Indian J. pure appl. Math., **34**(5) : 773-784, May 2003

FIXED POINT THEOREMS OF INCREASING OPERATORS AND APPLICATIONS TO NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS TERMS*

JINQING ZHANG

Institute for Financial Studies, Fudan University, Shanghai 200 433, *People's Republic of China*

(*Received* 29 *April* 1999; *accepted* 3 *September* 2002)

In this paper, we give some new existence theorems of the maximal and minimal fixed points for discontinuous increasing operators. As applications, we study the maximal and minimal solutions of the periodic boundary value problem for second order nonlinear integro-differential equations with discontinuous terms in Banach spaces. Our conclusions unify and improve many well-known results.

1. INTRODUCTION

Let E be a real Banach space with norm $\|\cdot\|$, $I = [a, b] \subset R^1$ with $a < b$. $C[I, E]$ denotes the set of all continuous functions defined on I with values in E. Clearly $C[I,E]$ is a Banach space with norm $\|x\|_C = \max_{t \in I} \|x(t)\|$. Let E be partially ordered by a cone P of E, i.e., $x \leq y$ if and only if $y - x \in P$. The partial ordering in $C[I, E]$ is given by the cone P in E as follows: for $u, v \in C[I, E]$, $u \leq v$ if and only if $u(t) \leq v(t)$ for any $t \in I$. In this paper, A is assumed an increasing operator which may be expressed as the form $\sum_{i=1}^{m} K_i F_i$, where D is an ordering operator interval of $C[I, E]$.. We first prove some new existence theorems of fixed points on A. The difference between this note and the well-known references (see[1-6]) is that we do not assume any continuity on A, and even relax the compactness conditions greatly. What is more important is that we only need to verify the compactness in E rather than in such spaces as $C[I, E]$ or $L_p[I, E]$ when E is a Banach space (see[1-7]), while it is easier to do directly in E, and even in many cases, the compactness in E is satisfied automatically (see § 3). As applications, we prove the existence of the minimal and maximal solutions of the periodic boundary value problem for second order nonlinear integro-differential equations in E with discontinuous terms.

For the sake of clarity, we give other notations and concepts. For any $p \geq 1$, set

$$L_p[I, E] = \{x(t) : I \to E \mid x(t) \text{ is strongly measurable and } \int_I \|x(t)\|^p \, dt < \infty\},$$

then $L_p[I, E]$ is a Banach space with norm $\|x\|_p = \left(\int_I \|x(t)\|^p \, dt \right)^{\frac{1}{p}}$. The ordering in $L_p[I, E]$ is also induced by the cone P as follows: $u, v \in L_p[I, E]$, $u \leq v$ iff $u(t) \leq v(t)$ for almost all $t \in I$. For details on cone theory and strongly measure functions, see 3 and 9 respectively.

*Research supported by the National Nature Science Foundation of China and State Education Commission Plan Foundation of China.

In addition, it is common knowledge that E and $C\,[I, E]$ are all ordered additive groups which are additive by the common addition and the orderings induced by the cone P in E, i.e., for $u_1, u_2, v_1, v_2 \in E$ (or $u_1, u_2, v_1, v_2 \in C\,[I, E]$), $u_1 \leq v_1$ and $u_2 \leq v_2$ imply $u_1 + u_2 \leq v_1 + v_2$.

2. Fixed Point Theorems

Let $u_0, v_0 \in C\,[I, E]$, $u_0 \leq v_0$, $D = [u_0, v_0] = \{u \in C\,[I, E] \mid u_0 \leq u \leq v_0\}$. For any i, $i = 1, 2, \ldots, m$, $1 \leq p_1, p_2, \ldots, p_m < +\infty$, let $Fi : D \to L_{p_i}\,[I, E]$ be an increasing operator, $D_i = [F_i u_0, F_i v_0] = \left\{ w \in L_{p_i}\,[I, E] \mid F_i u_0 \leq w \leq F_i v_0 \right\}$,. and $K_i : D_i \to C\,[I, E]$ an increasing operator. Define operator A by $A = \sum_{i=1}^{m} K_i F_i$, thus A is also an increasing operator from D into $C[I, E]$.

