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STOKES FLOW PAST A FLUID PROLATE SPHEROID

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The Stokes flow problem of a viscous incompressible fluid past a fluid prolate spheroid parallel to its axis of revolution is investigated. Explicit expressions are obtained for both the external and internal flow fields. The drag force experienced by the fluid prolate spheroid is evaluated. Special well-known cases and some new results are then deduced.

Key Words : Semiseparable Solution; Gegenbauer Function; Tangential Stress; Oblate Spheroid

1. INTRODUCTION

The solution of the problem of axisymmetric creeping flow past a solid prolate spheroid in an unbounded fluid was obtained by Payne & Pell[5] and is discussed in the book by Happel and Brenner[4]. Also, the problem of flow past a slightly deformed fluid sphere was studied by Ramkissoon[3]. Dassios, *et al*[1, 2] used an elegant method of semiseparation of variables for $E^4 \psi = 0$ to obtain a kind of semiseparable solution in prolate spheroidal coordinates and presented in terms of a complete full series expansion. This semiseparable solution is sufficiently general to handle internal and external fluid motions.

In this present work the problem of streaming flow past a fluid prolate spheroid parallel to its axis of revolution is examined. Explicit expressions are obtained for both the external and internal flow fields. The drag experienced by the fluid prolate spheroid is evaluated. Special well-known cases and some new results are then deduced.

2. STATEMENT AND SOLUTION OF THE PROBLEM

In prolate spheroidal coordinates (η, θ, ϕ), the complete solution of the equation,

$$E^4 \psi = E^2 (E^2 \psi) = 0, \qquad \text{... (2.1)}$$

where the operator E^2 as,

$$E^2 = \frac{1}{c^2 (\tau^2 - \zeta^2)} \left((\tau^2 - 1) \frac{\partial^2}{\partial \tau^2} + (1 - \zeta^2) \frac{\partial^2}{\partial \zeta^2} \right), \qquad \text{... (2.2)}$$

and $\tau = \cosh\eta,\ \zeta = \cos\theta,$ ... (2.3)

has been obtained by Dassios, G. *et al.* [1, 2] in semiseparable form as

$$\psi(\tau,\zeta) = \sum_{n=2}^{\infty} [g_n(\tau)\, G_n(\zeta) + h_n(\tau)\, H_n(\zeta)]. \quad \text{... (2.4)}$$

Here, the functions $g_n(\tau)$ and $h_n(\tau)$ are linear combinations of Gegenbauer functions of mix order $(n-2,\ n,\ n+2)$. $G_n(S)$ and $H_n(\zeta)$ denote Gogenbauer functions of the first and second kind, respectively, of degree (–1/2) and of order n. The expression (2.4) indicates that the equation of $E^4\psi = 0$ does not accept a complete separation of the τ and ζ variables in the usual sense. It is interesting that individual terms of the series are not solution of the eq. (2.1) but the full expansion is[2]. This is meant by the team "semiseparation". Since $G_n(\zeta)$ are regular on the z-axis, $H_n(\zeta)$ are singular on the z-axis and so only the terms $g_n(\tau)\, G_n(\zeta)$ occur in our solution. Furthermore,taking into account the symmetry of the ψ-field on the either side of equatorial plane ($\zeta = 0$), we retain only the even terms of the solution. Thus we have

$$\psi(\tau,\zeta) = \sum_{n=2}^{\infty} [g_n(\tau)\, G_n(\zeta)], \quad \text{... (2.5)}$$

where,

$$g_2(\tau) = A_1\, G_1(\tau) + C_2\, G_2(\tau) + D_2\, H_2(\tau) + A_4\, G_4(\tau) + B_4\, H_4(\tau), \quad \text{... (2.6a)}$$

$$g_n(\tau) = A_n\, G_{n-2}(\tau) + B_n\, H_{n-2}(\tau) + C_n\, G_n(\tau) + D_n\, H_n(\tau)$$
$$+ A_{n+2}\, G_{n+2}(\tau) + B_{n+2}\, H_{n+2}(\tau)$$

$$n = 4,\ 6,\ 8,\ \ldots. \quad \text{... (2.6b)}$$

Also, as provided an additional, equation by demanding that the solution for spheroid should tend to the corresponding solution for sphere as semifocal distance $c \to 0_+$, in such a way that no $r^{-3}\sin^2\theta$ terms appears. This condition fixes the values of the coefficient of $H_4(\tau)\, G_2(\zeta)$ in (2.6*a,b*) to nil, i.e., $B_4 = 0$.

