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ALMOST FIXED POINT THEOREMS OF THE FORT TYPE

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We obtain two generalizations of the almost fixed point theorems due to Fort and Smart. The first one is for an upper (or lower) semicontinuous multimap $T : \multimap \overline{X}$ with totally bounded range, where X is a convex subset of a locally convex topological vector space. The second one is for a large class of multimaps (so called the better admissible multimaps) defined on balls in normed vector spaces.

Key Words : KKM Principle; KKM Map; Locally Convex tv.s.; Fixed Point; Better Admissible Multimaps $\mathfrak{B}$; Almost Fixed Point

0. INTRODUCTION

In 1954, M. K. Fort, Jr.[2] showed that any open disk in the Euclidean plane has the almost fixed point property. In 1963, T. van der Walt[17] applied Fort's theorem to show that, for every continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a finite cover α of $\mathbb{R}^2$ by convex open sets, there exists a member $U \in \alpha$ such that $U \cap f(U) \neq \phi$. Moreover, in 1993, D. R. Smart[6] extended Fort's theorem to any normed vector spaces in several ways and suggested that it would hold for a suitable multimap.

Our aim in this paper is to show that Smart's results can be generalized to convex subsets of locally convex topological vector spaces or of metrizable topological vector spaces whose open balls are convex. Further, we show that Fort's theorem holds for a very large class of multimaps (so called the better admissible maps due to the present author[11-13] defined on balls in a normed vector space.

In Section 1, we obtain generalizations of the results of Fort and Smart to convex subsets of locally convex topological vector spaces. Our argument is based on the celebrated KKM theorem due to K. Fan[1].

Section 2 deals with a generalization of Kirk's theorem[6] and generalized versions of Fort's theorem for better admissible multimaps on balls of a normed vector space. We use one of the author's fixed point theorem on such type of multimaps in[11-13].

1. FOR LOCALLY CONVEX SPACES

In this section, we obtain generalized forms of almost fixed point theorems of Fort[2] and Smart[16] for convex subsets of locally convex topological vector spaces.

Recall that, for topological spaces X and Y, a multimap $T : X \multimap Y$ is said to be *closed* if its graph $Gr(T) = \{(x, y) : x \in X, y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the range $T(X)$ is contained in a compact subset of Y.

Our argument is based on the following Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem[7].

KKM Principle — Let D be the set of vertices of a simplex S and $F : D \multimap S$ a multimap with closed values such that

$$co\, N \subset F(N) \text{ for each } N \subset D.$$

Then

$$\bigcap_{z \in D} F(z) \neq \phi.$$

It is well-known that the following easily follows from the KKM principle; see Fan[1] and Park[14].

Lemma 1.1 — Let X be a subset of a topological vector space, D a nonempty subset of X such that $co\, D \subset X$, and $F : D \multimap X$ a *KKM* map with closed (resp. open) values. Then $\{F(z)\}_{z \in D}$ has the finite intersection property.

Note that a map $F : D \multimap X$ is called a *KKM* map of

$$co\, N \subset F(N) \text{ for each finite subset } N \text{ of } D.$$

From now on, t.v.s. means topological vector space.

From Lemma 1.1, we have our main result in this section as follows :

Theorem 1.2 — *Let X be a nonempty subset of a t.v.s. E, V a convex neighborhood of 0 in E, and $T : X \multimap E$ a multimap with convex values. Suppose that there is a finite subset $D := \{x_1, x_2, \ldots, x_n\}$ of X such that co $D \subset X$ and $T(X) \subset \bigcup_{i=1}^{n} (x_i + V)$. If one of the following conditions is satisfied* :

(1) T is upper semicontinuous and V is closed,

(2) T is lower semicontinuous and V is open,

then T has a V-fixed point; that is, there exists a point $x_0 \in X$ such that $T(x_0) \cap (x_0 + V) \neq \phi$.

PROOF : We will prove the result only for the case (1). A similar argument establishes the result for the case (2). Suppose that $T : X \multimap E$ is an upper semicontinuous multimap with convex values and V is a convex closed neighborhood of 0 in E. Define a multimap $F : D \multimap X$ by

$$F(x_i) := \{x \in X : T(x) \cap (x_i + V) = 0\} \text{ for each } x_i \in D.$$

Then F has open values in X since T is upper semicontinuous. Note that

$$= \bigcap_{i=1}^{n} F(x_i) : \left\{ x \in X : T(x) \cap \bigcup_{i=1}^{n} (x_i + V) = \phi \right\} = \phi .$$

