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ON VALUE DISTRIBUTION OF DIFFERENTIAL MONOMIAL OF ALGEBROID FUNCTIONS*

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In this paper we obtain the following result : Let $w(z)$ be a ν-valued algebroid function and $\Phi = (w')^{i_2} \ldots (w^{(n)})^{i_n} (w(z))^n$, and

$$(n - l_2 \nu)[n - l_2 \nu - 2\sigma(\nu - 1)]$$

$$\geq 2 l_0 (n - l_2 \nu) + l_2 (n - l_2 \nu + 1)$$

$$+ 2(\nu - 1) l_2 [l_0 (n - l_2 \nu) = 2(n - l_2 \nu + 1)].$$

Then Φ assumes all values except possibly zero infinitely often.

Key Words : Algebroid Functions; The value Distribution; Differential Polynomials

1. INTRODUCTION

We use the standard notation of Nevanlinna theory of meromorphic functions (see [1]).

In this paper, we will mainly consider the problem of possible Picard values of algebroid function and derivatives of the form

$$(w')^{i_2} \ldots (w^{(n)})^{i_n} (w(x))^n, \qquad \ldots (1)$$

where $w(z)$ is a ν-valued algebroid function.

We denote

$$\Phi = (w')^{i_2} \ldots (w^{(n)})^{i_n} (w(z))^n, \; l_0 = i_1 + \ldots + i_n,$$

$$l_2 = i_1 + 2i_2 + \ldots + ni_n, \; \sigma = i_1 + 3i_2 + \ldots + (2n-1)\, i_n.$$

For the case $(i_1, \ldots, i_n) = (1, 0, \ldots, 0)$, Hayman, Pang and Kari Katajamaki have proved the following theorems, respectively.

Theorem A[2] — *Suppose that* $w(z)$ *is a transcendental entire function and* $n \geq 2$. *Then* $w'(z)\, w(z)^n$ *assumes all values except possibly zero infinitely often.*

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***Theorem* B[3]** — *Let $w(z)$ be a ν-valued algebroid function and a be a finite complex number. Then*

$$w'(z)\, w(z)^n = a,\ n \geq 4\nu - 1$$

have infinite roots.

***Theorem* C[4]** — *Let $w(z)$ be a ν-valued transcendental entire algebroid function and set*

$$\phi(z) = w'(z)\, w(z)^n,$$

where $n \in N$ and $a \in C - \{0\}$. Then if $n \geq 8\nu - 6$, we have

$$\overline{N}\left(r, \frac{1}{\phi - b}\right) \neq S(r, w)$$

for each $b \in C$.

Our result is :

***Theorem* 1** — *Let $w(z)$ be a ν-valued algebroid function and*

$$\begin{aligned}&(n - l_2\nu)[n - l_2\nu - 2\sigma(\nu - 1)]\\ &\geq 2l_0(n - l_2\nu) + l_2(n - l_2\nu + 1) \qquad \ldots (2)\\ &\quad + 2(\nu - 1)\, l_2[l_0(n - l_2\nu) + 2(n - l_2\nu + 1)].\end{aligned}$$

Then Φ assumes all values except possibly zero infinitely often.

2. Some Lemmas

Lemma 1 — Let $w(z)$ be a ν valued algebroid function and Φ be as in (1), $a_j (\neq 0), j = 1, 2, \ldots, p$ be distinct complex numbers, then

$$pT(r, \Phi) \leq \overline{N}(r, \Phi) + \overline{N}\left(r, \frac{1}{\Phi}\right) + \sum_{k=1}^{p} \overline{N}(r, \Phi = a_k) + N_x(r, \Phi) + S(r, \Phi).$$

Proof See [4].

Lemma 2 — Let $w(z)$ and Φ be as above. Then

$$N_x(r, \Phi) \leq N_x(r, w).$$

Proof See [4]

Lemma 3 — Let $w(z)$ be a ν value algebroid function and Φ be as above. Then

$$\begin{aligned}&(l_2 + l_0)\, N(r, w) - N(r, \Phi) + N\left(r, \frac{1}{\Phi}\right) = l_2 N_1(r, w)\\ &\qquad - l_2 N_x(r, w) - (l_2 - l_0)\, N\left(r, \frac{1}{w}\right), \qquad \ldots (3)\end{aligned}$$

where $N_1(r, w)$ is the count function of all multiple points of $w(z)$ and every τ multiple point are counted only $\tau - 1$ times.