In the following, for $t \in I$, set $(F_i D)(t) = \{u(t) \in E \mid u \in F_i(D)\}$, $(K_i D_i)(t) = \{u(t) \in E \mid u \in K_i(D_i)\}$, obviously, $(F_i D)(t), (K_i D_i)(t) \subset E$, here $i = 1, 2, \ldots, m$.

Definition 1 — Let X be an ordered set. X is called a pseudo-separable set if for any complete ordered subset N in X, there exists a countable subset $\{u_n\}$ of N such that for any $u \in N$ and $u \neq \sup N$, there exists $u_{n_0} \in \{u_n\}$ which satisfies $u \leq u_{n_0}$.

Obviously, the separability of an ordered Banach space implies the pseudo-separability.

Lemma 1 — *Let* E be a Banach space, P a cone. Let $F_i : D \to L_{p_i}\,[I, E]$ ($i = 1, 2, \ldots, m$, which is the same sense in the following) be an increasing operator. Assume that

(i) for almost all $t \in I$, any complete ordered subset of $(F_i D)(t)$ are relatively weakly compact in E;

(ii) $F_i(D)$ are bounded sets in $L_{p_i}\,[I, E]$.

Then, for any monotonically increasing sequence $\{w_{i,n}\}$ of $F_i(D)$, there exists $w_i^* \in D_i = [F_i u_0, F_i v_0]$ such that for almost all $t \in I$,

$$(F_i u_0)(t) \leq w_{i,n}(t) \leq w_i^*(t) \leq (F_i v_0)(t),\; n = 1, 2, \ldots;\; w_{i,n}(t) \xrightarrow{w} w_i^*(t). \qquad \ldots (1)$$

Furthermore, let $K_i : D_i \to C\,[I, E]$ be an increasing operator, $A = \sum_{i=1}^{m} K_i F_i$, and $R = \{u \in A(D) \mid u \leq Au\}$. Assume

(iii) $u_0 \leq A_0$ and $A\,v_0 \leq v_0$.

Then any complete ordered sequence $\{u_n\}$of R has an upper bound in R.

Proof : Since $\{w_{i,n}\}$ is monotonically increasing sequence in $F_i(D)$,

$$(F_i u_0)(t) \leq w_{i,1}(t) \leq w_{i,2}(t) \leq \ldots \leq w_{i,n}(t) \leq \ldots \leq (F_i v_0)(t) \qquad \ldots (2)$$

holds for almost all $t \in I$. By condition (i), there exist $I_1 \subset I$ and $mes\ I_1 = 0$ such that for any $t \in I \setminus I_1$, $\{w_{i,n}(t)\}$ is relatively weakly compact and (2) holds, by which it is easy to show that there exists $w_t \in E$ such that

$$w_{i,n}(t) \xrightarrow{w} w_t,\ n \to \infty;\ w_{i,n}(t) \le w_t,\ n = 1, 2, \ldots, t \in I \setminus I_1. \quad \ldots (3)$$

Define $w_i^*: I \to E$ by $w_i^*(t) = w_t$ for $t \in I \setminus I_1$ and $w_i^*(t) = 0$ for $t \in I_1$. Since the closed convex set P is weakly closed, by (2), (3) we get

$w_{i,n}(t) \xrightarrow{w} w_i^*(t);\ (F_i u_0)(t) \le w_{i,n}(t) \le w_i^*(t) \le (F_i v_0)(t),\ n = 1, 2, \ldots, t \in I \setminus I_1$, by Pettis theorem and its proof in Chapter V of [9] $w_i^*(t)$ is also strongly measurable. By (4) and according to the weakly lower semi-continuity of norm, we have

$$\| w_i^*(t) \| \le \underline{\lim}_{n \to \infty} \| w_{i,n}(t) \|,\ \forall\ t \in I \setminus I_1.$$

By Fatou Lemma, we get

$$\int_I \| w_i^*(t) \|^p dt = \int_{I \setminus I_1} \| w_i^*(t) \|^p dt \le \int_I \underline{\lim}_{n \to \infty} \| w_{i,n}(t) \|^p dt$$

$$\le \underline{\lim}_{n \to \infty} \int_I \| w_{i,n}(t) \|^p dt,$$

which, by condition (ii) and (4), implies $w_i^* \in D_i \subset L_{p_i}[I, E]$. Then, observing (4) again, we know that (1) holds.

