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MIXED SYMMETRIC DUALITY IN NONDIFFERENTIABLE MATHEMATICAL PROGRAMMING[1]

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A mixed symmetric dual formulation is presented for a class of nondifferentiable nonlinear programming problems with multiple arguments. Weak, strong and converse duality theorems are established. The mixed symmetric dual formulation unifies the two exsting symmetric dual formulations in the literature

Key Words : Symmetric Duality; Nondifferentiable Nonlinear Programming; Generalized Convexity; Support Function

1. INTRODUCTION

Symmetric duality in nonlinear programming was introduced by Dorn in[7], More precisely, a mathematical programming problem and its dual are said to be symmetric if the dual of the dual is the original problem. In other words, when the dual is recast in the form of the primal, its dual is the primal problem. Subsequently, Dantzig, Eisenberg and Cottle[6] and Mond[9] formulated a pair of symmetric dual programs fora scalar-valued function $f(x, y)$ that is convex in the first variable and concave in the second variable. Then, Mond and Weir[10] gave a different pair of symmetric dual nonlinear programs in which a weaker convexity assumption was imposed on f.

Recently, Mond and Schechter[11] studied nondifferentiable symmetric duality (of both Wolfe and Mond-Weir types) for a case in which the objective function contains a support function. Chandra, *et al.*[5] presented a mixed symmetric dual formulation for a nonlinear programming problem. Motivated by their research, we propose a pair of new mixed symmetric dual nondifferentiable nonlinear programs in this paper. The pair can be reduced to that of Mond and Schechter[11] and that of Chandra *et al*[5] as special cases. We then obtain the weak and strong duality theorems for the new pair of mixed symmetric dual nondifferentiable nonlinear programs under a weaker F-convexity condition.

2. PRELIMINARIES

Let C be a compact convex set in $I\!R^n$. The support function of C is defined by

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$$s\,(x \mid C) := \max\left\{ x^T y : y \in C \right\}.$$

A support function, being convex and everywhere finite, has a subdifferential,. that is, there exists z such that $(s\,(y \mid C) \geq s\,(x \mid C) + z^T (y - x)$ for all $x \in C$. The subdifferential of $s\,(x \mid C)$ is given by

$$\partial s\,(x \mid C) := \left\{ z \in C : z^T x = s\,(x \mid C) \right\}.$$

For any set $S \subset I\!R^n$ the normal cone to S at a point $x \in S$ is defined by

$$N_S\,(x) := \left\{ y \in I\!R^n : y^T (z - x) \leq 0 \text{ for } all\ z \in \right\} S$$

It is readily verified that for a compact convex set C, y is in $N_C\,(x)$ if and only if $s\,(y \mid C) = x^T y$ if and only if $x \in \partial s\,(y \mid C)$.

Let $f(x,\ y)$ be a real-valued twice differentiable function defined on $I\!R^n \times I\!R^m$. Let $\nabla_1 f(x, y)$ and $\nabla_2 f(x, y)$ denote the partial derivatives of f with respect to x and y, respectively. Also let $\nabla^2_{11} f(x, y)$ denotes the Hessian matrix of f evaluated at $(x,\ y)$. The symbols $\nabla^2_{22} f(x, y)$, $\nabla^2_{12} f(x, y)$ and $\nabla^2_{21} f(x, y)$ are defined similarly.

We now introduce the following definitions, see Hanson and Mond[8].

Definition 1 — Let $X \subset I\!R^n$. A functional $F : X \times X \times I\!R^n \to I\!R$ is said to be sublinear with respect to its third argument if, for any $x, y \in X$

(A) $F\,(x, y; a_1 + a_2) \leq F\,(x, y; a_1) + F\,(x, y; a_2)$ for any $a_1, a_2 \in I\!R^n$;

(B) $F\,(x, y, \alpha a) = \alpha F\,(x, y; a)$, for any $\alpha F\,(x, y; a)$ for any $\alpha \in I\!R_+$ and $a \in I\!R^n$.

