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SOME IDENTITIES INVOLVING THE CENTRAL FACTORIAL NUMBERS AND RIEMANN ZETA FUNCTION

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In this paper, some identities involving the central factorial numbers and Riemann zeta function are given.

Key Words : Riemann Zeta Function; Bernoulli Numbers; Bernoulli Polynomials; Central Factorial Numbers

1. INTRODUCTION

Riemann zeta function $\zeta(x)$ (see [4]) and the central factorial numbers $t(n, k)$ (see [6]) are defined, respectively by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} (\text{Re}\,(x) > 1,\ x \in C) \qquad \dots (1)$$

and

$$(2 \log (t/2 + \sqrt{1 + t^2/4}))^k = k\,! \sum_{n=k}^{\infty} t\,(n, k)\, t^n/n\,! \quad (|\,t\,| < 1,\ \ t \in C). \qquad \dots (2)$$

By (2), we have

$$t\,(2n\,, 1) = 0\ (n \geq 1), t\,(2n+1, 1) = (-1)^n\,((2n)\,!)^2/(2^{4n}\,(n\,!)^2).$$

We denote

$$\Omega\,(f\,(v), k\,n) := \sum_{\substack{v_1 + v_2 + \dots + v_k = n \\ v_1 \geq 1,\, v_2 \geq 1,\, \dots,\, v_k \geq 1}} f\,(v_1)\,\zeta\,(2\,v_1) f\,(v_2)\,\zeta\,(2v_2) \dots f\,(v_k)\,\zeta\,(2\,v_k) \qquad \dots (3)$$

where the summation is over all k-dimensional positive integer coordinates $(v_1, v_2, \dots, v_k)$ such that $v_1 + v_2 + \dots + v_k = n$ and $n \geq k$ is any positive integer. Recently, several researchers have studied $\Omega\,(1, k, n)$. For example, see Liu[3], Sankaranarayanan[7], Sitaramachandrarao and Davis[8], and Zhang[9]. The main purpose of this paper is to study the calculating problem of $\Omega\,(2v - 1, k, n)$ and $\Omega\,(1 - 2^{-2v}, k, n)$. That is, we shall prove the following main conclusion.

Theorem 1 — *When* $n \geq k \geq 2$, *we have*

$$\Omega(2\nu - 1, k, n) = \frac{(-1)^k}{2^{k-1}} \sum_{i=1}^{k} \sum_{j=0}^{i-1} \binom{k}{i}$$

$$\frac{(-1)^{i-j}(2\pi)^{2j}\, t(2i, 2i-2j)\,(2n-2j-1)!\,\zeta(2n-2j)}{(2i-1)!\,(2n-2i)!}. \quad \dots (4)$$

Theorem 2 — *When* $n \geq k \geq 2$, *we have*

$$\Omega(1 - 2^{-2\nu}, k, n) = \sum_{i=1}^{[k/2]} \sum_{j=0}^{i-1} \binom{k}{2i}$$

$$\frac{(-1)^{k-j}(2\pi)^{2j}(2n-2j-1)!\, t(2i, 2i-2j)\,\zeta(2n-2j)}{2^{k-1}(2i-1)!\,(2n-2i)!}$$

$$\sum_{i=0}^{[(k-1)/2]} \sum_{j=0}^{i} \binom{k}{2i+1}$$

$$\frac{(-1)^{k-j}(2\pi)^{2j}(2n-2j-1)!\, t(2i+1, 2i+1-2j)\,(1 - 2^{-1-2n+2j})\,\zeta(2n-2j)}{2^{k-1}(2i)!\,(2n-2i-1)!} \quad \dots (5)$$

Here and in what follows $[x]$ denotes the greatest integer not exceeding x.