By Lemma 1.1, $F : D \multimap X$ is not a KKM map; that is, there exists a finite subset $A := \{y_1, y_2, ..., y_m\}$ of D such that co $A \not\subset F(A)$. Hence there is an $x_0 \in co\, A$ such that $x_0 \notin F(y_j)$ or $T(x_0) \cap (y_j + V) \neq 0$ for all $j = 1, 2, ..., m$. Let $x_0 = \sum_{j=1}^{m} r_j y_j$ with $0 \le r_j \le 1$ and $\sum_{j=1}^{m} r_j = 1$. Since $z_j \in T(x_0) \cap (y_j + V)$ for some z_j, $j = 1, ..., m$ and the sets $T(x_0)$ and V are convex, we conclude that

$$z_0 := \sum_{j=1}^{m} r_j z_j \in T(x_0) \cap (x_0 + V); \text{ that is, } T(x_0) \cap (x_0 + V) \neq \phi.$$

This completes the proof. □

Corollary 1.3 — Let X be a nonempty subset of a t.v.s. E, V a convex open (or closed) neighborhood of 0 in E, and $f: X \to E$ a continuous map. If there is a finite subset $D := \{x_1, x_2, ..., x_n\}$ of X such that co $D \subset X$ and $f(X) \subset \bigcup_{i=1}^{n} (x_i + V)$, then f has a V-fixed point $x_0 \in X$; that is, $f(x_0) \in x_0 + V$.

Corollary 1.4 — Let X be a nonempty convex subset of a t.v.s. E and $f: X \to X$ a continuous map. If X is totally bounded, then f has a U-fixed point $x_U \in X$ for any convex neighborhood U of 0 in E.

PROOF : Since X is totally bounded and $f(X) \subset X$, the conclusion follows immediately from Corollary 1.3. □

Theorem 1.5 — *Let X be a nonempty convex subset of a t.v.s E and $T: X \multimap \overline{X}$ an upper semicontinuous multimap with convex values such that T(X) is totally bounded. Then for any convex closed neighborhood U of 0 in E, T has a U-fixed point $x_U \in X$.*

Similarly, if T is lower semicontinuous, then T has a U-fixed point for any convex open neighborhood U of 0 in E.

PROOF : By symmetry, it suffices to show the result for the upper semicontinuous multimap T. Let $T: X \multimap \overline{X}$ be an upper semicontinuous multimap with convex values and U a convex closed neighborhood of 0 in E. Then there exists a neighborhood V of 0 in E such that $V + V \subset U$. Since $T(X)$ is totally bounded, there is a finite subset $\{y_1, y_2, ..., y_n\}$ of $T(X)$ such that $T(X) \subset \bigcup_{i=1}^{n} (y_i + V)$. For each $i \in \{1, 2, ..., n\}$ we can choose an $x_i \in X$ such that $y_i - x_i \in V$. From this it follows that

$$T(X) \subset \bigcup_{i=1}^{n} (y_i + V) \subset \bigcup_{i=1}^{n} (x_i + V + V) \subset \bigcup_{i=1}^{n} (x_i + U).$$

By Theorem 1.2, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \phi$. This completes the proof. □

Corollary 1.6 — Let X be a nonempty convex subset of a *t.v.s* E and $f: X \rightarrow \overline{X}$ a continuous map suchthat $f(X)$ is totally bounded. Then for any convex neighborhood U of 0 in E, there exists a point $x_U \in X$ such that $f(x_U) \in x_U + U$.

Example 1.7 — [1]If E is locally convex or a metrizable t.v.s. whose balls are convex, then Corollaries 1.4 and 1.6 hold for any neighborhood U of 0 in E.

Note that Smart[16] obtained Corollaries 1.4 and 1.6 for the following cases :

(2) X is an open ball in $\mathbb{R}^n$.

(3) X is an open ball in a normed vector space.

Earlier Fort[2] obtained Corollary 1.4 when

(4) X is an open disk in $\mathbb{R}^2$.

Corollary 1.6 does not guarantee the existence of fixed points of f.

Example 1.8 — Let $X = \{(x, y) : x^2 + y^2 < 1\}$ be the open unit disk in $\mathbb{R}^2$ and $f: X \rightarrow \overline{X}$ defined by $f(x, y) := (x, \sqrt{1 - x^2})$ for each $(x, y) \in X$. Then the continuous map f has no fixed point.

However, we have the following celebrated Himmelberg fixed point theorem from Theorem 1.5.

Theorem 1.9 — *Let X be a nonempty convex subset of a locally convex Hausdorff t.v.s. E and $T: X \multimap X$ a compact upper semicontinuous multimap with convex closed values. Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

PROOF : For any convex closed neighborhood U of 0 in E, by Theorem 1.5, there exists a point $x_U \in X$ such that $T(x_U) \cap (x_U + U) \neq \phi$, say $y_U \in T(x_U) \cap (x_U + U)$. Since T is compact and $y_U \in \overline{T(X)} \subset X$, we may suppose that the net $\{y_U\}$ converges to some point $x_0 \in X$. By the Hausdorffness of X, the net $\{x_U\}$ also converges to x_0. Since the graph of T is closed, we have $x_0 \in T(x_0)$. This completes the proof. □

Corollary 1.10 — Let X be a nonempty convex subset of a locally convex Hausdorff *t.v.s.* E and $f: X \rightarrow X$ a compact continuous map. Then f has a fixed point.

