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FUNCTIONAL BOUNDARY VALUE PROBLEM FOR FIRST ORDER IMPULSIVE DIFFERENTIAL EQUATIONS AT VARIABLE TIMES[1]

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In this paper, we discuss the functional boundary value problem for first order impulsive differential equations at variable times. Some existence results for extremal solutions are obtained.

Key Words : Impulsive Differential Equations at Variable Times; Functional Boundary Value Problem; Extremal Solutions; Upper and Lower Solutions

1. INTRODUCTION

Recently, there exist some investigation about first order impulsive differential equations with variables times[2-5]. They give existence results for the initial value problems ([2, 4,5] and the periodic boundary value problems [3, 5].

In this paper, we consider the functional boundary value problems of the type

$$x'(t) = f(t, x(t)), \quad t \neq \tau(x(t)), t \in J,$$

$$\Delta x(t) = I(x(t)), \quad t = \tau(x(t)),$$

$$B(x(0), x) = 0 \qquad \text{... (I)}$$

where $J = [0, T], f \in C(J \times R, R), I \in C'(R, R), \tau \in C'(R, R)$ and $B \in C(J \times PC(J, R), R)$.

This general setting makes it possible to study different types of conditions under a unique notation. By the following choices for the appropriate definitions of B : for initial condition, $x(0) = x_0 \in R$, we consider

$$B(a, \zeta) \equiv B(a) = a - x_0, \text{ for all } a \in R$$

and for the periodic conditions we take

$$B(a, \zeta) \equiv \zeta(T) - a, \text{ for all } (a, \zeta) \in R \times PC(J, R)$$

The results in this paper are inspired by R. L. Pouso[6]. We obtain existence results for extremal solutions of (I). The results of [2, 3] are extended.

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2. Initial Value Problems

Consider the problem

$$x'(t) = f(t, x(t)), \quad t \neq \tau(x(t)), t \in J,$$

$$\Delta x(t) = I(x(t)), \quad t = \tau(x(t)),$$

$$x(0) = x_0, \qquad \text{... (II)}$$

which corresponds to (I) with

$$B(a, \zeta) \equiv B(a) = a - x_0 \text{ for all } a \in R.$$

Let us introduce a notation and a definition before proceeding further.

Definition 2.1 — By a function $h \in PC^1(J, R)$, we mean that h is discontinuous at points $\bar{t} : \bar{t} = \tau(h(\bar{t}))$ but elsewhere continuous and differentiable.

Definition 2.2 — A function $\upsilon \in PC(J, R)$ is said to be a lower solution of (II), if it satisfies the following inequalities

$$\upsilon'(t) \neq f(t, \upsilon), \quad t \neq \tau(\upsilon(t)), t \in J,$$

$$\Delta \upsilon(t) \leq I(\upsilon(t)), \quad t = \tau(\upsilon(t)),$$

$$\upsilon(0) \leq x_0. \qquad \text{... (2.1)}$$

Analogously, we define the upper solution of (II) by reversing the inequalities in (2.1).

Theorem 2.3 — *Let $\upsilon, w \in PC^1(J, R)$ be lower, upper solutions of (II) such that $\upsilon(t) \leq w(t)$ on J, and w(t) hit the surface $S : t = \tau(x)$ at $t = t_0$, $t_0 \in (0, T]$. Assume that the following conditions are satisfied :*

(*i*) $\tau(x)$ is increasing and $\tau(\upsilon(0^+)) > 0$;

(*ii*) $\tau_x(x + sI(x)) I(x) < 0, 0 \leq s \leq 1, t = \tau(x), \upsilon(t) \leq x \leq w(t), t \in J$;

(*iii*) $I^*(x) = x + I(x)$ is increasing and $(I^*)'(x) f(t, x) > f(t, I^*(x))$.

Then, there exists a solution $x(t)$ of (II) such that $x \in [\upsilon, w]$ on J.