We take up here the problem of streaming flow of an incompressible fluid of viscosity μ_c past a fluid prolate spheroid of viscosity μ_i, parallel to its axis of revolution. The prolate spheroid is asusmed to remain at rest while fluid streams past with velocity U in the direction of the negative z-axis.

In view of the axial symmetry and incompressibility of the flow we can express the components of velocity in terms of the Stokes stream function $\psi(\tau,\zeta)$ via.

$$v_\tau(\tau,\zeta) = \frac{1}{c^2\sqrt{(\tau^2-\zeta^2)}\,\sqrt{(\tau^2-1)}}\ \frac{\partial\psi(\tau,\zeta)}{\partial\zeta}, \quad \text{... (2.6c)}$$

$$v_\zeta(\tau,\zeta)=\frac{1}{c^2\sqrt{(\tau^2-\zeta^2)}\sqrt{(1-\zeta^2)}}\;\frac{\partial\psi(\tau,\zeta)}{\partial\tau}. \qquad \text{... (2.6d)}$$

There are two distinct fluid motions, namely the internal motion within the spheroid and the external motion of the flow past the spheroid. We shall use superscript (i) and (e) to distinguish between these separate motions occurring inside and outside the fluid spheroid respectively. We take the stream function in the exterior of spheroid to be,

$$\frac{\psi^{(e)}}{Uc^2}=[\{A_1G_1(\tau)+C_2G_2(\tau)+D_2H_2(\tau)+A_4G_4(\tau))\}G_2(\zeta)+\sum_{n=4,6,\ldots}^{\infty}$$

$$\{A_nG_{n-2}(\tau)+B_nH_{n-2}(\tau)+C_nG_n(\tau)+D_nH_n(\tau)+A_{n+2}G_{n+2}(\tau)$$

$$\{+B_{n+2}H_{n+2}(\tau)\}G_n(\zeta)\Big), \qquad \text{... (2.7)}$$

while in the interior of spheroid we take it as,

$$\frac{\psi^{(i)}}{Uc^2}=[\{A_1'G_1(\tau)+C_2'G_2(\tau)+D_2'H_2(\tau)+A_4'G_4(\tau)\}G_2(\zeta)+\sum_{n=4,6,\ldots}^{\infty}]$$

$$\{A_n'G_{n-2}(\tau)+B_n'H_{n-2}(\tau)+C_n'G_n(\tau)+D_n'H_n(\tau)\}$$

$$\Big\{+B_{n+2}'H_{n+2}(\tau)\Big\}G_n(\zeta)\Big]. \qquad \text{... (2.8)}$$

Now, using the condition,

$$\psi^{(e)}(\tau,\zeta)\to\frac{1}{2}Uc^2(\tau^2-1)(1-\zeta^2),\ \text{as}\ \tau\to\infty \qquad \text{... (2.9)}$$

and the fact that the components of velocity at the origin must be finite, the above representation (2.7) and (2.8) takes the form,

$$\frac{\psi^{(e)}(\tau,\zeta)}{Uc^2}=\Big[(-2G_2(\tau)+D_2H_2(\tau)+A_1G_1(\tau))G_2(\tau)$$

$$+\sum_{n=4,6,\ldots}^{\infty}(B_nH_{n-2}^{(t)}+D_nH_n(\tau)+B_{n+2}H_{n+2}(\tau))G_n(\zeta)\Big], \qquad \text{... (2.10)}$$

$$\frac{\psi^{(i)}(\tau,\zeta)}{Uc^2}=\Big[(C_2'G_2(\tau)+A_4'G_4(\tau))G_2(\zeta)+\sum_{n=4,6,\ldots}^{\infty}(A_n'G_{n-2}(\tau)$$

$$+C_n'G_n(\tau)+A_{n+2}'G_{n+2}(\tau))G_n(\zeta)\Big]. \qquad \text{... (2.11)}$$

To evaluate the unknown coefficients appearing in (2.10) & (2.11), we use the following boundary conditions. Since, the surfaces τ = constant are prolate spheroids, so we designate a particular spheroid by τ_0.

