = f_{\omega}$, and $e_{\omega^{\infty}} = \infty$. As a special case if we take $D = \boldsymbol{R}$, then D^n has dimension 4^n over its center. On the other hand, assume D is the classical quaternions $\boldsymbol{R}$ (–1, –1) = $\{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \boldsymbol{R}\}$ where $i^2 = -1, j^2 = -1, ij = -ji = k$. The valuation of $\boldsymbol{R}$ can be extended to an invariant *-valuation ω on D with residue degree $f_{\omega} = 4$. Hence, D^n has dimension 4^{n+1} over its center with residue degree 4 and ramification index 4^n.

REFERENCES

1. R. Wiegandt, *Vietnam J. Math.*, **21** (1993), 1-12.
2. A. I. Lichtman, *J. Algebra*, **177** (1995), 870-98.
3. P. Morandi, *J. Algebra*, 122, **1** (1989), 232-43.
4. S. S. Holland, *J. Algebra*, **101** (1986), 16-46.
5. J. Dieudonné, *Trans. Amer. Math. Soc.*, **75** (1953), 665-78.
6. J. Gräter, *Arch. Math.*, Vol. **66**, (1996), 13-18.
7. P. Cohn, *Skew field constructions*, London Math. Soc. Lecture Notes No. 27, Cambridge University Press, London, 1977.