Taking $k = 2$ in Theorem 1 and Theorem 2, we may immediately deduce the following :

Corollary 1 — For any positive integer $n \geq 2$, we have

$$\sum_{i=1}^{n-1} (2i-1)(2n-2i-1)\,\zeta(2i)\,\zeta(2n-2i)$$

$$= \frac{1}{6}(4n^3 - 12n^2 - n + 3)\,\zeta(2n) + \frac{1}{3}\pi^2(2n-3)\,\zeta(2n-2). \quad \dots (6)$$

Corollary 2 — For any positive integer $n \geq 2$, we have

$$\sum_{i=1}^{n-1} (1 - 2^{1-2i})(1 - 2^{1-2n+2i})\,\zeta(2i)\,\zeta(2n-2i) = \left(n - \frac{3}{2} + 2^{1-2n}\right)\zeta(2n). \quad \dots (7)$$

2. Definitions and Lemmas

Definition 1 — The kth order Bernoulli numbers $B_n^{(k)}$ and kth order Bernoulli polynomials $B_n^{(k)}(x)$ are defined, respectively, by (see [2], [4], [5])

$$\left(\frac{t}{e^t - 1}\right)^k = \sum_{n=0}^{\infty} B_n^{(k)}\, t^n/n!, \quad \left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\, t^n/n!. \quad \dots (8)$$

Clearly $B_n^{(k)} = B_n^{(k)}(0)$, and the usual Bernoulli numbers $B_n = B_n^{(1)}$, Bernoulli polynomials $B_n(x) = B_n^{(1)}(x)$, $B_0^{(k)} = 1, B_1 = -1/2$, $B_{2n-1}^{(k)}(k/2) = B_{2n+1} = 0\ (n \geq 1)$.

Definition 2 — The generalized binomial coefficients $\sigma_k(x_1, x_2, ..., x_n)$ is given by the following expansion formula

$$(x+x_1)(x+x_2)\ldots(x+x_n) = \sum_{k=0}^{n} \sigma_k(x_1, x_2, ..., x_n)\, x^{n-k}. \quad ...\ (9)$$

By (9), we have $\sigma_0(x_1, x_2, \ldots, x_n) = 1$, when

$$1 \le k \le n, \quad \sigma_k(x_1 x_2, ..., x_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k}$$

and

$$\sigma_n(x_1, x_2, \ldots, x_n) = x_n\, \sigma_{n-1}(x_1, x_2, ..., x_{n-1}); \quad ...\ (10)$$

$$\sigma_k(x_1, x_2, ..., x_n) = x_n\, \sigma_{k-1}(x_1, x_2, \ldots, x_{n-1}) + \sigma_k(x_1, x_2, \ldots, x_{n-1})\ (1 \le k < n). \quad ...\ (11)$$

Lemma 1 — We denote $\sigma(n, x, k) := \sigma_k(1-x, 2-x, 3-x, ..., n-1-x)$, then

$$\sigma(n, x, k) = \sigma(n-2, x-1, k) + (n-2x)\,\sigma(n-2, x-1, k-1)$$
$$+ (1-x)(n-1-x)\,\sigma(n-2, x-1, k-2). \quad ...\ (12)$$

PROOF : By (9) we have

$$\sum_{k=0}^{n-1} (\sigma(n-2, x-1, k) + (n-2x)\,\sigma(n-2, x-1, k-1)$$
$$+ (1-x)(n-1-x)\,\sigma(n-2, x-1, k-2))\, t^{n-1-k}$$
$$= t^2 \sum_{k=0}^{n-3} \sigma(n-2, x-1, k)\, t^{n-3-k} + (n-2x) \sum_{k=0}^{n-3} \sigma(n-2, x-1, k)\, t^{n-2-k}$$
$$+ (1-x)(n-1-x) \sum_{k=0}^{n-3} \sigma(n-2, x-1, k-2))\, t^{n-3-k}$$
$$= (t^2 + (n-2x)\,t)(t+2-x)(t+3-x)\ldots(t+n-2-x)$$
$$+ (1-x)(n-1-x)(t+2-x)(t+3-x)\ldots(t+n-2-x)$$
$$= (t+1-x)(t+2-x)\ldots(t+n-1-x) = \sum_{k=0}^{n-1} \sigma(n, x, k)\, t^{n-1-k}. \quad ...\ (13)$$

Comparing the coefficient of t^{n-1-k} on both sides of (13), we immediately obtain (12).□

Lemma 2 —

$$(t, n, k) = \sigma(n, n/2, n-k)\ \ (n \ge k). \quad ...\ (14)$$

PROOF : We prove (14) by using mathematical induction.