The kinematical condition of mutual impenetrability at the surface requires that we take $v_\tau = 0$ on $\tau = \tau_0$, i.e.,

$$\psi^{(e)}_{(\tau, \zeta)} = 0 \text{ at } \tau = \tau_0, \quad \text{... (2.12)}$$

$$\psi^{(i)}_{(\tau, \zeta)} = 0 \text{ at } \tau = \tau_0. \quad \text{... (2.13)}$$

Also, we assume that the tangential velocity is continuous across the surface. Hence,

$$\frac{\partial \psi^{(e)}(\tau, \zeta)}{\partial \tau} = \frac{\partial \psi^{(i)}(\tau, \zeta)}{\partial \tau} \text{ at } \tau = \tau_0. \quad \text{... (2.14)}$$

We further assume that the theory of interfacial tension is applicable to our problem. This means that the presence of interfacial tension only produces a discontinuity in the normal stress $T_{\tau\tau}$ and does not in anyway affect the tangential stress $T_{\tau\zeta}$. The latter is therefore continuous across the surface and so,

$$T^{(e)}_{\tau\zeta}(\tau, \zeta) = T^{(i)}_{\tau\zeta}(\tau, \zeta) \text{ at } \tau = \tau_0. \quad \text{... (2.15)}$$

where tangential stress in prolate spheroid co-ordinate is,

$$T_{\tau\zeta} = \frac{\mu}{c^3 \sqrt{1-\zeta^2}\sqrt{\tau^2-1}\,(\tau^2-\zeta^2)} \left[\left\{ (\tau^2-1)\frac{\partial^2 \psi}{\partial \tau^2} - (1-\zeta^2)\frac{\partial^2 \psi}{\partial \zeta^2} \right\} - \frac{2}{(\tau^2-\zeta^2)} \left\{ \tau(\tau^2-1)\frac{\partial \psi}{\partial \tau} + \zeta(1-\zeta^2)\frac{\partial \psi}{\partial \zeta} \right\} \right] \quad \text{... (2.16)}$$

Because of B.C. (2.12), (2.13); (2.15) can be written as,

$$\mu_e \left[(\tau^2-\zeta^2)\frac{\partial^2 \psi^{(e)}}{\partial \tau^2} - 2\tau \frac{\partial \psi^{(e)}}{\partial \tau} \right] = \mu_i \left[(\tau^2-\zeta^2)\frac{\partial^2 \psi^{(i)}}{\partial \tau^2} - 2\tau \frac{\partial \psi^{(i)}}{\partial \tau} \right]$$

at $\tau = \tau_0$. ... (2.17)

The boundary conditions (2.12)-(2.14) and (2.17) lead to the following values of the constants for $n = 2$,

$$A_1 = \frac{2\,[10\,\lambda\,\tau_0^2\,(\tau_0^2-1) + (15\,\tau_0^4+1)]}{\Delta_2}, \quad \text{... (2.18)}$$

$$D_2 = \frac{2\,[12\,\lambda\,\tau_0^2\,(\tau_0^2-1) + (15\,\tau_0^4+1)\,(1+\tau_0^2)]}{\Delta_2}, \quad \text{... (2.19)}$$

$$C_2' = \frac{4\,\lambda\,\tau_0\,(5\,\tau_0^2 - 1)^2}{5\,(\tau_0^2 - 1)\,\Delta_2}, \qquad \text{... (2.20)}$$

$$A_4' = -\frac{16\,\lambda\,\tau_0\,(5\,\tau_0^2 - 1)}{5\,(\tau_0^2 - 1)\,\Delta_2}, \qquad \text{... (2.21)}$$

where,

$$\Delta_2 = [4\,\lambda\,\tau_0^2\ \ \{3\,(\tau_0^2 - 1)\coth^{-1}\tau_0 + 2\,\tau_0\} + (15\,\tau_0^4 + 1)\,\{-\tau_0 + (\tau_0^2 + 1)\coth^{-1}\tau_0\}] \qquad \text{... (2.22)}$$