Next, we show that any complete ordered sequence $\{u_n\}$ of R has an upper bound in R. By $Au_0 \in R$, $R \ne \phi$. Consider the sequence

$$v_1 = u_1,\ v_n = \max\left\{ v_{n-1}, u_n \right\},\ n = 2, 3, \ldots. \quad \ldots (5)$$

Since $\{u_n\}$ is a complete ordered set, v_n makes sense. Obviously,

$$v_n \in \left\{ u_n \right\} \subset R\ (n = 1, 2, \ldots),$$

and

$$v_1 < v_2 \le \ldots \le v_n \le \ldots \quad \ldots (6)$$

In the following, we still let $w_{i,n} = F_i v_n$ for convenience. The monotonicity of $F_i: D \to L_{p_i}[I, E]$ and the relation (6) imply that $\{w_{i,n}\}$ is a monotonically increasing sequence in $F_i(D)$. By the above proof, there exists $w_i^* \in D_i$ such that (1) holds. Let $u^* = \sum_{i=1}^{m} K_i w_i^*$. Since the relation $w_i^* \in D$ holds and $C[I\ E]$ is an ordered additive group, both $K_i w_i^*$ and $\sum_{i=1}^{m} K_i w_i^*$ make sense. Thus it follows from (5), (6) and (1) that $F_i u_n \le F_i v_n = w_{i,n} \le w_i^*$ for any n. On account of the relation $u_n \in R$ and the monotonicity of K_i, it is easy to show that

$$u_n \le Au_n \le A\, v_n = \sum_{i=1}^{m} K_i F_i v_n = \sum_{i=1}^{m} K_i w_{i,n} \le \sum_{i=1}^{m} K_i w_i^* = u^*,$$

$$n = 1, 2, 3, \ldots, \qquad \ldots (7)$$

i.e., u^* is an upper bound of $\{u_n\}$.

Finally, we show $u^* \in R$. Since $F_i u_0 \le w_i^* \le F_i v_0$, we have

$$u_0 \le Au_0 = \sum_{i=1}^{m} K_i F_i u_0 \le \sum_{i=1}^{m} K_i w_i^* \le \sum_{i=1}^{m} K_i F_i v_0 = A\, v_0 \le v_0,$$

so $u^* = \sum_{i=1}^{m} K_i w_i^* \in D = [u_0, v_0]$. Observing (7) and $v_n \in R$ (i.e., $v_n \le A\, v_n = \sum_{i=1}^{m} K_i F_i v_n$, we get $v_n \le u^*$ $(n = 1, 2, \ldots)$, hence $w_{i,n} = F_i v_n \le F_i u^*$, i.e.,

$$w_{i,n}(t) \le (F_i u^*)(t)\ (n = 1, 2, 3, \ldots) \qquad \ldots (8)$$

for almost all $t \in I$. By (8) and the second formula of (1) and in view of the weak closeness of P, it follows that for almost all $t \in I$, $w_i^*(t) < (F_i u^*)(t)$, i.e., $w_i^* \le F_i u^*$. By the monotonicity of K_i, $u^* = \sum_{i=1}^{m} K_i w_i^* \le \sum_{i=1}^{m} K_i F_i u^* = Au^*$, hence $u^* \in R$. The proof is completed. □

***Theorem* 1** — *Let E be a Banach space, P a cone. Let $F_i : D \to L_{p_i}[I, E]$ and $K_i : D_i \to C[I, E]$ be increasing operators (i = 1, 2, ..., m, which is the same sense in the following), and $A = \sum_{i=1}^{m} K_i F_i$. Assume that*

(i) for almost all $t \in I$, any complete ordered subset of $(F_i D)(t)$ is relatively weakly compact in E;

(ii) for all $t \in I$, $(K_i D_i)(t)$ are pseudo-separable in E;

(iii) $F_i(D)$ are bounded sets in $L_{p_i}[I, E]$.

Then A has at least one fixed point in D.