1^0 $\sigma(n, n/2, n-1) = \left(1-\frac{n}{2}\right)\left(2-\frac{n}{2}\right)\cdots\left(n-1-\frac{n}{2}\right)$

$$= \begin{cases} 0, & n = 2m, \\ \dfrac{(-1)^m\,((2m)\,!)^2}{2^{4m}\,(m\,!)^2}, & n = 2m+1. \end{cases}$$

We, therefore, have $t(n, 1) = \sigma(n, n/2, n-1)$.

2^0 Taking $x = n/2$ in (12), we have

$$\sigma(n, n/2, k) = \sigma(n-2, (n-2)/2, k) - \frac{1}{4}(n-2)^2\,\sigma(n-2)/2, k-2), \qquad \text{... (15)}$$

We denote

$$f_1(t) := \sum_{n=k+1}^{\infty} \sigma(n, n/2, n-k-1)\frac{t^n}{n\,!}, \qquad \text{... (16)}$$

$$f_2(t) := \sum_{n=k+1}^{\infty} \sigma(n-2, (n-2)/2, n-k-1)\frac{t^n}{n\,!}$$

$$= \sum_{n=k-1}^{\infty} \sigma(n, n/2, n-k+1)\frac{t^{n+2}}{(n+2)\,!}, \qquad \text{... (17)}$$

$$f_3(t) := \sum_{n=k+1}^{\infty} (n-2)^2\,\sigma\left(n-2, \frac{n-2}{2}, n-k-3\right)\frac{t^n}{n\,!}$$

$$= \sum_{n=k+1}^{\infty} n^2\,\sigma(n, n/2, n-k-1)\frac{t^{n+2}}{(n+2)\,!} \qquad \text{... (18)}$$

Suppose (14) is true for some natural number k. By the supposition and (15) ~ (18), we have

$$f_1(t) = f_2(t) - \frac{1}{4}f_3(t), \quad f_2(t) = \sum_{n=k-1}^{\infty} t(n, k-1)\frac{t^{n+2}}{(n+2)\,!},$$

thus,

$$\frac{d^2}{dt^2}f_1(t) = \frac{d^2}{dt^2}f_2(t) - \frac{1}{4}\frac{d^2}{dt^2}f_3(t)$$

$$= \sum_{n=k-1}^{\infty} t(n, k-1)\frac{t^n}{n\,!} - \frac{1}{4}\left(t\frac{d}{dt}f_1(t) + t^2\frac{d^2}{dt^2}f_1(t)\right)$$

$$= \frac{1}{(k-1)!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{1}{4}t^2}\right)\right)^{k-1} - \frac{1}{4}\left(t\frac{d}{dt}f_1(t)+t^2\frac{d^2}{dt^2}f_1(t)\right). \quad \dots (19)$$

By (19), we have

$$\frac{d^2}{dt^2}\left(f_1(t)-\frac{1}{(k+1)!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{1}{4}t^2}\right)\right)^{k+1}\right)$$

$$= -\frac{t}{4+t^2}\frac{d}{dt}\left(f_1(t)-\frac{1}{(k+1)!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{1}{4}t^2}\right)\right)^{k+1}\right),$$

thus,
$$f_1(t)=\frac{1}{(k+1)!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{1}{4}t^2}\right)\right)^{k+1}.$$

i.e.
$$\sum_{n=k+1}^{\infty}\sigma(n, n/2, n-k-1)\frac{t^n}{n!}=\frac{1}{(k+1)!}\left(2\log\left(\frac{t}{2}+\sqrt{1+\frac{1}{4}t^2}\right)\right)^{k+1}. \quad \dots (20)$$

By (2) and (20), we have

$$\sigma(n, n/2, n-k-1) = t(n, k+1). \quad \dots (21)$$

and (21) shows that (14) is also true for the natural number $k + 1$. From 1^0 and 2^0, we know that (14) is true. □

Lemma 3 — Let $(a_k \mid k=1,2,\dots)$, $(x_k \mid k=1,2,\dots)$, $(F_n^{(1)} \mid n=0,\pm 1,\pm 2,\dots)$ be any three sequence, and the kth order sequence $F_n^{(k)}$ $(k=1,2,\dots)$ satisfy the recurrence relation

$$F_n^{(k+1)} = a_k x_k F_{n-1}^{(k)} + a_k F_n^{(k)}. \quad \dots (22)$$

For $k \ge 2$, $1 \le s \le k-1$, we have

$$F_n^{(k)} = \left(\prod_{t=s}^{k-1} a_i\right)\sum_{j=0}^{k-s}\sigma_j(x_s, x_{s+1}, x_{s+2}, \dots, x_{k-1})F_{n-j}^{(s)}. \quad \dots (23)$$

PROOF : We prove Lemma 3 by using mathematical induction

1^0 — When $k = 2$, (23) is clearly true.