and $\lambda = \mu_c/\mu_i$ is viscosity ratio. For $n = 4$,

$$D_4 = \frac{1}{\Delta_4}\begin{bmatrix} 0 & H_6(\tau_0) & 0 & 0 \\ -A_4' G_2(\tau_0) & 0 & G_4(\tau_0) & G_6(\tau_0) \\ A_4' G_2'(\tau_0) & H_6'(\tau_0) & -G_4'(\tau_0) & -G_6'(\tau_0) \\ \begin{matrix}[(4/5)\,[\lambda\,(2 + D_2 H_2''(\tau_0) \\ +(C_2' + A_4' G_4''(\tau_0)] \\ +A_4'(\tau_0^2 + 7/35)]\end{matrix} & \begin{matrix}\lambda\,[(\tau_0^2 - 7/35)\,H_6''(\tau_0) \\ -2\,\tau_0 H_6'(\tau_0)]\end{matrix} & \begin{matrix}-[(\tau_0^2 - 7/35)\,G_4''(\tau_0) \\ -2\,\tau_0 G_4'(\tau_0)]\end{matrix} & \begin{matrix}-[(\tau_0^2 - 7/35)\,G_6''(\tau_0) \\ -2\,\tau_0 G_6'(\tau_0)]\end{matrix} \end{bmatrix} \qquad \text{... (2.23)}$$

$$B_6 = \frac{1}{\Delta_4}\begin{bmatrix} H_4(\tau_0) & 0 & 0 & 0 \\ 0 & -A_4' G_2(\tau_0) & G_4(\tau_0) & G_6(\tau_0) \\ H_4'(\tau_0) & A_4' G_2'(\tau_0) & -G_4'(\tau_0) & -G_6'(\tau_0) \\ \begin{matrix}\lambda\,[(\tau_0^2 - 7/35)\,H_6''(\tau_0) \\ -2\,\tau_0 H_6'(\tau_0)]\end{matrix} & \begin{matrix}[(4/5)\,[\lambda\,(2 + D_2 H_2''(\tau_0) \\ +(C_2' + A_4' G_4''(\tau_0)] \\ +A_4'(\tau_0^2 + 7/35)]\end{matrix} & \begin{matrix}-[(\tau_0^2 - 7/35)\,G_4''(\tau_0) \\ -2\,\tau_0 G_4'(\tau_0)]\end{matrix} & \begin{matrix}-[(\tau_0^2 - 7/35)\,G_6''(\tau_0) \\ -2\,\tau_0 G_6'(\tau_0)]\end{matrix} \end{bmatrix} \qquad \text{... (2.24)}$$

$$C_4^1 = \frac{1}{\Delta_4}\begin{bmatrix} H_4(\tau_0) & H_6(\tau_0) & 0 & 0 \\ 0 & 0 & -A_4' G_2(\tau_0) & G_6(\tau_0) \\ H_4'(\tau_0) & H_6'(\tau_0) & A_4' G_2'(\tau_0) & -G_6'(\tau_0) \\ \begin{matrix}\lambda\,[(\tau_0^2 - 7/35)\,H_4''(\tau_0) \\ -2\,\tau_0 H_4'(\tau_0)]\end{matrix} & \begin{matrix}\lambda\,[(\tau_0^2 - 7/35)\,H_6''(\tau_0) \\ -2\,\tau_0 H_6'(\tau_0)]\end{matrix} & \begin{matrix}[(4/5)\,[\lambda\,(2 + D_2 H_2''(\tau_0) \\ +(C_2' + A_4' G_4''(\tau_0)] \\ +A_4'(\tau_0^2 + 7/35)]\end{matrix} & \begin{matrix}-[(\tau_0^2 - 7/35)\,G_6''(\tau_0) \\ -2\,\tau_0 G_6'(\tau_0)]\end{matrix} \end{bmatrix} \qquad \text{... (2.25)}$$

$$A_6' = \frac{1}{\Delta_4}\begin{bmatrix} H_4(\tau_0) & H_6(\tau_0) & 0 & 0 \\ 0 & 0 & G_4(\tau_0) & -A_4' G'(\tau_0) \\ H_4'(\tau_0) & H_6'(\tau_0) & -G_4'(\tau_0) & A_4' G_2'(\tau_0) \\ \begin{array}{l}\lambda[(\tau_0^2 - 7/35) H_4''(\tau_0) \\ -2\tau_0 H_4'(\tau_0)]\end{array} & \begin{array}{l}\lambda[(\tau_0^2 - 7/35) H_6''(\tau_0) \\ -2\tau_0 H_6'(\tau_0)]\end{array} & \begin{array}{l}-[(\tau_0^2 - 7/35) G_4''(\tau_0) \\ -2\tau_0 G_4'(\tau_0)]\end{array} & \begin{array}{r}[(4/5)[\lambda(2 + D_2 H_2''(\tau_0) \\ +(C_2' + A_4' G_4''(\tau_0)] \\ +A_4'(\tau_0^2 + 7/35)]\end{array} \end{bmatrix} \quad \ldots (2.26)$$