Definition 2 — Let $X \subset I\!R^n$, $Y \subset I\!R^m$ and $F : X \times Y \times I\!R^n \to I\!R$ be sublinear with respect to its third component. $f(\cdot, y)$ is said to be F-convex at $\bar{x} \in X$, for fixed $y \in Y$, if

$$f(x, y) - f(\bar{x}, y) \geq F\,(x, \bar{x}; \nabla_1 f(\bar{x}, y)), \quad \forall x \in X.$$

Definition 3 — Let $X \subset I\!R^n$, $Y \subset I\!R^m$ and $f : X \times Y \to I\!R$. Let $F : X \times Y \times I\!R^n \to I\!R$ be sublinear with respect to its third component. $f(x, \cdot)$ is said to be F-concave at $\bar{y} \in Y$, for fized $x \in X$ if

$$f(x, \bar{y}) - f(x, y) \geq F\,(y, \bar{y}; -\nabla_2 f(x, \bar{y})), \quad \forall y \in Y.$$

Definition 4 — Let $X \subset I\!R^n$, $Y \subset I\!R^m$ and $F : X \times Y \times I\!R^n \to I\!R$ be sublinear with respect to its third component. $f(\cdot, y)$ is said to be F-pseudoconvex at $\bar{x}$, for fixed $y \in Y$, if

$$F\,(x, \bar{x}; \nabla_1 f(\bar{x}, y)) \geq 0 \Rightarrow f(x, y) \geq f(\bar{x}, y), \quad \forall x \in X.$$

Definition 5 — Let $X \subset I\!R^n$, $Y \subset I\!R^m$ and $f: X \times Y \to I\!R$. Let $F: X \times Y \times I\!R^n \to I\!R$ be sublinear with respect to its third component. $f(x, \cdot)$ is said to be F-pseudoconcve at $\bar{y}$, for fixed $x \in X$, if

$$F(y, \bar{y}; \nabla_2 f(\bar{x}, \bar{y})) \geq 0 \Rightarrow f(x, \bar{y}) \geq f(\bar{x}, y), \ \forall \ y \in Y.$$

3. Mixed Type Symmetric Duality

For $N = \{1, 2, ..., n\}$ and $M = \{1, 2, ..., m\}$ let $J_1 \subset N, K_1 \subset M$ and $J_2 = N \setminus J_1, K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in the subject J_1. The other numbers $|J_2|, |K_1|$ and $|K_2|$ are defined similarly. It is clear that $x \in I\!R^n$ can be written as $x = (x^1, x^2), x^1 \in I\!R^{|J_1|}$, $x^2 \in I\!R^{|J_2|}$ Similarly, $y \in I\!R^m$ can be wriite as $y = (y^1 . y^2)$, $y^1 \in I\!R^{|K_1|}$, $y^2 \in I\!R^{|K_2|}$. Let $f: I\!R^{|J_1|} \times I\!R^{|K_1|} \to I\!R$ and $g: I\!R^{|J_2|} \times I\!R^{|K_2|} \to I\!R$ be twice differentiable. Here, if $J_1 = 0$, then $J_2 = N, |J_1| = 0$ and $|J_2| = n$. So, in this $I\!R^{|J_1|}$ and $I\!R^{|J_2|}$ are 0 and $I\!R^n$, respectively. The other cases $K_1 = 0, K_2 = 0$ and $J_2 = 0$ are defined similarly.

We now state the following pair of non-differentiable programs and discuss their duality results.