Corollary 1.10 is due to Hukuhara[3] with different proof, and includes fixed point theorem due to Brouwer (for an n-simplx X), Schauder (for a normed vector space E), and Tychonoff (for a compact convex subset X).

We have one more

Corollary 1.11 — Let X be a nonempty convex subset of a metrizable *t.v.s.* E whose balls are convex and $f: X \rightarrow X$ a compact continuous map. Then f has a fixed point.

If X itself is compact, then Corollary 1.11 reduces to a result of Rassias[15]..

Note that, since the KKM principle is equivalent to the Brouwer fixed point theorem, each of Corollaries 1.3, 1.4, 1.6, 1.10 and 1.11 is also equivalent to the Brouwer theorem,

2. For Normed Vector Spaces

In this section, we first obtain a generalization of a theorem due to Kirk[6] as an application of Corollary 1.3.

Theorem 2.1 — *Let X be a convex subset of a normed vector space E and $f: X \to N_\varepsilon(X) := \{ y \in E : \inf\{ \| x - y \| : x \in X \} \le \varepsilon \}$ a continuous map which has totally bounded range, where ε is a positive real number. Then $\inf \{\| x - f(x) \| : x \in X\} \le \varepsilon$.*

PROOF : For any natural n, $V_n := \{z \in E : \| z \| \le \varepsilon + 1/n \}$. Since $f(X)$ is covered by the family $\{x + V_n : x \in X\}$ and totally bounded, there exists a finite subset $\{x_1, x_2, \ldots, x_k\}$ of X such that $f(X) \subset \bigcup_{i=1}^{k} (x_i + V_n)$. Therefore, by Corollary 1.3, f has V_n-fixed point $\overset{n}{x_0} \in X$; that is, $f(\overset{n}{x_0}) \in \overset{n}{x_0} + V_n$ or $\| \overset{n}{x_0} - f(\overset{n}{x_0}) \| \le \varepsilon + 1/n$. Therefore, we have the conclusion. □

Note that Kirk[6] obtained Theorem 2.1 for the case when X is a closed convex subset of a Banach space.

Finally, we obtain generalized versions of Fort's theorem for better admissible multimaps on balls of normed vector spaces.

Let X be a nonempty convex subset of a t.v.s. E and Y a topological space. A *polytope* P in X is any convex hull of a nonempty finite subset of X, or a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

The "*better*" admissible class $\mathcal{B}$ of multimaps is defined as follows[8-14]

$F \in \mathcal{B}(X, Y) \Leftrightarrow F : X \multimap Y$ is a multimap such that for any polytope P in X and any continuous map $f: F(P) \to P, f \circ (F|_P) : P \multimap P$ has a fixed point.

Subclasses of $\mathcal{B}$ are classes of continuous functions $\mathbb{C}$, the Kakutani maps (upper semicontinuous with nonempty compact convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (upper semicontinuous with R_δ values), the acyclic map $\mathbb{V}$ (upper semicontinmuous with compact acyclic values), the Powers maps $\mathbb{V}_c$ (finite compositions of acyclic maps), the O'Neill maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are uniform spaces), admissible maps of Górniewicz, σ-selectionable maps of Haddad and Larsy, permissible maps of Dzedzej, the class $\mathbb{K}_c^+$ of Lassonde, the class $\mathbb{V}_c^+$ of Park *et al.*, and approximable maps of Ben-El-Mechaiekh and Idzik, and many others. Those subclasses are examples of the admissible class $\mathfrak{U}_c^\kappa$ due to the author. Some examples of maps in $\mathcal{B}$ not belonging to $\mathfrak{U}_c^\kappa$ were given recently. For details, see[8-14].

Lemma 2.2 — Let X be a convex subset of a locally convex Hausdorff *t.v.s.* E and $\Phi \in \mathcal{B}(X, X)$. If Φ is closed and compact, then Φ has a fixed point.

This result is due to the second author[11], and contains fixed point theorems due to Himmelberg, Lassonde, Park, Park-Singh-Watson, Chang and Yen, and many others; see[10-14].