where,

$$\Delta_4 = \begin{vmatrix} H_4(\tau_0) & H_6(\tau_0) & 0 & 0 \\ 0 & 0 & G_4(\tau_0) & G_6(\tau_0) \\ H_4'(\tau_0) & H_6'(\tau_0) & -G_4'(\tau_0) & -G_6'(\tau_0) \\ \begin{array}{l}\lambda[(\tau_0^2 - 7/35) H_4''(\tau_0) \\ -2\tau_0 H_4'(\tau_0)]\end{array} & \begin{array}{l}\lambda[(\tau_0^2 - 7/35) H_6''(\tau_0) \\ -2\tau_0 H_6'(\tau_0)]\end{array} & \begin{array}{l}-[(\tau_0^2 - 7/35) G_4''(\tau_0) \\ -2\tau_0 G_4'(\tau_0)]\end{array} & \begin{array}{l}-[(\tau_0^2 - 7/35) G_6''(\tau_0) \\ -2\tau_0 G_6'(\tau_0)]\end{array} \end{vmatrix} \quad \ldots (2.27)$$

In general, for $n = 6, 8, 10, \ldots$

$$D_n = \frac{1}{\Delta_n}\begin{bmatrix} -B_n H_{n-2}(\tau_0) & H_{n+2}(\tau_0) & 0 & 0 \\ -A_n' G_{n+2}(\tau_0) & 0 & G_n(\tau_0) & G_{n+2}(\tau_0) \\ [A_n' G_{n-2}'(\tau_0) - B_n H_{n-2}'(\tau_0)] & H_{n+2}'(\tau_0) & -G_n'(\tau_0) & -G_{n+2}'(\tau_0) \\ \begin{array}{l}[((\tau_0^2 - \gamma_n) G_{n-2}''(\tau_0) - 2(\tau_0) G_{n-2}(\tau_0)] A_n' \\ -\lambda[(\tau_0^2 \gamma_n) H_{n-2}''(\tau_0 + B_n) \\ +\beta_{n-2}[\lambda(B_{n-2} H_{n-4}''(\tau_0) + D_{n-2} H_{n-2}''(\tau_0) + B_n H_n'' \tau_0] \\ -(A_{n-2}'(G_{n-4}''(\tau_0) + C_{n-2}' G_{n-2}''(\tau_0) + A_n' G_n''(\tau_0))]\end{array} & \begin{array}{l}\lambda[(\tau_0^2 - 2\tau_0 H_{n-2}^1 \gamma_n) H_{n+2}''(\tau_0) \\ -2\tau_0 H_{n+2}'(\tau_0)]\end{array} & \begin{array}{l}[(\tau_0^2 - \gamma_n) G_n''(\tau_0) \\ -2\tau_0 G_n'(\tau_0)]\end{array} & \begin{array}{l}[(\tau_0^2 - \gamma_n) G_{n+2}''(\tau_0) \\ -2\tau_0 G_{n+2}'(\tau_0)]\end{array} \end{bmatrix}$$