Proof : First we prove that $\upsilon(t)$ hits the surface S exactly once. For it, set $p(t) = t - \tau(\upsilon(t))$. If $\upsilon(t)$ do not hit the surface S, then $\upsilon \in C(R, R)$ and $p(t) \neq 0, t \in J$. By the conditions (*i*) $p(0) = -\tau(\upsilon(0)) < 0$. As a sequence, $p(t) < 0, t \in J$. But, $p(t_0) = t_0 - \tau(\upsilon(t_0)) \geq t_0 - \tau(w(t_0)) = 0$, this is a contradiction. Thus $\upsilon(t)$ hits the surface S at $s_0 \in (0, T)$, that is, $s_0 = \tau(\upsilon(s_0))$. Further since $s_0 = \tau(\upsilon(s_0))$ and $\tau(\upsilon(s_0^+)) \leq \tau(\upsilon(s_0) + I(\upsilon(s_0)))$, then from condition (ii) we get

$$\tau(\upsilon(s_0^+)) - \tau(\upsilon(s_0)) \leq \tau(\upsilon(s_0) + I(\upsilon(s_0))) - \tau(\upsilon(s_0))$$

$$= \int_0^1 \tau_x'(v(s_0) + sI(v(s_0)))\, ds\, I(v(s_0)) < 0, 0 \le s \le 1.$$

Thus we obtain τ_x'

$$p(s_0^+) = s_0 - \tau(v(s_0^+)) \ge s_0 - \tau(v(s_0)) = 0,$$

$$p'(t) \ge 1 - \tau'(v(t)) f(t, v(t)) > 0.$$

Further, $p(t) > 0$, $t > s_0$. This implies that $v(t)$ does not hit S for $t > s_0$.

We now use the method of lower and upper solution in the theory of ordinary differential equations. This result gives the existence of the solution $x(t)$ of (II) in $[0, s_0]$, such that

$$v(t) \le x(t) \le w(t), \quad 0 \le t \le s_0.$$

Since $v(s_0^+) \le v(s_0)$, then we get

$$v(s_0^+) \le v(s_0) \le x(s_0) \le w(s_0).$$

Now using the above relation and the theorem of method of lower and upper solutions in the theory of ordinary differential equations, the solution $x(t)$ of (II) exists in $(s_0, t_1]$, where $t_1 = \tau(x(t_1))$ (similar to $v(t)$, we have that $x(t)$ hits the surface S exactly once, that is, there exists a $t_1 \in (0, T)$ such that $t_1 = \tau(x(t_1))$) such that

$$v(t) \le x(t) \le w(t), \quad s_0 < t \le t_1.$$

Then, condition (ii) implies that $\tau(x(t_1^+)) < \tau(x(t_1))$ and the increasing nature of $\tau(x)$ further gives

$$x(t_0^+) < x(t_1) \le w(t_1).$$

Now we shall show that

$$x(t_1^*) = I^*(x(t_1)) \ge v(t_1). \qquad \text{... (2.2)}$$

Suppose that (2.2) is not true. Since $v(s_0) \le x(s_0)$ and I^* is increasing we have

$$v(s_0^+) \le I^*(v(s_0)) \le I^*(x(s_0)).$$

Then there exists one point $s \in (s_0, t_1)$ such that $v(s) = I^*(x(s))$ and $v(t) > I^*(x(t))$ for $s < t \le t_1$. Thus, using the condition (iv), we obtain

$$v'(s) \ge (I^*)'(x(s))\, x'(s) = (I^*)'(x(s)) f(s, x(s))$$

$$> f(s, I^*(x(s))) = f(s, v(s)).$$

But this is a contradiction and hence $x(t_1^+) \geq \upsilon(t_1)$. Hence, starting with t_1 as the initial point, using previous arguments, we get that the solution $x(t)$ of (II) exists until $w(t)$ hits the surface at $t = t_0$. Thus, we have that

$$\upsilon(t) \leq x(t) \leq w(t), \quad t_1 < t \leq t_0.$$

Now we shall show that

$$x(t_0) \leq w(t_0^+). \qquad \text{... (2.3)}$$

Suppose that (2.3) is not true, we have $x(t_0) > w(t_0^+) \geq I^*(w(t_0))$. Since $x(t_1) \leq w(t_1)$ and I^* is increasing, we get

$$x(t_1^+) = I^*(x(t_1)) \leq I^*(w(t_1)).$$

Thus there exists one point $\eta \in (t_1, t_0)$ such that $x(\eta) = I^*(w(\eta))$ and $x(t) > I^*(w(t))$ for $t_1 < \eta \leq t_0$. So, using the condition (iv), we obtain