Primal Problem (MP) — Minimize $f(x^1, y^1) + g(x^2, y^2) + s(x^1 | C_1) + s(x^2 | C_2) - (y^1)^T \nabla_2 f(x^1, y^1) - (y^2)^T z^2$ subject to $(x^1, x^2, y^1, y^2, z^1, z^2) \in I\!R^{|J_1|} \times I\!R^{|J_2|}\, I\!R^{|K_1|} \times I\!R^{|K_2|} \times I\!R^{|K_1|} \times I\!R^{|K_2|}$

$$\nabla_2 f(x^1, y^1) - z^1 \leq 0, \quad \text{... (1)}$$

$$\nabla_{2g}(x^2, y^2) - z^2 \leq 0, \quad \text{... (2)}$$

$$(y^2)^T (\nabla_2 g(x^2, y^2) - z^2) \geq, \quad \text{... (3)}$$

$$x^1 \geq 0, x^2 \geq 0, \quad \text{... (4)}$$

$$z^1 \in D_1, z^2 \in D_2 \quad \text{... (5)}$$

Dual problem (MD)
Maximize :

$$f(u^1, v^1) + g(u^2, v^2) - s(v^1 | D_1) - s(v^2 | D_2) - (u^1)^T \nabla_1 f(u^1, v^1) + (u^2)^T w^2$$

subject to : $(u^1, u^2, v^1, v^{2,} w^1, w^2) \in I\!R^{|J_1|} \times I\!R^{|J_2|} \times I\!R^{|K_1|} \times I\!R^{|K_2|} \times I\!R^{|K_1|} \times I\!R^{|K_2|}$.

$$\nabla_1 f(u^1, v^1) + w^1 \geq 0, \quad \text{... (6)}$$

$$\nabla_{1g}(u^2, v^2) + w^2 \geq 0, \quad \text{... (7)}$$

$$(u^1)^T (\nabla_1 f(u^1, v^1) + w^1) \leq 0, \quad \text{... (8)}$$

$$v^1 \geq 0, v^2 \geq 0, \qquad \dots (9)$$

$$w^1 \in C_1, w^2 \in C_2, \qquad \dots (10)$$

where C_1, C_2, D_1 and D_2 are compact and convex sets of $I\!R^{|J_1|}$, $I\!R^{|J_2|}$, $I\!R^{|K_1|}$ and $I\!R^{|K_2|}$ respectively.

Theorem 1 — *(Weak Duality) :Let F_1, F_2, G_1 and G_2 be sublinear functionals, and let $(x^1, x^2, y^1, y^2, z^1, z^2)$ be feasible for problem (MP) and $(u^1, u^2, v^1, v^2, w^1, w^2)$ be feasible for problem (MD).* If $f(\cdot, y^1)$ is F_1-convex for fixed $y^1, f(x^1, \cdot)$ is F_2-concave for fixed x^1, $g(\cdot, y^2) + (\cdot)^T w^2$ is G_1-pseudoconvex for fixed y^2 and $g(x^2, \cdot) - (\cdot)^T z^2$ is G_2-pseudoconcave for fixed x^2, and the following conditions are satisfied:

(*i*) $F_1 (x^1, u^1; \nabla_1 f(u^1, v^1)) + (u^1)^T \nabla_1 f(u^1, v^1) + (x^1)^T w^1 \geq 0;$

(*ii*) $G_1 x^2, u^2 \nabla_{1g} (u^2, v^2) + w^2) + (u^2)^T (w^2 + \nabla_{1g} (u^2, v^2)) \geq 0;$

(*iii*) $F_2 (y^1, v^1) \nabla_2 f(x^1, y^1)) + (y^1)^T \nabla_2 f(x^1, y^1) - (v^1)^T z^1 \leq 0;$ and

(*iv*) $G_2 (y^2, v^2) \nabla_2 f(x^2, y^2)) + (y^1)^T \nabla_2 f(x^2, y^2) - z^2 \leq 0,$

then inf $(MP) \geq$ sup (MD).