***Theorem* 2.3** — *Let E be a normed vector space and $B = \{x \in E : \|x\| < d\}$ for some $d > 0$. Let $\Phi \in \mathcal{B}(B, \overline{B})$ be a closed map. If Φ maps each smaller concentric ball to a compact set in $\overline{B}$, then for any $\varepsilon > 0$, there exists an $x_0 \in B$ and a $y_0 \in \Phi(x_0)$ such that $\|x_0 - y_0\| \le \varepsilon$.*

PROOF : Let $\varepsilon > 0$ be given. We may assume $\varepsilon < d$. Let $C = \{x \in B : \|x\| \le d - \varepsilon\}$, and define a retraction $r : B \to C$ by

$$r(x) = \begin{cases} (d-\varepsilon)\, x/\|x\| & \text{for } x \in \overline{B} \setminus C \\ x & \text{for } x \in C \end{cases}$$

Then $\Phi' := r \circ \Phi|_C \in \mathcal{B}(C, C)$ and Φ' is compact and closed since r is continuous and $\Phi|_C$ is upper semicontinuous. Therefore, by Lemma 2.2, there exists a point $x_0 \in C \subset B$ such that $x_0 \in \Phi'(x_0) = r \circ \Phi(x_0)$. Hence, there exists a $y_0 \in \Phi(x_0) \subset \overline{B}$ such that $r(y_0) = x_0$. Since $\|r(x) - x\| \le \varepsilon$ for all $x \in \overline{B}$ and $y_0 \in \overline{B}$, we have $\|x_0 - y_0\| = \|r(y_0) - y_0\| \le \varepsilon$. This completes the proof. □

Corollary 2.4 — Let $B^n = \{x \in I\!R^n : \|x\| < d\}$ for some $d > 0$, and let $f : B^n \to \overline{B}^n$ be continuous. Then for each $\varepsilon > 0$, there exists a point $x \in B^n$ such that $\|x - f(x)\| \le \varepsilon$.

Example 2.5 — (1) For $n = 2$ and $f : B^2 \to B^2$, Corollary 2.4 is first obtained by Fort[2] with different proof.

(2) For $f : B^n \to B^n$, Corollary 2.4 is obtained by van der Walt[17] who applied his result to show that the Euclidean plane $I\!R^2$ has the almost fixed property with respect to continuous maps and finite covers by convex open sets (that is, for every continuous $f : I\!R^2 \to I\!R^2$ and a finite cover α of $I\!R^2$ by convex open sets, there exists a member $U \in \alpha$ such that $U \cap f(U) \ne \phi$). This fact was extended by Hazewinkel and van de Vel[4] to any locally convex t.v.s. instead of $I\!R^2$. Some related results can be seen in Idzik[5].

(3) Smart[16] suggested that Corollary 2.4 would hold for suitable multimaps.

Finally, note that Corollary 2.4 follows also from Corollary 1.6.

REFERENCES

1. K. Fan, *Math. Ann.*, **142** (1961), 305-10.
2. M. K. Fort, Jr., *J. Indian Math. Soc.*, **18** (1954), 24-26.
3. M. Hukuhara, *Jap. J. Math.*, **20** (1950), 1-4.
4. M. Hazewinkel and M. van de Vel, *Canad. J. Math.*, **30** (1978), 673-99.
5. A. Idzik, *Proc. Amer. Math. Soc.*, **104** (1988), 779-84.
6. W. A. Kirk, Approximate fixed points for mappings in Banach spaces,. Nonlinear Functional Analysis and Its Applications (S.P. Singh, Ed.), D. Reidel Publ. Co, Dordrecht, 1986, pp. 299-303.
7. B. Knaster, K. Kuratowski and S. Mazurkiewicz, *Fund. Math.*, **14** (1929), 132-37.

8. S. Park, *C. R. Math. Rep. Acad. Sci., Canada* **15** (1993), 125-30.
9. S. Park, *J. Korean Math. Soc.*, **31** (1994), 493-519.
10. S Park, Eighty years of the Brouwer fixed point theorem, Antipodal Points and Fixed Points (ed. J. Jaworowski, W. A. Kirk, and S. Park), Lect. Notes Ser. 28, RIM-GARC, Seoul Nat. Univ., 1995, pp 55-97.
11. S. Park, *Nonlinear Anal.*, **303** (1997), 4183-91.
12. S. Park, *Math. Sci. Res. Hot-Line*, **1**(9) (1997), 1-6.
13. S. Park, *J. Korean Math. Soc.*, **35** (1998), 803-29. Corrections, ibid. **36** (1999), 829-32.
14. S. Park, *Vietnam J. Math.*, **27** (1999), 187-22.
15. T. M. Rassias, *Tamkang J. Math.*, **8** (1997), 233-37.
16. D. R. Smart, *Exposition. Math.*, **11** (1993), 81-90.
17. T. van der Walt., *Fixed and Almost Fixed Points*, Mathematisch Centrum, Amsterdam, 1963.