2^0 — Suppose (23) is true for some natural number k. By the supposition and (22), we have

$$F_n^{(k+1)} = a_k x_k F_{n-1}^{(k)} + a_k F_n^{(k)} = \left(\prod_{i=s}^{k} a_i\right)$$

$$\sum_{j=0}^{k-s} x_k \sigma_j (x_s, x_{s+1}, x_{s+2}, \ldots, x_{k-1}) F_{n-1-j}^{(s)}$$

$$+ \left(\prod_{t=s}^{k} a_i \right) \sum_{j=0}^{k-s} \sigma_j (x_s, x_{s+1}, x_{s+2}, \ldots, x_{k-1}) F_{n-j}^{(s)}$$

$$= \left(\prod_{t=s}^{k} a_i \right) \left(\sum_{j=1}^{k+-s} x_k \sigma_{j-1} (x_s, x_{s+1}, x_{s+2}, \ldots, x_{k-1}) F_{n-j}^{(s)} \right.$$

$$\left. + \sum_{j=0}^{k-s} \sigma_j (x_s, x_{s+1}, x_{s+2}, \ldots, x_{k-1} F_{n-j}^{(s)} \right)$$

$$= \left(\prod_{i=s}^{k} a_i \right) x_k \sigma_{k-s} (x_s, x_{s+1}, x_{s+2}, \ldots, x_{k-1}) F_{n-k+s-1}^{(s)} + \left(\prod_{i=s}^{k} a_i \right) F_n^{(s)}$$

$$+ \left(\prod_{i=s}^{k} a_i \right) \sum_{j=1}^{k-s} (x_k \sigma_{j-1} (x_s, x_{s+1}, x_{s+2}, \ldots, x_{k-1})$$

$$+ \sigma_j (x_s, x_{s+1}, x_{s+2}, \ldots, x_{k-1})) F_{n-j}^{(s)}$$

$$= \left(\prod_{i=s}^{k} a_i \right) \sigma_{k+1-s} (x_s, x_{s+1}, x_{s+2}, \ldots, x_k) F_{n-k-1+s}^{(s)}$$

$$+ \left(\prod_{i=s}^{k} a_i \right) \sum_{j=1}^{k-s} \sigma_j (x_s, x_{s+1}, x_{s+2}, \ldots, x_k) F_{n-j}^{(s)}$$

$$+ \left(\prod_{i=s}^{k} a_i \right) F_n^{(s)} = \left(\prod_{i=s}^{k} a_i \right) \sum_{j=0}^{k+1-s} \sigma_j (x_s, x_{s+1}, x_{s+2}, \ldots, x_k) F_{n-j}^{(s)}, \quad \ldots (24)$$

and (24) shows that (23) is also true for the natural number $k + 1$. From 1^0 and 2^0, we know that (23) is true. □

Lemma 4 — When $n \geq k \geq 2,\ 1 \leq s \leq k-1$, we have

$$B_n^{(k)}(x) = \frac{(-1)^{k-s} k}{s} \binom{n}{k} \sum_{j=0}^{k-s}$$

$$\sigma_i (s-x, s+1-x, s+2-x, \ldots, k-1-x) B_{n-j}^{(s)}(x) \Big/ \binom{n-j}{s} \quad \ldots (25)$$