PROOF : Let $R = \{u \in A(D) \mid u \leq Au\}$. By $u_0 \in R, R \neq \phi$. In view of the separability of $I = [a, b]$, there exists a countable set $\{t_n\} \subset I$ such that $\{t_n\}$ is dense in I, where $n = 1, 2, \ldots$ Define operators $A_n : R \to \left\{ \sum_{i=1}^{m} (K_i F_i u)(t_n) \in \mid u \in D \right\}$ by

$$A_n u = \sum_{i=1}^{m} (K_i F_i u)(t_n), \quad \forall u \in R, n = 1, 2, \ldots \qquad \ldots (9)$$

Since E is also an ordered additive group $\sum_{i=1}^{m} (K_i F_i u)(t_n)$ makes sense. It is obvious that A_n are increasing ($n = 1, 2, \ldots$). For $x \in R$, set $R(x) = \{y \in R \mid x \leq y\}$, $\Sigma = \{M \subset R(x) \mid M$ is a complete ordered set, and $A_1 u \neq A_1 v$ for $u, v \in M$ and $u \neq v\}$. Since $\{x\} \in \Sigma$, Σ is non-empty. Define a partial order in Σ by the inclusion relation of sets as follows : for $M_1, M_2 \in \Sigma$, M_1 M_2 holds if and only if $M_1 \subset M_2$. So Σ is a ordered set. For any complete ordered set Γ of Σ, let $N = \bigcup_{M \in \Gamma} M$. Clearly N is an upper bound of Σ. Now we show that $N \in \Sigma$ is a complete ordered set in $R(x)$. For any $u, v \in N$, by the definition of N there exist $M_u \in \Gamma$ and $M_v \in \Gamma$ such that $u \in M_u$ and $v \in M_v$. Since Γ is a complete ordered set of Σ, by the definition of either $M_u \subset M_v$ or $M_v \subset M_u$, thus either $u, v \in M_u$ or $u, v \in M_v$. Observing that both M_u and M_v are completed ordered sets of $R(x)$, we obtain either $u \leq v$ or $v \leq u$. Moreover, taking $M_u, M_v \in \Gamma \subset \Sigma$ and the definition of Σ into account, we can have $A_1 u \neq A_1 v$ for $u \neq v$. Since $u, v \in N$ are arbitrary, by the above discussions N is a complete ordered set in $R(x)$ and $N \in \Sigma$ holds, thus N is an upper bound of Γ in Σ. It follows from Zorn's lemma that Σ has a maximal element M^*.

Since $M^* \in \Sigma$ is a complete ordered set of $R(x)$ and F_i, K_i are increasing, $(K_i F_i M^*)(t_1)$ are complete ordered sets in $(K_i D_i)(t_1)$ are complete ordered sets in $(K_i D_i)(t_1)$. By condition (ii) and Definition 1, there exists a countable subset $\{u_{i,n}\} \subset M^*$ such that for any $u \in M^*$, there exists $u_{i,n_i} \in \{u_{i,n}\}$ satisfying

$$(K_i F_i u)(t_1) \leq (K_i F_i u_{i,n_i})(t_1). \qquad \ldots (10)$$

Let $u_{n_0} = \max\{u_{i,n_i} \mid i = 1, 2, \ldots, m\}$, $\{u_n\} = \bigcup_{i=1}^{m} \{u_{i,n}\}$. Since M^* is a complete ordered set and $\{u_{i,n_i} \mid i = 1, 2, \ldots,\} \subset M^*$, u_{n_0} makes sense. Obviously, $\{u_n\}$ is a countable subset in M^* and

$u_{n_0} \in \{u_n\}$. In view of the monotonicity of F_i, K_i and the definition of u_{n_0}, it follows from (10) that

$$(K_i F_i u)(t_1) \leq (K_i F_i u_{n_0})(t_1). \qquad \text{... (11)}$$

Hence, for any $u \in M^*$, there exists $u_{n_0} \in \{u_n\}$ such that (11) holds, thus (9) and (11) imply $\sum_{i=1}^{m} (K_i F_i u)(t_1) \leq \sum_{i=1}^{m} (K_i F_i u_{n_0})(t_1)$, i.e.,

$$A_1 u \leq A_1 u_{n_0}. \qquad \text{... (12)}$$

By using (12), we now show

$$u \leq u_{n_0}. \qquad \text{... (13)}$$

Observing that M^* is complete ordered again and $u, u_{n_0} \in M^*$, we get either $u \leq u_{n_0}$ or $u_{n_0} \leq u$. If (13) is not true, then $u_{n_0} \leq u$ and $u \neq u_{n_0}$. Thus, by the definition of Σ, we have $A_1 u_{n_0} \leq A_1 u$ and $A_1 u_{n_0} \neq A_1 u$, which contradicts (12). The contradiction implies that (13) holds. Evidently, $\{u_n\} \subset M^* \subset R(x)$ indicates that $\{u_n\}$ is a complete ordered sequence in $R(x)$. By Lemma 1, there exists $u_1^* \in R(x)$ such that u_1^* is an upper bound of $\{u_n\}$. By virtue of (13), we know that u_1^* is also an upper bound of M^*