PROOF : Let w_i, $i = 1, \ldots, \lambda$ be the branches of $w(z)$, $w_i(z_0) = a$. Then by

$$w(z) - a = (z - z_0)^{\tau/\lambda} w_0(z),\ w^{(k)}(z) = (z - z_0)^{(\tau - k\lambda)/\lambda} \overline{w}_0(z).$$

We know that z_0 is a zero of $w^{(k)}(z)$ with multiplicity $\tau - k\lambda$ if $\tau - k\lambda > 0$; z_0 is a pole of $w^{(k)}(z)$ with multiplicity $k\lambda - \tau$, if $\tau - k\lambda < 0$. Thus

$$
\begin{aligned}
&(l_2 + l_0)\, n(r, w) - n(r\, \Phi) + n\left(r, \frac{1}{\Phi}\right) \\
&= (l_2 + l_0) \sum_{w=\infty} \tau - \left\{ \sum_{w=\infty} (l_0 \tau + l_2 \lambda) + \sum_{w \neq \infty} (l_2 \lambda - l_0 \tau)^+ \right\} \\
&+ \sum_{w \neq \infty} (l_0 \tau - l_2 \lambda)^+ \\
&= \sum_{w=\infty} (l_0 \tau + l_2 + l_2 \tau - l_2) - \left\{ \sum_{w=\infty} (l_0 \tau + l_2 + l_2 \lambda - l_2) \right. \\
&+ \sum_{w \neq \infty,\, l_2 \lambda - l_0 \tau > 0} [l_2 \lambda - l_2 - (l_2 \tau - l_2] \\
&+ \sum_{w \neq \infty,\, l_2 \lambda - l_0 \tau < 0} [l_2 \tau - l_2 - (l_2 \lambda - l_2)] \\
&- \sum_{w \neq \infty,\, l_2 \lambda - l_0 \tau > 0} (l_2 - l_0)\, \tau - \sum_{w \neq \infty,\, l_2 \lambda - l_0 \tau < 0} (l_2 - l_0)\, \tau \\
&= l_2 \Sigma (\tau - 1) - l_2 \Sigma (\lambda - 1) - (l_2 - l_0) \Sigma \tau \\
&= l_2 n_1(r, w) - l_2 n_x(r, w) - (l_2 - l_0)\, n\left(r, \frac{1}{w}\right).
\end{aligned}
$$

Integrating logarithmically we obtain (3).

3. PROOF OF THEOREM 1

We assume conversely that $\overline{N}\left(r, \frac{1}{\Phi - b}\right) = S(r, w)$ for some $b \in C - \{0\}$.

First, we prove that

$$T(r, \Phi) \geq (n - l_2 v)\, T(r, w) - \sigma N_x(r, w),\ (r \to \infty, r \notin I). \quad \ldots (4)$$

In fact,

$$
\begin{aligned}
(n + l_1)\, T(r, w) = T(r, (w)^{n + l_1}) &= T\left(\frac{\Phi_w\, l_1}{w^{i_0} (w')^{i_2} \ldots (w^{(n)})^{i_n}} \right) \\
&\leq T(r, \Phi) + T\left(r, \frac{w^{i_0} (w')^{i_2} \ldots (w^{(n)})^{i_n}}{w_{l_1}} \right) + O(1)
\end{aligned}
$$

$$= T(r, \Phi) + T\left(r, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w'}{w}\right)^{i_n}\right) + O(1)$$

$$\leq T(r, \Phi) + l_1\, m(r, w) + N\left(r, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right) + S(r, w). \qquad \ldots (5)$$

We estimate $N\left(r, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right)$. The poles of $\left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}$ may arise only from one of the following cases :

Case (i) — the zeros of $w(z)$;

Case (ii) — the poles of $w(z)$;

Case (iii) — the brances point of $w(z)$.

Case (i) — If z_0 is a zero of $w(z)$, then its contribution to

$$N\left(r, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right) \text{ is } l_1 N\left(r, \frac{1}{w}\right).$$

Case (ii) — If z_0 is a pole of $w(z)$, then

$$w(z) = (z - z_0)^{-\frac{\tau}{\lambda}} w_0(z),\ w_0(z_0) \neq 0, \infty,$$

$$w^{(k)} = C(z - z_0)^{-\frac{\tau + k\lambda}{\lambda}} w_k(z),\ w_k(z_0) \neq 0, \infty.$$

Thus

$$\tau\left(z_0 \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right) \leq i_1 \lambda + \ldots + n i_n \lambda = l_2 \lambda \leq l_2 \nu.$$

Its contribution to

$$N\left(r, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right) \text{ is } l_2\, \nu \overline{N}(r, w).$$

Case (iii) — Let $w_i(z)$, $i = 1, 2, \ldots, \lambda$ be branches of $w(z)$ such that $w(z_0) = a, a \neq 0, \infty$. Then in the neighbourhood of z_0, we have

$$w(z) = a + (z - z_0)^{\frac{\tau}{\lambda}} w_0(z),\ w_0(z_0) \neq 0, \infty,$$

$$w^{(k)} = C(z - z_0)^{\frac{\tau - k\lambda}{\lambda}} w_k(z),\ w_k(z_0) \neq 0, \infty.$$

It easy to see that z_0 is a pole when $\tau - k\lambda < 0$.