$$x'(\eta) \geq (I^*)'(w(\eta))\, w'(\eta) > I^{*\prime}(w(\eta)) f(t, x(\eta))$$

$$> f(t, I^*(w(\eta))) = f(t, x(\eta)).$$

This is a contradiction, and hence

$$\upsilon(t_0) \leq x(t_0) \leq w(t_0^+).$$

If $w(t)$ does not hit S for $t > t_0$, using the above inequality and proceeding as before, we get the existence of solution on J such that $\upsilon(t) \leq x(t) \leq w(t)$ and the proof is complete. If there exists one point $t_2 \in (t_0, T)$ such that $t_2 = \tau(w(t_2))$. Since $\upsilon(t_0) \leq x(t_0) \leq w(t_0^+)$, we obtain $\upsilon(t) \leq x(t) \leq w(t)$ for $t \in [t_0, t_2]$. Proceeding as before we get

$$\upsilon(t_2) \leq x(t_2) \leq w(t_0^+)$$

Employing the same procedures successively, we may conclude that $\upsilon(t) \leq x(t) \leq w(t)$ for all $t \in J$, and the proof is now complete.

From the proof of Theorem 2.3 we have the following corollary.

Corollary 2.4 — Suppose that all the condition of Theorem 2.3 hold. For any $x_0 \in [\upsilon(0), w(0)]$, the solution $x(t)$ of (II) hits the surface S only once.

Combine Theorem 2.3 with Corollary 2.2.1 [1, P_{64}], we also get the next corollary.

Corollary 2.5 — Suppose that all the conditions of Theorem 2.3 hold. If IVP (III) :

$$x' = f(t, x(t)), \quad t \in J,$$

$$x(0) = x_0 \qquad \text{... (II)}$$

exists a unique solution, then so does (II).

For every $A \in [v(0), w(0)]$, we shall denote by (II_A) the following initial value problem :

$$x' = f(t, x(t)), \quad t \neq \tau(x(t)), \quad t \in J,$$
$$\Delta x(t) = I(x(t)), \quad t = \tau(x(t)), \quad \ldots (\mathrm{II}_A)$$
$$x(0) = A.$$

Theorem 2.6 — *Suppose that all the conditions of Corollary 2.5 hold. Then for any* $A_i \in [v(0), w(0)]$, $A_1 \leq A_2$ *implies* $x(t, A_1) \leq x(t, A_2)$ *and* $s_1 \leq s_2$, *where* $x(t, A_i)$ *be the solution of* (II_{A_i}) *and* $s_i \in J$ *such that* $s_i = \tau(x(s_i, A_i)), i = 1, 2.$

PROOF : By Corollary 2.5, problem (II_{A_i}) has a solution $x_i(t) = x(t, A_i) \in [v, w]$, for every $A_i \in [v(0), w(0)]$, and $x_i(t)$ hits the surface S only once at $t = s_i$, thus $s_i = \tau(x_i(s_i))$, $i = 1, 2$. Since $\tau(x)$ is increasing and $A_1 \leq A_2$, then $x_1(t)$ hits the surface S at first, that is, $s_1 \leq s_2$.

It is obviously that $x_1(t) \leq x_2(t)$ on $[0, s_i]$. Further, from condition (iii), we see that $x_1(s^+) \leq x_1(s)$, and hence $x_1(t) \leq x_2(t)$ for $t \in [s_1, s_2]$.

We shall show that

$$x_1(s_2) \leq x_2(s_2^+). \quad \ldots (2.4)$$

If it is not true, and hence $x_2(s_2^+) = I^*(x_2(s_2)) < x^1(s_2)$. Since $x_1(s_1) \leq x_2(s_2)$ and I^* is increasing, we have $x_1(s_1^+) = I^*(x_1(s_1)) \leq I_*(x_2(s_1))$, Then there exists a point $s \in (s_1, s_2)$ such that $x_1(s) = I^*(x_2(s))$ and $x_1(t) > I^*(x_2(t))$ for $s < t \leq s_2$. Using condition (iv), we obtain

$$x_1'(s) \geq (I^*)'(x_2(s))\, x_2'(s) = (I^*)'(x_2(s))\, f(t, x_2(s))$$
$$> f(t, I^*(x_2(s))) = f(t, x_1(s)).$$

This is a contradiction. Hence (2.4) holds. Using (2.4) and proceeding as before, we get that $x_1(t) \leq x_2(t)$ on J. Thus the proof is complete.