PROOF : Taking

$$a_k = 1/k,\ x_k = k - x,\ F_n^{(1)} = B_n(x)/n \quad (n \geq 1),$$

$$\sum_{n=0}^{\infty} (-1)^{k-1} F_n^{(k)} t^n/(n-k)! = g_k(x, t) \ (n \geq k)$$ in Lemma 3, then

$$g_1(x, t) = \sum_{n=0}^{\infty} F_n^{(1)} t^n/(n-1)! = \sum_{n=0}^{\infty} B_n(x) t^n/n! = (t/(e^t - 1)) e^{xt}, \qquad \dots (26)$$

and by (22), we have

$$g_k(x, t) = (t(x - k + 1)/(k-1))\, g_{k-1}(x, t),$$

$$+ (t/(1-k)) \frac{d}{dt} g_{k-1}(x, t) + g_{k-1}(x, t) \qquad \dots (27)$$

By (26) and (27),

$$g_k(x, t) = (t/(e^t - 1))^k e^{xt}. \qquad \dots (28)$$

By (28) and (8), $B_n^{(k)}(x) = (-1)^{k-1} (n!/(n-k)!) F_n^k$. It follows from Lemma 3 that

$$B_n^{(k)}(x) = \frac{(-1)^{k-s} k}{s} \binom{n}{k} \sum_{j=0}^{k-s}$$

$$\sigma_j(s - x, s + 1 - x, s + 2 - x, \dots, k - 1 - x)\, B_{n-j}^{(s)}(x) \Big/ \binom{n-j}{s}. \qquad \square$$

Remark 1 : Taking $s = k - 1$ in Lemma 4, we may immediately deduce the following (see [5])

$$B_n^{(k)}(x) = \frac{k-1-n}{k-1} B_n^{(k-1)}(x) - \frac{n(k-1-x)}{k-1} B_{n-1}^{(k-1)}(x). \qquad \dots (29)$$

Lemma 5 —

$$B_n^{(k)}(x) = (-1)^{k-1} k \binom{n}{k} \sum_{j=0}^{k\ 1} \sigma(k, x, j) \frac{B_{n-j}(x)}{n-j} \ (n \geq k). \qquad \dots (30)$$

PROOF : Take $s = 1$ in Lemma 4.

Lemma 6 — (*i*) When $n \geq m \geq 1$, we have

$$B_{2n}^{(2m)}(m) = -2m \binom{2n}{2m} \sum_{j=0}^{m-1} t(2m, 2m - 2j) \frac{B_{2n-2j}}{2n-2j}; \qquad \dots (31)$$

(*ii*) When $n \geq m+1 \geq 1$, we have

$$B_{2n}^{(2m+1)}\left(m+\frac{1}{2}\right)=(2m+1)\binom{2n}{2m+1}\sum_{j=0}^{m}$$

$$t(2m+1, 2m+1-2j)\,(2^{1-2n+2j}-1)\,\frac{B_{2n-2j}}{2n-2j}. \qquad \text{... (32)}$$

PROOF : (*i*) By Lemma 5, Lemma 2 and Definition 2, we have

$$B_{2n}^{(2m)}(m)=-2m\binom{2n}{2m}\sum_{j=0}^{2m-1} t(2m, 2m-2j)\,\frac{B_{2n-j}(m)}{2n-j}$$

$$=-2m\binom{2n}{2m}\sum_{j=0}^{2m-1} t(2m, 2m-j)\left(\frac{B_{2n-j}}{2n-j}+\sum_{i=0}^{m-1} i^{2n-j-1}\right)$$

and
$$\sum_{j=0}^{2m-1} t(2m, 2m-j)\sum_{i=0}^{m-1} i^{2n-j-1}=\sum_{i=0}^{m-1} i^{2n-2m}\sum_{j=0}^{2m-1}\sigma(2m, m, j)\, i^{2m-1-j}$$

$$=\sum_{i=0}^{m-1} i^{2n-2m}\sum_{j=0}^{2m-1}\sigma_j(1-m, 2-m, \dots, m-2, m-1)\, i^{2m-1-j}$$

$$=\sum_{i=0}^{m-1} i^{2n-2m}(i+1-m)(i+2-m)\dots(i+m-2)(i+m-1)$$

$$=\sum_{i=0}^{m-1} i^{2n-2m+1}(i^2-1^2)(i^2-2^2)\dots(i^2-(m-1)^2)=0.$$

We, therefore, have

$$B_{2n}^{(2m)}(m)=-2m\binom{2n}{2m}\sum_{j=0}^{2m-1} t(2m, 2m-j)$$

$$\frac{B_{2n-j}}{2n-j}=-2m\binom{2n}{2m}\sum_{j=0}^{m-1} t(2m, 2m-2j)\,\frac{B_{2n-2j}}{2n-2j}.$$

