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A NOTE ON QUASI POWER INCREASING SEQUENCES

H. S. ÖZARSLAN

Department of Mathematics, Erciyes University, 38039, *Kayseri, Turkey*
E-mail: seyhan@erciyes.edu.tr

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In this paper a theorem of Özarslan[8] has been proved under weaker conditions by using a quasi β-power increasing sequence instead of an almost increasing sequence.

Key Words : Absolute Summability; Quasi Power Increasing Sequence

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let Σa_n be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} P_v \to \infty \text{ as } n \to \infty, \ (P_{-i} = p_{-i} = 0, i \geq 1). \qquad \text{... (1)}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \qquad \text{.. (2)}$$

defines the sequence (σ_n) of the $(\overline{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [5]). The series Σa_n is said to be summable $|\overline{N}, p_n|_k, k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |\Delta \sigma_{n-1}|^k < \infty \qquad \text{... (3)}$$

and it is said to be summable $|\overline{N}, p_n ; \delta|_k, k \geq 1$ and $\delta \geq 0$, if (see [3])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta_k + k-1} |\Delta \sigma_{n-1}|^k < \infty, \qquad \text{... (4)}$$

where
$$\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \ n \geq 1. \qquad \text{... (5)}$$

In the special case when $\delta = 0$ (resp. $p_n = 1$ for all values of n) $|\overline{N}, p_n; \delta|_k$ summability is the same as $|\overline{N}, p_n|_k$ (resp. $|C, 1; \delta|_k$) summability.

Quite recently Özarslan[8] proved the following theorem for $|\overline{N}, p_n; \delta|_k$ summability factors of infinite series.

Thereom A — *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \text{ as } n \to \infty. \qquad \text{... (6)}$$

Let (X_n) be an almost increasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that

$$|\Delta \lambda_n| \le \beta_n , \qquad \text{... (7)}$$

$$\beta_n \to 0 \text{ as } n \to \infty, \qquad \text{... (8)}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \qquad \text{... (9)}$$

$$|\lambda_n| X_n = O(1) \text{ as } n \to \infty. \qquad \text{... (10)}$$

If
$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right\} \qquad \text{... (11)}$$

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |s_n|^k = O(X_m) \text{ as } m \to \infty, \qquad \text{... (12)}$$

then the series $\Sigma a_n \lambda_n$ is summable $|\overline{N}, p_n; \delta|_k$ for $k \ge 1$ and $0 \le \delta < 1/k$.

Remark : It may be noted that, if we take (X_n) as a positive non-decreasing sequence and $\delta = 0$ in this theorem, then we get a result of Bor[4] on $|\overline{N}, p_n|_k$ summability factors.

2. The Main Result

The aim of this paper is to prove Theorem A under weaker conditions. For this we need the concept of quasi β-power increasing sequence. A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \ge 1$ such that

$$K n^{\beta} \gamma_n \ge m^{\beta} \gamma_m \qquad \text{... (13)}$$

holds for all $n \ge m \ge 1$ (see[6]). It should be noted that the class of almost increasing sequences is a strict subclass of the quasi β-power increasing sequences if $\beta > 0$. So that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of the theorem replacing an almost increasing sequence by a quazi β-power increasing sequence.

Now, we shall prove the following theorem :

Theorem — *Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If all the conditions from (6) to (12) are satisfied, then the series $\Sigma a_n \lambda_n$ is summable $|\bar{N}, p_n; \delta|_k$ for $k \geq 1$ and $0 \leq \delta < 1/k$.*

We need the following lemma for the proof of our theorem.

Lemma 7 — Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (9) is satisfied :

$$n \beta_n X_n = O(1) \text{ as } n \to \infty, \qquad \text{... (14)}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \qquad \text{... (15)}$$

PROOF OF THE THEOREM

Let (T_n) denotes the $(\bar{N}, p_n)$ mean of the series $\Sigma a_n \lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i \lambda_i = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v .$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v.$$

By Abel's transformation, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta (P_{v-1} \lambda_v) s_v + \frac{p_n}{P_n} s_n \lambda_n$$

$$= - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \Delta \lambda_v + \frac{p_n}{P_n} s_n \lambda_n$$

$$= T_{n,1} + T_{n,2} + T_{n,3}, \text{ say.}$$

Since

$$| T_{n,1} + T_{n,2} + T_{n,3} |^k \leq 3^k (| T_{n,1} |^k + | T_{n,2} |^k + | T_{n,3} |^k),$$

to complete the proof of the Theorem, it is enough to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} | T_{n,r} |^k < \infty \text{ for } r = 1, 2, 3. \qquad \text{... (16)}$$