$$\ldots (2.28)$$

$$B_{n+2} = \frac{1}{\Delta_n}\begin{bmatrix} H_n(\tau_0) & -B_n H_{n-2}(\tau_0) & 0 & 0 \\ 0 & -A_n' G_{n-2}(\tau_0) & G_n(\tau_0) & G_{n+2}(\tau_0) \\ H_n'(\tau_0) & [A_n' G_{n-2}'(\tau_0) - B_n H_{n-2}'(\tau_0)] & -G_n'(\tau_0) & -G_{n+2}'(\tau_0) \\ \begin{array}{l}\lambda[(\tau_0^2 - \gamma_n) H_n''(\tau_0) \\ -2\tau_0 H_n'(\tau_0)]\end{array} & \begin{array}{l}[((\tau_0^2 - \gamma_n) G_{n-2}''(\tau_0) - 2\tau_0 G_{n-2}'(\tau_0)] A_n' \\ -\lambda[(\tau_0^2 - \gamma_n) H_{n-2}''(\tau_0) H_{n-2}'(\tau_0)] B_n \\ +\beta_{n-2}[\lambda(B_{n-2} H_{n-4}''(\tau_0) + D_{n-2} H_{n-2}''(\tau_0) + B_n H_n'' \tau_0] \\ -(A_{n-2}' G_{n-4}''(\tau_0) + C_{n-2}' G_{n-2}''(\tau_0) + A_n' G_n''(\tau_0))]\end{array} & \begin{array}{l}-[(\tau_0^2 - \gamma_n) G_n''(\tau_0) \\ -2\tau_0 G_n'(\tau_0)]\end{array} & \begin{array}{l}[(\tau_0^2 - \gamma_n) G_{n+2}''(\tau_0) \\ -2\tau_0 G_{n+2}'(\tau_0)]\end{array} \end{bmatrix}$$

$$\ldots (2.29)$$

$$
C'_{n+2}=\frac{1}{\Delta_n}\begin{bmatrix}
H_n(\tau)_0 & H_{n+2}(\tau_0) & -B_n H_{n-2}(\tau_0) & 0 \\
0 & 0 & -A'_n G_{n-2}(\tau_0) & G_{n+2}(\tau)_0 \\
H'_n(\tau)_0 & H'_{n+2}(\tau_0) & [A'_n G'_{n-2}(\tau_0)-B_n H'_{n-2}(\tau_0)] & -G'_{n+2}(\tau)_0 \\
\begin{array}{l}\lambda[(\tau_0^2-\gamma_n)H''_n(\tau)_0 \\ -2\tau_0 H'_n(\tau)_0]\end{array} &
\begin{array}{l}\lambda[(\tau_0^2-\gamma_n)H''_{n+2}(\tau_0) \\ -2\tau_0 H'_{n+2}(\tau_0)]\end{array} &
\begin{array}{l}[((\tau_0^2-\gamma_n)G'_{n-2}(\tau_0)-\tau_0-2\tau_0 G'_{n-2}(\tau_0)]A'_n \\ -\lambda[(\tau_0^2-\gamma_n)H''_{n-2}(\tau_0)-2\tau_0 H''_{n-2}(\tau_0)]B_n) \\ +\beta_{n-2}[\lambda(B_{n-2}H''_{n-4}(\tau_0)+D_{n-2}H''_{n-2}(\tau_0)+B_n H''_n(\tau_0)] \\ -(A'_{n-2}G''_{n-4}(\tau_0)+C'_{n-2}G''_{n-2}(\tau_0)+A'_n G''_n(\tau_0))]\end{array} &
\begin{array}{l}-[(\tau_0^2-\gamma_n)G_n(\tau)_0 \\ -2\tau_0 G_n(\tau)_0]\end{array}
\end{bmatrix}
$$

... (2.30)

$$
A'_{n+2}=\frac{1}{\Delta_n}\begin{bmatrix}
H_n(\tau_0) & H_{n+2}(\tau_0) & 0 & -B_n H_{n-2}(\tau_0) \\
0 & 0 & G_n(\tau_0) & -A'_n G_{n-2}(\tau_0) \\
H'_n(\tau_0) & H'_{n+2}(\tau_0) & -G'_n(\tau_0) & [A'_n G'_{n-2}(\tau_0)-B_n H'_{n-2}(\tau_0)] \\
\begin{array}{l}\lambda[(\tau_0^2-\gamma_n)H''_n(\tau_0) \\ -2\tau_0 H'_n(\tau_0)]\end{array} &
\begin{array}{l}\lambda[(\tau_0^2-\gamma_n)H''_{n+2}(\tau_0) \\ -2\tau_0 H'_{n+2}(\tau_0)]\end{array} &
\begin{array}{l}-[(\tau_0^2-\gamma_n)G''_n(\tau_0) \\ -2\tau_0 G'_n(\tau_0)]\end{array} &
\begin{array}{l}[((\tau_0^2-\gamma_n)G''_{n-2}(\tau_0)-2\tau_0 G'_{n-2}(\tau_0)]A'_n \\ -\lambda[(\tau_0^2-\gamma_n)H''_{n-2}(\tau_0)-2\tau_0 H''_{n-2}(\tau_0)]B_n) \\ +\beta_{n-2}[\lambda(B_{n-2}H''_{n-4}(\tau_0)+D_{n-2}H''_{n-2}(\tau_0)+B_n H''_n(\tau_0)] \\ -(A'_{n-2}G''_{n-4}(\tau_0)+C'_{n-2}G''_{n-2}(\tau_0)+A'_n G''_n(\tau_0))]\end{array}
\end{bmatrix}
$$