In the following, we show that for any $y \in R(u_1^*)$,

$$A_1 u_1^* = A_1 y \qquad \text{... (14)}$$

holds. Since $y \in R(u_1^*)$ means $u_1^* \leq y$, we have that if (14) is not true, then

$$A_1 u_1^* \leq A_1 y, A_1 u_1^* \neq A_1 y. \qquad \text{... (15)}$$

Similar to (13), by (15) we can prove

$$u_1^* \leq y, u_1^* \neq y. \qquad \text{... (16)}$$

Noting that u_1^* is an upper bound of M^* and taking $M^* \subset R(x)$ into account, we get that for any $u \in M^*, x \leq u \leq u_1^* \leq y$ holds, so by (16) $u \leq y$ and $u \neq y$, which implies that $M^* \cup \{y\}$ is a complete ordered set of $R(x)$ and $y \notin M^*$. Thus $M^* \cup \{y\} \in \Sigma$ and $M^* \neq M^* \cup \{y\}$, which contradicts that M^* is a maximal element in Σ. The contradiction shows that (14) holds. Similarly, there exists $u_2^* \in R(u_1^*)$ such that $A_2 u_2^* = A_2 y$ for any $y \in R(u_2^*)$. It is obvious that $u_1^* \le u_2^*$ and $R(u_2^*) \subset R(u_1^*) \subset R(x)$, hence by (14) $A_1 u_2^* = A_1 y$ for any $y \in R(u_2^*)$. Using the same arguments and going on with the above process, we get a monotonically increasing sequence $\{u_n^*\} \subset R(x)$ such that

$$\ldots \subset R(u_n^*) \subset R(u_{n-1}^*) \subset \ldots \subset R(u_1^*) \subset R(x) \subset R, \qquad \ldots (17)$$

and for any $y \in R(u_n^*)$,

$$A_i y = A_i y_n^*, \; i = 1, 2, ..., n. \qquad \ldots (18)$$

Since the monotonically increasing sequence $\{u_n^*\}$ is a complete ordered sequence in $R(x) \subset R$, by Lemma 1 there exists $u^* \in R$ such that u^* is an upper bound of $\{u_n^*\}$, thus by (17) $u^* \in \bigcap_{n=1}^{\infty} R(u^*) \subset R$ holds. By (18), we have

$$A_n y = A_n u^*, \; \forall y \in R(u^*), \; n = 1, 2, \ldots . \qquad \ldots (19)$$

Obseving that the relation $u^* \in R$ indicates $u^* \le A \overset{*}{u}$, we get $u^* \le A \overset{*}{u} \le A(A \overset{*}{u})$, i.e., $Au^* \in R(u^*) \subset R$. So by (19),

$$A_n (Au^*) = A_n u^*, n = 1, 2, \ldots \qquad \ldots (20)$$

On account of (9) and (20), we show

$$(A(Au^*))(t_n) = \sum_{i=1}^{m} (K_i F_i (Au^*))(t_n) = A_n (Au^*) = A_n u^* (Au^*)(t_n), n = 1, 2, \ldots \qquad \ldots (21)$$

Since $A(Au^*)$, $Au^* \in C[I, E]$ and $\{t_n\}$ is dense in I, the relation (21) implies, for any $t \in I, A(Au^*)(t) = (Au^*)(t)$, i.e., $A(Au^*) = A u^*$. Therefore, Au^* is a fixed point of A. The proof is completed.

Theorem 2 — *If the conditions in Theorem 1 are satisfied, then A has the minimal fixed point and the maximal fixed point in D.*

PROOF : Set Fix $= \{u \in D \mid u = Au\}$. By Theorem 1, $Fix\ A \ne \phi$. Set

$$S = \{u \in A(D) \mid u \le Au, u \le \bar{u}, \; \forall \bar{u} \in Fix A\}$$

Evidently $S \ne \phi$ due to $Au_0 \in S$. By the same arguments as in the proof of Theorem 1, we can show that A has a fixed point u^* in S. From the relation $u^* \in S$ and the definition of S it follows that u^* is a minimal fixed point in D. Similarly, A has a maximal fixed point in D. The proof is completed.