Now, when $k>1$ applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{\delta k+k-1}|T_{n,1}|^k\le\sum_{n=2}^{m+1}(P_n/p_n)^{\delta k-1}(P_{n-1})^{-k}$$

$$\left\{\sum_{v=1}^{n-1}p_v|s_v||\lambda_v|\right\}^k$$

$$\le\sum_{n=2}^{m+1}(P_n/p_n)^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v|s_v|^k|\lambda_v|^k\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$=O(1)\sum_{v=1}^{m}p_v|s_v|^k|\lambda_v|^k\sum_{n=v+1}^{m+1}(P_n/p_n)^{\delta k-1}\frac{1}{P_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}(P_v/p_v)^{\delta k-1}|s_v|^k|\lambda_v||\lambda_v|^{k-1}$$

$$=O(1)\sum_{v=1}^{m}(P_v/p_v)^{\delta k-1}|s_v|^k|\lambda_v|$$

$$=O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}(P_r/p_r)^{\delta k-1}|s_r|^k$$

$$+O(1)|\lambda_m|\sum_{v=1}^{m}(P_v/p_v)^{\delta k-1}|s_v|^k$$

$$=O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m$$

$$=O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)|\lambda_m|X_m$$

$$=O(1)\text{ as }m\to\infty,$$

bu virtue of the hypotheses of the Theorem and Lemma.

Since $v\beta_v=O(1/X_v)$ by (14), using the fact that $P_v=O(vp_v)$ by (6), and $|\Delta\lambda_n|\le\beta_n$ by (7), and after applying Hölder's inequality again, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} \,|\, T_{n,2} \,|^k$$

$$\leq \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} (P_{n-1})^{-k} \left\{ \sum_{v=1}^{n-1} P_v \,|\, \Delta \lambda_v \,|\,|\, s_v \,| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} (P_{n-1})^{-k} \left\{ \sum_{v=1}^{n-1} v p_v \beta_v \,|\, s_v \,| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} (v \beta_v)^k p_v \,|\, s_v \,|^k$$

$$\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (v \beta_v)^k p_v \,|\, s_v \,|^k \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} (v \beta_v)^k (P_v/p_v)^{\delta k-1} \,|\, s_v \,|^k$$

$$= O(1) \sum_{v=1}^{m} (v \beta_v)^{k-1} (v \beta_v) (P_v/p_v)^{\delta k-1} \,|\, s_v \,|^k$$

$$= O(1) \sum_{v=1}^{m} \Delta (v \beta_v) \sum_{r=1}^{v} (P_r/p_r)^{\delta k-1} \,|\, s_r \,|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta (v \beta_v) \sum_{r=1}^{v} (P_r/p_r)^{\delta k-1} \,|\, s_r \,|^k$$

$$+ O(1)\, m \beta_m \sum_{v=1}^{m} (P_v/p_v)^{\delta k-1} \,|\, s_v \,|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\, \Delta (v \beta_v) \,|\, X_v + O(1)\, m \beta_m X_m$$

$$= O(1) \sum_{v=1}^{m-1} v \,|\, \Delta \beta_v \,|\, X_v + O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_{v+1} + O(1)\, m \beta_m X_m$$

$$= O(1) \text{ as } m \to \infty,$$

by virtue of the hypotheses of the Theorem and Lemma.

Finally using the fact that $P_\upsilon = O(\upsilon p_\upsilon)$ by (6), as in $T_{n,1}$, we have that

$$\sum_{n=1}^{m} (P_n/p_n)^{\delta k + k - 1} | T_{n,3} |^k = O(1) \sum_{n=1}^{m} (P_n/p_n)^{\delta k - 1} | s_n |^k | \lambda_n |$$

$$= O(1) \text{ as } m \to \infty.$$

Therefore, we get (16) and this completes the proof of the theorem.

If we take $p_n = 1$ for all values of n in this theorem, then we get a new result concerning the $| C, 1; \delta |_k$ summability factors.

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