... (2.31)

$$
\Delta_n=\begin{vmatrix}
H_n(\tau_0) & H_{n+2}(\tau_0) & 0 & 0 \\
0 & 0 & G_n(\tau_0) & G_{n+2}(\tau_0) \\
H'_n(\tau_0) & H'_{n+2}(\tau_0) & -G'_n(\tau_0) & -G'_{n+2}(\tau_0) \\
\begin{array}{l}\lambda[(\tau_0^2-\gamma_n)H''_n(\tau_0) \\ -2\tau_0 H'_n(\tau_0)]\end{array} &
\begin{array}{l}\lambda[(\tau_0^2-\gamma_n)H''_{n+2}(\tau_0) \\ -2\tau_0 H'_{n+2}(\tau_0)]\end{array} &
\begin{array}{l}-[(\tau_0^2-\gamma_n)G''_n(\tau_0) \\ -2\tau_0 G'_n(t_0)]\end{array} &
\begin{array}{l}[(\tau_0^2-\gamma_n)G''_{n+2}(\tau_0) \\ -2\tau_0 G'_{n+2}(\tau_0)]\end{array}
\end{vmatrix}, \qquad \text{... (2.32)}
$$

where lower order terms are known from previous step and the dashes in the functions denote the order of derivatives w.r.t. to τ. Also, we have used the following recurrence relations :

$$\zeta^2 G_2(\zeta)=\frac{1}{5}G_2(\zeta)+\frac{4}{5}G_4(\zeta), \qquad \text{... (2.33 } a)$$

$$\zeta^2 G_2(\zeta)=\alpha_n G_{n-2}(\zeta)+\gamma_n G_n(\zeta)+\beta_n G_{n+2}(\zeta), \qquad \text{... (2.33}b)$$

$$\alpha_n=\frac{(n-3)(n-2)}{(2n-3)(2n-1)}, \quad n\geq 4, \qquad \text{... (2.34}a)$$

$$\beta_n=\frac{(n+1)(n+2)}{(2n-1)(2n+1)}, \quad n\geq 4, \qquad \text{... (2.34b)}$$

$$\gamma_n = \frac{(2n^2 - 2n - 3)}{(2n+1)(2n-3)} \qquad n \geq 4, \qquad \text{... (2.34c)}$$

We have thus determined the stream functions for both the external and internal flow fields (2.10), (2.11) respectively and the constants have been all determined.

We now propose to examine a feature of this flow which is of most practical significance - the force experienced by the prolate spheroid. Thus, the force exerted by the fluid on the prolate spheroid can be determined by applying the simple elegant formula[5].

$$F_z = 8\,\pi\,\mu_e\,c \lim_{\tau \to \infty} \left(\frac{\tau(\psi^{(e)} - \psi^{(\infty)})}{\overline{\omega}^2} \right) \qquad \text{... (2.35)}$$

Thus, for the present case, provides

$$F_z = -\,4\,\pi\,\mu_e\,cUA_1, \qquad \text{... (2.36)}$$

where A_1 is given by (2.18).

The following special cases can be deduced immediately:

[a] A RIGID PROLATE SPHEROID ($\lambda = \mu_e/\mu_i = 0$):

For this case, we have

$$F_z = -\frac{8\,\pi\,\mu_e\,cU}{[(\tau_0^2 + 1)\coth^{-1}\tau_0 - \tau_0]}, \qquad \text{... (2.37)}$$

where semifocal distance $c = \sqrt{a^2 - b^2}$, a & b being the polar and equatorial radii, respectively and

$$\tau_0 = \cosh\eta_0 = a/c = [1 - (b/a)^2]^{-1/2}, \qquad \text{... (2.37a)}$$

so that (2.37) becomes

$$F_z = -\,6\,\pi\,\mu_e\,b\,U\,K_R, \qquad \text{... (2.38)}$$

where

$$K_R = \frac{1}{(3/4)\sqrt{(\tau_0^2 - 1)}\,[(\tau_0^2 + 1)\coth^{-1}\tau_0 - \tau_0]}. \qquad \text{... (2.38a)}$$

When $c = \sqrt{a^2 - b^2} \to 0_+$ i.e. $b = a\ (\tau_0 \to \infty)$ and then $K_R = 1$ and hence

$$F_z = -\,6\,\pi\,\mu_e\,bU. \qquad \text{... (2.39)}$$

This result was reported by Happel & Brenner[4] for the flow past a rigid sphere.