Remark 1 : It is easy to see from the proof of the above theorems that if I is a measurable closed subset of non-zero measures in R^n, these conclusions still hold.

3. APPLICATIONS

Consider the periodic boundary value problem (PBVP) for second order nonlinear integro-differential equation in E :

$$-u''=f_1(t,u)+\int_0^1 k(t,s)f_2(s,u(s))\,ds,\ t\in I;\ u(0)=u(1),\ u'(0)=u'(1), \quad \text{... (22)}$$

where $I=[0,1]$, $f_i(t,u):I\times E\to E$ (we do not suppose that $f_i(t,u)$ are continuous, $i=1,2$), and $k:I\times I\to R^1$ is nonnegative and continuous. By Theorem 1.5.6 in [4], we have that the periodic boundary value problem (22) is equivalent to the equation

$$u(t)=\int_0^1 G(t,s)\left[(F_1u)(s)+\int_0^1 k(s,\tau)f_2(\tau,u(\tau))\,d\tau\right]ds \quad \text{... (23)}$$

if $f_1(t,u)$ is continuous, where

$$G(t,s)=\begin{cases}\dfrac{1}{2\sqrt{M}(e^{\sqrt{M}}-1)}\left(e^{\sqrt{M}(s-t)}+e^{\sqrt{M}(1-s+t)}\right), & t\le s,\\[2mm] \dfrac{1}{2\sqrt{M}(e^{\sqrt{M}}-1)}\left(e^{\sqrt{M}(t-s)}+e^{\sqrt{M}(1-t+s)}\right), & t>s,\end{cases} \quad \text{... (24)}$$

$$(F_1u)(t)=f_1(t,u(t))+Mu(t), \quad \text{... (25)}$$

and $M>0$ is a constant. Therefore, when $f_1(t,u)$ is not continuous, we define the solutions of the integral eq. (23) as the solutions of the eq. (22).

We list for convenience the following assumptions :

(H_1) E is a Banach space, P a normal cone in E, and any complete ordered set of bounded sets in E is relatively weakly compact;

(H_2) There exist $u_0, v_0\in C^2[I,E]$, $u_0\le v_0$, $p_i\ge 1$ $(i=1,2)$ such that

$$\|f_i(t,u_0(t))\|,\ \|f_i(t,v_0(t))\|\in L_{p_i}[I,R^1], \quad \text{... (26)}$$

$$-u_0''(t)\le f_1(t,u_0(t))+\int_0^1 k(t,s)f_2(s,u_0(s))\,ds,\ t\in I;$$

$$u_0(0)=u_0(1),\ u_0'(0)\ge u_0'(1), \quad \text{... (27)}$$

$$-v_0''(t)\le f_1(t,v_0(t))+\int_0^1 k(t,s)f_2(s,v_0(s))\,ds,\ t\in I;$$

$$v_0(0)=u_0(1),\ v_0'(0)\le v_0'(1). \quad \text{... (28)}$$

(H_3) For $t \in I, u \in C[I, E], f_i(t, u(t))$ $(i = 1, 2)$ are strongly measurable functions, there exists $M > 0$ such that

$$f_1(t, x) - f_1(t, y) \geq -M(x, y)) \;\; \forall \; x, y \in E, x \geq y,$$

and $f_2(t, x)$ is increasing on x for $t \in I$.

Remark 2 : The assumption (H_1) is reasonable and useful.' For example, in many widely used spaces such as Hilbert spaces, reflexive spaces, L_1 spaces and even sequentially weakly complete spaces, the assumption (H_1) is satisfied naturally (see [5, 7-9].

Theorem 3 — *Suppose that the assumptions (H_1) - (H_3) are fulfilled. Then PBVP (22) has the maximal solution and the minimal solution in* $D = [u_{0,} \, v_0]$.

To prove theorem 3, we first show a lemma.

Lemma 2 — Let E be a Banach space, P a cone and Y a subset in E. If any complete ordered subset of Y is relatively weakly compact, then Y is pseudo-separable in E.