(*ii*) By Lemma 5, Lemma 2 and Definition 2, we have

$$B_{2n}^{(2m+1)}(m+1/2)=(2m+1)\binom{2n}{2m+1}$$

$$\sum_{j=0}^{2m} t(2m+1, 2m+1-j) \frac{B_{2n-j}(m+1/2)}{2n-j}$$

$$= (2m+1)\binom{2n}{2m+1} \sum_{j=0}^{2m} t(2m+1, 2m+1-j)$$

$$\left(\frac{B_{2n-j}\left(\frac{1}{2}\right)}{2n-j} + \sum_{i=0}^{m-1} \left(i+\frac{1}{2}\right)^{2n-j-1} \right)$$

and

$$\sum_{j=0}^{2m} t(2m+1, 2m+1-j) \sum_{i=0}^{m-1} \left(i+\frac{1}{2}\right)^{2n-j-1}$$

$$= \sum_{i=0}^{m-1} \left(i+\frac{1}{2}\right)^{2n-1-2m} \sum_{j=0}^{2m} \sigma\left(2m+1, m+\frac{1}{2}, j\right)\left(i+\frac{1}{2}\right)^{2m-1}$$

$$= \sum_{i=0}^{m-1} \left(i+\frac{1}{2}\right)^{2n-1-2m} \sum_{j=0}^{2m} \sigma_j\left(\frac{1}{2}-m, \frac{3}{2}-m, \ldots, m-\frac{3}{2}, m-\frac{1}{2}\right)\left(i+\frac{1}{2}\right)^{2m-j}$$

$$= \sum_{i=0}^{m-1} \left(i+\frac{1}{2}\right)^{2n-1-2m} (i+1-m)(i+2-m) \ldots (i+m-1)(i+m)$$

$$= \sum_{i=0}^{m-1} \left(i+\frac{1}{2}\right)^{2n-1-2m} i(i+m)(i^2\ 1^2)(i^2-2^2) \ldots (i^2-(m-1)^2) = 0.$$

We, therefore, have

$$B_{2n}^{(2m+1)}(m+1/2) = (2m+1)\binom{2n}{2m+1} \sum_{j=0}^{2m} t(2m+1, 2m+1-j) \frac{B_{2n-j}(1/2)}{2n-j}$$

$$= (2m+1)\binom{2n}{2m+1} \sum_{j=0}^{m} t(2m+1, 2m+1-2j) \frac{B_{2n-2j}(1/2)}{2n-2j}. \quad \ldots (33)$$

By (33) and $B_{2n}(1/2) = (2^{1-2n}-1) B_{2n}$ (see [1, p.805]), we immediately obtain (32).

Remark 2 : Taking $m = 1$ in Lemma 6 (*i*), we have

$$B_{2n}^{(2)}(1) = (1-2n) B_{2n}. \quad \ldots (34)$$

3. Proof of Theorem 1 and Theorem 2

Proof of Theorem 1

By $\sum_{n=k}^{\infty}\left(\sum_{\substack{v_1+v_2+\dots+v_k=n\\ v_1\geq 1, v_2\geq 1,\dots,v_k\geq 1}}\frac{B_{2v_1}^{(2)}B_{2v_2}^{(2)}(1)\dots B_{2v_k}^{(2)}(1)}{(2v_1)!(2v_2)!\dots(2v_k)!}\right)$

$$t^{2n}=\left(\sum_{n=1}^{\infty}B_{2n}^{(2)}(1)\frac{t^{2n}}{(2n)!}\right)^k$$

$$=\left(\left(\frac{t}{e^t-1}\right)^2 e^t-1\right)^k=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\left(\frac{t}{e^t-1}\right)^{2t}e^{it}=(-1)^k$$

$$\sum_{n=0}^{\infty}\sum_{i=0}^{k}(-1)^i\binom{k}{i}B_n^{(2i)}(i)\frac{t^{2n}}{n!},$$

we have
$$\sum_{\substack{v_1+v_2+\dots+v_k=n\\ v_1\geq 1, v_2\geq 1,\dots,v_k\geq 1}}\frac{B_{2v_1}^{(2)}(1)B_{2v_2}^{(2)}(1)\dots B_{2v_k}^{(2)}(1)}{(2v_1)!(2v_2)!\dots(2v_k)!}=\frac{(-1)^k}{(2n)!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}B_{2n}^{(2i)}(i)\quad\dots(35)$$