Thus

$$\tau\left(z_0, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right) \le \sum_{\alpha=1}^{n} i_\alpha [\alpha\lambda - \tau]^+ \le \sum_{\alpha=1}^{n} i_\alpha [\alpha\lambda - 1]$$

$$\le (\lambda - 1) \sum_{\alpha=1}^{n} i_\alpha [2\alpha - 1]$$

$$= \sigma(\lambda - 1).$$

Its contribution to

$$N\left(r, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right) \text{ is } \sigma N_x(r, w).$$

Combining the cases (*i*)-(*iii*), we get

$$N\left(r, \left(\frac{w'}{w}\right)^{i_1} \cdots \left(\frac{w^{(n)}}{w}\right)^{i_n}\right) \le l_1 N\left(r, \frac{1}{w}\right) + l_2\, v \overline{N}(r, w) + \sigma N_x(r, w). \quad \text{.. (6)}$$

Substituting the inequality (6) into the inequality (5), we obtain the inequality (4). Secondly, by Lemma 1 and Lemma 2, we have

$$T(r, \Phi) \le \overline{N}(r, \Phi) + \overline{N}\left(r, \frac{1}{\Phi}\right) + N_x(r, w) + S(r, w). \quad \text{... (7)}$$

Since the poles of Φ must be the pole of $w(z)$ or a branch point of $w(z)$, we get

$$\overline{N}(r, \Phi) \le \overline{N}(r, w) + N_x(r, w). \quad \text{... (8)}$$

Let $w(z) - a = c(z - z_0)^{\tau/\lambda} w_0(z),\ w_0(z) \ne 0, \infty$. Then

$$\Phi = C(z - z_0)^{\frac{(l_0) r - l_2 \lambda}{\lambda}} \Phi_0(z) \ne 0, \infty.$$

Thus

$$\overline{n}\left(r, \frac{1}{\Phi}\right) \le \overline{n}\left(r, \frac{1}{w}\right) + \sum_{w \ne 0} \frac{(l_0 \tau - l_2 \lambda)^+}{\lambda}$$

$$\le \overline{n}\left(r, \frac{1}{w}\right) + l_0 \sum_{w \ne 0} (\tau - 1) \quad \text{... (9)}$$

$$= \overline{n}\left(r, \frac{1}{w}\right) + l_0\, n_1(r, w)$$

But
$$(n-l_2 v)\,\bar{n}\left(r,\frac{1}{w}\right)\le \sum_{w\neq 0}(l_0\tau-l_2\lambda-l_1)\le n\left(r,\frac{1}{\Phi}\right)-l_0\bar{n}\left(r,\frac{1}{\Phi}\right)$$
$$<n\left(r,\frac{1}{\Phi}\right)-\bar{n}\left(r,\frac{1}{\Phi}\right),$$

that is

$$\bar{n}\left(r,\frac{1}{w}\right)<\frac{1}{n-l_2 v}\left[n\left(r,\frac{1}{\Phi}\right)-\bar{n}\left(r,\frac{1}{\Phi}\right)\right]. \qquad \text{... (10)}$$

It follows from (9) and (10)

$$\bar{N}\left(r,\frac{1}{\Phi}\right)<\frac{1}{n-l_2 v}\left[n\left(r,\frac{1}{\Phi}\right)-\bar{n}\left(r,\frac{1}{\Phi}\right)\right]+l_0 N_1(r,w). \qquad \text{... (11)}$$

By Lemma 3, we have

$$N_1(r,w)\le 2T(r,w)+N_x(r,w). \qquad \text{... (12)}$$

Combining (11) and (12), we get

$$\bar{N}\left(r,\frac{1}{\Phi}\right)<\frac{1}{n-l_2 v+1}N\left(r,\frac{1}{\Phi}\right)\frac{2l_0(n-l_2 v)}{(n-l_2 v+1)}T(r,w)$$
$$+\frac{l_0(n-l_2 v)}{n-l_2 v+1}N_x(r,w). \qquad \text{... (13)}$$

Combining (7), (8) and (13), we get

$$\left(1-\frac{1}{n-l_2 v+1}\right)T(r,\Phi)<\frac{2l_0(n-l_2 v)+l_2(n-l_2 v+1)}{(n-l_2 v+1)}T(r,w)$$
$$+\frac{l_0(n-l_2 v)+2(n-l_2 v+1)}{n-l_2 v+1)}N_x(r,w). \qquad \text{... (14)}$$

Substituting the inequality (4) into the inequality (14), we get

$$(n-l_2 v)\,[(n-l_2 v)-2\sigma(v-1)]$$
$$<2\,l_0(n-l_2 v)+l_2(n-l_2 v+1)$$
$$+\ 2(v-1)\,l_2\,[l_0(n-l_2 v)+2(n-l_2 v+1)].$$

This is a contradiction. This proves Theorem 1.

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