PROOF : Let M be any complete ordered subset in Y. Then M is relatively weakly compact, and so the weak colosure $\overline{M}^{\omega}$ on M is a weakly compact set. For $x \in M$, let $B(x) = \{ y \in \overline{M}^{w} \mid x \leq y \}$. Clearly $B(x)$ is a weakly closed set in E. Take any finite members $\{B(x_i) \mid x_i \in M, \; i = 1, 2, ..., n\}$ and let x^* max $\{x_i \mid i = 1, 2, ..., n\}$. Since M is a complete ordered set, x^* makes sense. Observing $x^* \in \bigcap_{i=1}^{n} B(x_i)$, thus $\bigcap_{i=1}^{n} B(x_i) \neq \phi$. By the weakly compactness of $\overline{M}^{w}$ and according to the finite intersection property of compact sets (see [9], Chapter 1), we set $\bigcap_{x \in M} B(x) \neq \phi$. Let $\tilde{y} \in \bigcap_{x \in M} B(x)$, then by the definition of $B(x)$ we have

$$u \leq \tilde{y}, \; \forall u \in M. \qquad \text{... (29)}$$

Since $\tilde{y} \in \bigcap_{x \in M} B(x) \subset \overline{M}^{w}$, there exists a sequence $\{x_n\}$ of M such that

$$x_n \xrightarrow{w} \tilde{y}, n \to \infty. \qquad \text{... (30)}$$

If $\tilde{y} \in M$, it is obvious that the sequence $\{u_n\}$ defined by $u_n = \tilde{y}$ $(n = 1, 2, ...)$ satisfies Defintion 1. If $\tilde{y} \notin M$, define $u_n = x_n$ $(n = 1, 2, ...)$, we prove that $\{u_n\}$ alsosatisfies Definition 1. If otherwise, there exists $x_0 \in M$ such that

$$u_n \leq x_0, u_n \neq x_0, n = 1, 2, \ldots . \qquad \text{... (31)}$$

By the weak closeness of the cone P and on account of (30), (31), we have $\tilde{y} \leq x_0$, which, together with the relation $x_0 \in M$ and (29), indicates $x_0 = \tilde{y}$. Thus $\tilde{y} \in M$, which contradicts $\tilde{y} \notin M$. The contradiction means that $\{u_n\}$ satisfies Definitin 1. Hence Y is pseudo-separable. The proof is completed.

PROOF OF THEOREM 3 : Similar to Lemma 1.5.4. in [4], it is easy to show that for any $h \in L_{p_1}[I, E]$, the linear periodic boundary value problem

$$-u'' = h - Mu, \; u(0) = u(1), \; u'(0) = u'(1) \qquad \dots (32)$$

has a unique solution in $C[I, E]$. So we may define a mapping $K_1 : L_{p_1}[I, E] \to C[I, E]$ by

$$u_h = K_1 h = \int_0^1 G(t, s) h(s)\, ds, \qquad \dots (33)$$

where $G(t, s)$ is given by (24), u_h is the unique solution of (32) corresponding to h. Now we prove that K_1 is increasing. Suppose that $h_1, h_2 \in L_{p_1}[I, E], h_1 \leq h_2 \cdot \phi \in P^* = \{\phi \in E^* \mid \phi(x) \geq 0, \; \forall x \in P\}$, set $m(t) = \phi((K_1 H_2)(t) - (K_1 h_1)(t))$. it is easy to see from (24) that $G(t, s)$ is nonnegative, hence by (33) we have

$$m(t) = \phi((K_1 h_2)(t) - (K_1 h_1)(t))$$

$$= \int_0^1 g(t, s)\, [\phi(h_2(s) - h_1(s))]\, ds \geq 0, \quad \forall t \in I.$$

Since $\phi \in P^*$ is arbitrary, $K_1 h_2 \geq K_1 h_1$ by Theorem 1.4.1 in[3], i.e, K_1 is an increasing operator. Let

$$(F_2 u)(t) = f_2(t, u(t)), \; Ku = \int_0^1 k(t, s)\, u(s)\, ds. \qquad \dots (34)$$