[b] A GASEOUS PROLATE SPHEROIDAL BUBBLE ($\mu_c \gg \mu_i$ i.e. $\lambda \to \infty$):

This gives rise to a new result

$$F_z = -\,4\pi\mu_e bUK_G, \qquad \text{... (2.40)}$$

where

$$K_G = \frac{1}{(1/5)\sqrt{(\tau_0^2 - 1)}\,[3\coth^{-1}\tau_0 + 2\tau_0/(\tau_0^2 - 1)]}$$

Whence $c \to 0_+$, i.e., $b = a$ or $(\tau_0 \to \infty)$ and then $K_G = 1$ and hence

$$F_z = -4\pi\mu_e bU, \qquad \text{... (2.41)}$$

which agrees with the result for a gaseous spherical bubbles as reported by Happel & Brenner[4].

Now, using the transformation,

$$\tau \to i\lambda, \qquad c \to -i\bar{c}, \qquad \text{... (2.42)}$$

we can also obtain the result for a oblate spheroid.

[c] A RIGID OBLATE SPHEROID ($\lambda = \mu_e/\mu_i = 0$):

From (2.37), we find

$$F_z = \frac{8\pi\mu_e \bar{c}\, U}{[\lambda_0 - (\lambda_0^2 - 1)\cot^{-1}\lambda_0]}, \qquad \text{... (2.43)}$$

where $\bar{c} = \sqrt{a^2 - b^2}$ and $\lambda_0 = b/c = [(a/b)^2 - 1]^{-1/2}$, so that we can write (2.43) as

$$F_z = -6\pi\mu_e aUK_R', \qquad \text{... (2.44)}$$

where,

$$K_R' = \frac{1}{3/4\sqrt{(\lambda_0^2 + 1)}\,[\lambda_0 - (\lambda_0^2 - 1)\cot^{-1}\lambda_0]}. \qquad \text{... (2.44a)}$$

This result has been also reported by Happel & Brenner4 for flow past a rigid oblate spheroid.

[d] A GASEOUS OBLATE SPHEROIDAL BUBBLE ($\mu_e > \mu_i$, $\lambda \to \infty$):

$$F_z = -4\,\pi\,\mu_e\, a\, UK_G', \qquad \text{... (2.45)}$$

where,

$$K_G' = \frac{1}{(1/5)\sqrt{(\lambda_0^2 + 1)}\,[3\cot^{-1}\lambda_0 + 2\,\lambda_0/(\lambda_0^2 + 1)]}. \qquad \text{... (2.46)}$$

When semifocal distance $\bar{c} \to 0_+$, i.e., $(\lambda_0 \to \infty)$ then $K_G' = 1$ and hence $F_z = -4\,\pi\,\mu_e\, aU$, a result reported by Happel and Brenner[4] for the case of a gaseous spherical bubble.

The result (2.45) may be accepted with reservation, since because of the boundary condition (2.17), higher order terms ignored, as in Dassios[1], seem also to contribute.

REFERENCES

1. G. Dassios, M. Hadjinicolaou, F. A. Coutelieris and A. C. Payatakes, *Int. J. Engng. Sci.*, Vol. **33**, No. 10, 1465-90 (1995).
2. G. Dassios, M. Hadjinicolaou and A. C. Payatakes, *Quart. J. Appl. Math.*, Vol. **52**, No. 1, 157-91 (1994)
3. H. Ramkissoon, *ZAMP*, Vol. **37** 859-66 (1986)
4. J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics* Prentice Hall, Engle wood cliffs, N.J. (1965).
5. L. E. Payne and W. H. Pell, *J. Fluid Mech.*, Vol. **7**, 529-49, (1960)