3. FUNCTIONAL BOUNDARY CONDITIONS

In this section we shall prove a result on the existence of extremal solutions between given upper and lower a solution of (I).

Definition 3.1 — A function $v \in PC[J, R]$ is said to be a lower solution of (I), if it satisfies the following inequalities

$$v' \leq f(t, v),$$
$$v(t^+) \leq v(t + I(v(t)), \quad \ldots (3.1)$$
$$B(v(0), v) \leq 0.$$

Analogously, we define the upper solution of (I) by reversing the inequalities in (3.1).

Remark 3.2 : Let $\upsilon, w \in PC^1(J, R)$ be lower and upper solutions of (I) such that $\upsilon(t) \le w(t)$. Then, υ, w are lower and upper solutions of (II) for any $x_0 \in [\upsilon(0), w(0)]$.

Theorem 3.3 — *Let $\upsilon, w \in PC^1(J, R)$ be lower and upper solutions of (I) such that $\upsilon(t) \le w(t)$ on J, and $w(t)$ hit the surface $S: t = \tau(x)$ at $t = t_0$, $t_0 \in (0, T]$. Suppose IVP′ (III) exists a unique solution. Assume that the following conditions are satisfied :*

(*i*) $\tau(x)$ is increasing and $\tau)(\upsilon(0^+)) > 0$;

(*ii*) $\tau_x(x + sI(x))\, I(x) < 0, 0 \le s \le 1, t = \tau(x), \upsilon(t) \le x \le w(t), t \in J$;

(*iii*) $I^*(x) = x + I(x)$ is increasing and $(I^*)'(x) f(t, x) > f(t, I^*(x))$.

(*iv*) for each $A \in R$, the function $B(A, \cdot)$ is nonincreasing on $[\upsilon, w]$.

Then, problem (I) has the extremal solution in $[\upsilon, w]$.

PROOF : For any $A \in [\upsilon(0), w(0)]$, by Remark 3.2, $\upsilon, w \in PC^1(J, R)$ be lower and upper solutions of (II_A). If $\upsilon(0) = w(0)$, then, by the properties of B, any solution of (II_A) in $[\upsilon, w]$ is a solution in of (I).

Let us suppose that $\upsilon(0) < w(0)$. First, for every $A \in [\upsilon(0) < w(0)]$ by Theorem 2.6, Problem (II_A) has a solution in $[\upsilon, w]$. Define

$$Z = \{A \in (\alpha(0), \beta(0) : \exists\, x \in [\upsilon, w] \text{ solution of } (II_A) \text{ such that } B(x(0), x) \le 0\}.$$

The set Z is not empty. In fact, $\upsilon(0) \in Z$ because if $x \in [\upsilon, w]$ is a solution of $(\mathrm{II}_{\upsilon_{(0)}})$,then, since B is nonincreasing with respect to its second variable, we have

$$B(x(0), x) = B(\upsilon(0), x) \le B(\upsilon(0), \upsilon) \le 0$$

Let $A^* = \sup Z$ and let $\{A_n\}_{n \in N} \subset Z$ be a nondecreasing sequence such that $\lim_{n \to \infty} A_n = A^*$.

For every $n \in N$ we take $x_n \in [\upsilon, \omega]$ to a solution of (II_{A_n}) such that $B(x_n(0), x_n) \le 0$, which is possible by the definition of Z. By Theorem 2.6, we have that $x_n(t)$ hits the surface S only once at $t = s_n$ and

$$\upsilon \le x_1 \le x_2 < \ldots \le x_n \le \ldots \le w \text{ on } J,$$

$$s_0 \le s_1 \le s_2 \le \ldots \le s_n \le T$$

where $\quad s_0 = \tau(\upsilon(s_0))$.