For any $u \in D = [u_0, v_0]$, by (H_3), (25) and (34), we get $F_i u_0 \leq F_i u \leq F_i v_0$ $(i = 1, 2)$. Hence for almost all $t \in I$, $0 \leq (F_i u)(t) - (F_i u_0)(t) \leq (F_i v_0)(t) - (F_i u_0)(t)$. By virtue of the normality of P, there exists constants $\lambda_i > 0$ such that for almost all $t \in I$, $\| (F_i u)(t) - (F_i u_0)(t) \| \leq \lambda_i \| (F_i v_0(t) - (F_i u_0)(t) \|$, i.e.,

$$\| (F_i u)(t) \| \leq \| (F_i u_0)(t) \| + \lambda_i \| (F_i v_0)(t) - (F_i u_0)(t) \|. \qquad \dots (35)$$

Because of (26) and the definitions of $F_i (i = 1, 2)$, (35) implies $F_i u \in L_{p_i}[I, E]$, and by

(H_3), F_i are increasing operators from $[u_0, v_0]$ into $L_{p_i}[I, E]$. Then, by using of (34) and the nonnegativity of $k(t, s)$, it is easy to show that K is also increasing from $L_{p_2}[I, E]$ into $C[I, E]$. Set

$$K_2 = K_1 K, A = K_1 F_1 + K_2 F_2,$$

where F_1, K_1, F_2 and K are given by (25), (33) and (34) respectively. By the above discussions, we know that $K_2 : L_{p_2}[I, E] \times C[I, E]$ and $A : [u_0, v_0] \to C[I, E]$ are increasing.

Similar to (35), we can show that $F_i(D)$ are bounded sets in $L_{p_i}[I, E]$. Thus, for almost all $t \in I$, $(F_i D)(t) \{ w(t) \mid w \in F_i(D) \}$ are bounded sets in E, and hence by (H_1), any complete ordered set of $(F_i D)(t)$ is relatively weakly compact. Let $D_i = \{ w \in L_{p_i}[I, E] \mid F_i u_0 \leq w \leq F_i v_0]$. Clearly $F_i(D) \subset D_i$. By the same arguments, we can show that for all $t \in I$, any complete ordered set of $(K_i D_i)(t) = \{ u(t)(\mid u \in K_i(D_i) \}$ is also relatively weakly compact in E. So by Lemma 2, we have that for all $t \in I$, $(K_i D_i)(t)$ are pseudo-separable. According to the above discussions, we know that the conditions (i), (ii) and (iv) in Theorem 2 are satisfied.

We now show that condition (iii) in Theorem 2 is fulfilled. Let $u_1 = Au_0$. Observing (23) and the definitions of K_1, A, we have

$$u_1 = A u_0 = \int_0^1 g(t, s) \left[(F_1 u_0)(s) + \int_0^1 k(s, \tau) f_2(\tau, u_0(\tau)) \, d\tau \right] ds. \quad \text{... (36)}$$

In view of (27) and by direct proof, it is easy to follow that

$$u_0 \leq \int_0^1 G(t, s) \left[(F_1 u_0)(s) + \int_0^1 k(s, \tau) f_2(\tau, u_0(\tau)) \, d\tau \right] ds. \quad \text{... (37)}$$

For any $\phi \in P^*$, set $n(t) = \phi[u_0(t) - (Au_0)(t)]$. By (36) and (37), we get $n(t) \leq 0$ $(\forall\, t \in I)$. Since $\phi \in P^*$ is arbitrary, $u_0 \leq Au_0$ holds. Similarly we can prove $A v_0 \leq v_0$.

By the facts just discussed above, we show that all conditions in Theorem 2 are satisfied. By Theorem 2, A has the minimal fixed point and the maximal fixed point in D. Observing that fixed points of A are equivalent to solutions of eq. (23), and eq. (23) is equivalent to eq. (22), the conclusions of Theorem 3 holds.

REFERENCES

1. H. Aman, *SIMA Rev.*, **18** (1976), 620-709.
2. K. Deiming, *Nonlinear Functional Analysis*, Springer-Verlag, New York/Berlin, 1985.

3. D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
4. V. Lakshmikanthn, D. Guo and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publisher, Boston/London, 1996.
5. J. Sun, CHin. *Ann. of Math.*, (Chinese), **11A** (1990), 408-12.
6. J. Sun and Y. Sun, *Appl. Anal.*, **175** (1993), 33-45.
8. R. E. Edwards, *Functional Analysis*, New York. 1965.
9. Kosaku Yosida, *Functional Analysis*, Springer, Berlin, 1978.