By (35) and Lemma 6 (i), we have

$$\sum_{\substack{v_1+v_2+\dots+v_k=n\\ v_1\geq 1, v_2\geq 1,\dots,v_k\geq 1}}\frac{B_{2v_1}^{(2)}(1)B_{2v_2}^{(2)}(1)\dots B_{2v_k}^{(2)}(1)}{(2v_1)!(2v_2)!\dots(2v_k)!}=(-1)^{k-1}\sum_{i=0}^{k}$$

$$\left(\sum_{j=0}^{i-1}\binom{k}{i}\frac{(-1)^i t(2i,2i-2j)}{(2i-1)!(2n-2i)!}\right)\frac{B_{2n-2jj}}{2n-2j}.\quad\dots(36)$$

By (36), (34), and $B_{2n}=(-1)^{n-1}2(2n)!\zeta(2n)/(2\pi)^{2n}$ $(n\geq 1)$ 9see [4, p. 19]), we immediately obtain Theorem 1. □

Proof of Theorem 2

By $\sum_{n=k}^{\infty}\left(\sum_{\substack{v_1+v_2+\dots+v_k=n\\ v_1\geq 1, v_2\geq 1,\dots,v_k\geq 1}}\frac{B_{2v_1}^{(2)}(1)B_{2v_2}^{(2)}(1)\dots B_{2v_k}^{(2)}(1)}{(2v_1)!(2v_2)!\dots(2v_k)!}\right)t^{2n}$

$$=\left(\sum_{n=1}^{\infty}B_{2n}(1/2)\frac{t^{2n}}{(2n)!}\right)^k$$

$$= \left(\left(\frac{t}{e^t - 1} \right)^2 e^{\frac{1}{2}t} - 1 \right)^k = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \left(\frac{t}{e^t - 1} \right)^i e^{\frac{it}{2}}$$

$$= (-1)^k \sum_{n=0}^{\infty} \sum_{i=0}^{k} (-1)^i \binom{k}{i} B_n^{(i)} \left(\frac{1}{2} \right) \frac{t^n}{n!},$$

and lemma 6 (*ii*), we have

$$\sum_{\substack{v_1 + v_2 + \ldots + v_k = n \\ v_1 \geq 1, v_2 \geq 1, \ldots, v_k \geq 1}} \frac{B_{2v_1}(1/2) B_{2v_2}(1/2) \ldots B_{2v_k}(1/2)}{(2v_1)!(2v_2)! \ldots (2v_k)!} = \frac{(-1)^k}{(2n)!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} B_{2n}^{(2i)} \left(\frac{i}{2} \right)$$

$$= \frac{(-1)^k}{(2n)!} \sum_{i=1}^{[k/2]} \binom{k}{2i} B_{2n}^{(2i)}(i) - \frac{(-1)^k}{(2n)!} \sum_{i=0}^{[(k-1)/2]} \binom{k}{2i+1} B_{2n}^{2i+1)} \left(\frac{2i+1}{2} \right)$$

$$= (-1)^{k-1} \sum_{i=1}^{[k/2]} \sum_{j=0}^{i-1} \binom{k}{2i} \frac{t(2i, -2i-2j) B_{2n-2j}}{(2i-1)!(2n-2i)!(2n-2j)}$$

$$- (-1)^k \sum_{i=1}^{[(k-1)/2]} \sum_{j=0}^{i} \binom{k}{2i+1} \frac{t(2i+1, 2i+1-2j) B_{2n-2j}}{(2i)!(2n-2i-1)!(2n-2j)}. \quad \ldots (37)$$

By (37) and $B_{2n}(1/2) = (2^{1-2n} - 1) B_{2n}$, we immediately obtain Theorem 2. □

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