Next, we aim to show that $x_n \to \rho$. Since $x_n(t)$ is a monotonic bounded sequence and hence there exists $\rho(t)$ such that $x_n(t) \to \rho(t)$, pointwise, $t \in J$. Also, the sequence $\{s_n\}$ the points of discontinuity of $x_n(t)$, is increasing in J, thus $s_n \to t^*$. Now, we claim that $\rho(t)$ hits the surface S

at $t = t^*$. If not, suppose there exists a $\bar{t} < t^*$, such that $\bar{t} = t(\rho(\bar{t}))$. Since $s_n \to t^*$, there exists a $m \in N$ such that $\bar{t} < s_m < t^*$ and $s_m = \tau(x_m(x_m))$. But $x_m(t) \le \rho(t), 0 \le t \le T$ and $\tau(x)$ is an increasing function. This implies that $x_m(t)$ hits the surface S first at $t = s_m$. This is a contradiction. This show that $\bar{t} = t^*$.

Further, we claim that

$$x_n(s_n) \to \rho(t^*) \text{ and } x_n(s_n^+) \to \rho(t^{*+}).$$

Consider $t_m \neq s_n, m, n \ge 0$, such that $t_m \to t^*$. Then for any n, $t_m > s_n$ eventually and hence $x_n(t_m) \to x_n(t^*)$. So we obtain that either $x_n(t^*) \to \rho(t^*)$ or $x_n(t^*) \to \rho(t^{*+})$. Thus we can find a diagonalizing sequence $\{t_{mn}\}$ such that $x_n(t_{mn}) \to \rho(t^*)$ or $x_n(t_{mn}) \to \rho(t^{*+})$.

Suppose that $t_{mn} < s_n, n \ge 0$. Because s_n is a point of discontinuity for $x_n(t), x_n(t)$ is continuous in $[0, s_n]$ for each n, so as $n \to \infty$, we get $x_n(t_{mn}) \to p(t^*)$ so we consider

$$| x_n(s_n) - \rho(t^*) | \le | x_n(s_n) - x_n(t_{mn}) | + | x_n(t_{mn}) - \rho(t^*) |$$

the latter part of right-hand side (rhs) goes to zero as $n \to \infty$ and we estimate the former part as follows

$$| x_n(s_n) - x_n(t_{mn}) |$$

$$= \left| \left(A_n + \int_0^{s_n} f(s, x_n(s))\, ds \right) - \left(A_n + \int_0^{t_{mn}} f(s, x_n(s))\, ds \right) \right|$$

$$= \left| \int_{t_{mn}}^{s_n} f(s, x_n(s))\, ds \right| \le \| f(s, x_n(s)) \|_\infty (s_n - t_{mn})$$

(where $\| y \|_\infty = \max_{t \in J} | y(t) |$). Now as $n \to \infty$, $s_n \to t^*$ and $t_{mn} \to t^*$, thus $rhs \to 0$. This yields that $x_n(s_n) \to \rho(t^*)$.

Let $t_{mn} > s_n$, for all $n \in N$, then, the fact that at $t = s_n, x_n(t)$ has the impulse effect, yields that $x_n(t_{mn}) \to \rho(t^{*+})$. Consider

$$| x_n(s_n^+) - \rho(t^{*+}) | \le | x_n(s_n^+) - x_n(t_{mn}) | + | x_n(t_{mm}) - p(t^{*+}) |,$$

the second term of the *rhs* goes to zero as $n \to \infty$, and we estimate the first term below.

$$|x_n(s_n^+) - x_n(t_{mn})| = \left| x_n(s_n^+) - (x_n(s_n^+) + \int_{s_n}^{t_{mn}} f(s, x_n(s))\, ds) \right|$$

$$\leq \| f(t, x_n(t)) \|_\infty (t_{mn} - s_n)$$

as $n \to \infty$, $t_{mn} \to t^*$, $s_n \to t^*$ and we obtain $|x_n(s_n^+) - x_n(t_{mn})| \to 0$. Thus, $x_n(s_n^+) \to \rho(t^{*+})$.

If $\{t_{mn}\}$ is any diagonalizing sequence, then we can always extract a sequence $\{t_{mnk}\}$ such that $t_{mnk} < s_{nk}$ and $t_{mnk} > s_{nr}$, $s_{nk} \neq s_{nr}$, $k, r \geq 0$. Then convergence follows from the preceding cases.

Since $v(t), w(t) \in PC^1[J, R], f \in C[J \times R, R]$, there exists a $M > 0$ such that

$$|x_n'(t)| = |f(t, x_n(t))| \leq \max_{v(t) \leq x \leq w(t),\, t \in J} |f(t, x)| = M.$$

Thus, using the Ascoli-Arzela theorem, together with the monotonicity of the given sequence, we conclude that $\{x_n\}_{n \in N}$ converges uniformly on J to ρ. Let $t \in J$, then

$$x_n(t) = A_n + \int_0^t f(s, x_n(s))\, ds + \psi_n(t)\, I(x_n(s_n))$$

where $\psi_n(t)$ is the characteristic function such that

$$\psi_n(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq s_n, \\ 1, & \text{if } s_n < t \leq T \end{cases}$$

Letting $n \to \infty$, on both sides, we get

$$\rho(t) = A^* + \int_0^t f(s, \rho(s))\, ds + \psi(t)\, I(\rho(t))$$

where $$\psi(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq t^*, \\ 1, & \text{if } t^* < t \leq T \end{cases}.$$

On the other hand, $B(x_n(0), x_n) \leq 0$ for all $n \in N$ and $\{x_n\}$ is nondecreasing. Hence, using the property of B, we have

$$B(\rho(0), \rho) = \lim_{Z \to \rho(0)} B(Z, \rho) = \lim_{n \to \infty} B(x_n(0), P)$$

$$\leq \lim_{n \to \infty} B(x_n(0), x_n) \leq 0$$

Now, if $w(0) = A^*$, we can prove that there exists a solution $x(t, A^*) \in [\rho, w]$ of (II_{A^*}). By the properties of B we obtain

$$0 \geq B(\rho(0), \rho) \geq B(x(0), x) \geq B(w(0), w) \geq 0.$$

Thus x is a oslution of (I) in $[\upsilon, w]$.

If $A^* < w(0)$ we take a nonincreasing sequences $\{B_n\}_{n \in N}$ in $(A^*, w(0)]$ such that $\lim_{n \to \infty} B_n = A^*$. Analogous arguments to the previous ones show that for every $n \in N$ there exists $y_n \in [\rho, w]$, a solution of (II_{B_n}) such that $B(y_n(0), y_n) > 0$ and the sequences $\{y_n\}_{n \in N}$ converges uniformly on J to some $r \in PC^1[J, R]$. Furthermore, we can prove that the sequence is nonincreasing.

Repeating the arguments we used concerning the sequence $\{x_n\}_{n \in N}$, we can prove that $r(t)$ hits the surface $S : t = \tau(x)$ at $t = \bar{t}^*$ and

$$r(t) = A^* + \int_0^t f(s, r(s))\, ds + \overline{\psi}(t)\, I(r(t^*)),$$

$$B(r, (0), r) \geq 0$$

where
$$\overline{\psi}(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \bar{t}^* \\ 1, & \text{if } \bar{t}^* < t \leq T \end{cases}$$

By Theorem 2.3, problem (II_{A^*}) has at least one solution $x \in [\rho, r]$. Now, using the properties of B, we have

$$0 \leq B(r(0), r) \leq B(x(0), x) \leq B(\rho(0), \rho) \leq 0.$$

Hence x is a solution of (I) in $[\rho, r] \subset [\upsilon, w]$.

Define

$$D = \{x \in [\upsilon, w] : x \text{ be a solution of (I)}\},$$

$$D_1 = \{l \in [\upsilon(0), w(0)], x \text{ be a solution of (I)}, x(0) = l\}$$

and
$$s_* = \inf_{l \in D_1} l, \quad s^* = \sup_{l \in D_1} l \quad .$$

Proceeding as before we can prove that $x(t, s_*) = \rho_0(t) \in D$, and $x(t, s^*) = r_0(t) \in D$, and $\rho_0(t), r_0(t)$ are minimal and maximal solutions of (I), respectively. Thus the proof is completed.

Remark 3.4 : Theorem 3.3 generalizes the main results of [3, Theorem 3.1], and improves the result of [2, Theorem 3.1].

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