PROOF : Suppose that $(x^1, x^2, y^1, y^2, z^1, z^2)$ and $(u^1, u^2, v^1, v^2, w^1, w^2)$ are feasible for problems (*MP*) and (*MD*), respectively. Then using the F_1-convexity of $f(\cdot, v^1)$ and F_2-concavity of function $f(x^1, \cdot)$, we have

$$f(x^1, v^1) - f(u^1, v^1) \geq F_1 (x^1, u^1; \nabla_1 f(u^1, v^1)),$$

and
$$f(x^1, v^1) - f(x^1, y^1) \geq F_2 (v^1, y^1; \nabla_2 f(x^1, y^1)).$$

Rearranging the above two inequalities, and by using conditions (i) and (iii), we obtain

$$f(x^1, y^1) - f(u^1, v^1) \geq - (u^1)^T \nabla_1 f(u^1, v^1)$$
$$- (x^1)^T w^1 + (y^1)^T \nabla_2 f(x^1, y^1) - (v^1)^T z^1.$$

Using $(v^1)^T z^1 \leq s (z^1 \mid D_1$ and $(x^1)^T w^1 \leq s (x \mid C_1)$ we have

$$f(x^1, y^1) + s (x^1 \mid C_1) - (y^1)^T \nabla_2 f(x^1, y^1) \geq f(u^1, v^1)$$
$$- s (v^1 \mid D_1) - (u^1)^T \nabla_1 f(u^1, v^1) \qquad \dots (11)$$

From condition (ii) and (8), we have

$$G_1 (x^2, u^2; \nabla_{1g} (u^2, v^2) + w^2) \geq - (u^2)^T (w^2 + \nabla_{1g} (u^2, v^2)) \geq 0.$$

By G_1-pseudoconvexity of $g(\cdot, y^2) + (\cdot)^T w^2$, we get

$$g(x^2, v^2) + (x^2)^T w^2 \geq g(u^2, v^2) + (u^2)^T w^2. \quad \text{... (12)}$$

In a similar fashion, from condition (iv) and (3), we have

$$G_2(y^2, v^2; \nabla_2 f(x^2, y^2) - z^2) \leq -(y^2)^T (\nabla_2 f(x^2, y^2) - (z^2)^T \leq 0.$$

By G_2-pseudoconcavity of $g(x^2, \cdot) - (\cdot)^T z^2$, we get

$$g(x^2, v^2) - (v^2)^T z^2 \leq g(x^2, y^2) - (y^2)^T z^2. \quad \text{... (13)}$$

From eq. (12) and (13) we can conclude that

$$g(x^2, v^2) + (x^2)^T w^2 - (y^2)^T z^2 \geq g(u^2, v^2) - (v^2)^T z^2 + (u^2)^T w^2. \quad \text{... (14)}$$

Using $(x^2)^T w^2 \leq s(x^2 + | C_2)$ and $(v^2)^T z^2 \leq s(v^2 | D_2)$, we have

$$g(x^2, v^2) + s(x^2 | C_2) - (y^2)^T \geq g(u^2, v^2) - s(v^2 | D_2) + (u^2)^T w^2. \quad \text{... (15)}$$

Finally, (11) and (15) give

$$f(x^1, y^1) + g(x^2, v^2) + s(x^1 | C_1) + s(x^2 | C_2) - (y^1)^T \nabla_2 f(x^1, y^1) - (y^2)^T z^2$$

$$\geq f(u^1, v^1) + g(u^2, v^2) - s(v^1 | D_1) - s(v^2 | D_2) - (u^1)^T \nabla_1 f(u^1, v^1) + (u^2)^T w^2. \quad \text{... (16)}$$

Thus, inf $(MP) \geq$ sup (MD). □

***Theorem* 2** — *(Strong duality) Suppose that* $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2)$ *is optimal for problem (MP) and that the Hessian matrix* $\nabla_2^2 f(\bar{x}^1, \bar{y}^1)$ *is nonsingular, that* $\nabla_2^2 g(\bar{x}^2, \bar{y}^2)$ *is positive definite or negative definite and that* $\nabla_2 g(\bar{x}^2, \bar{y}^2) \neq \bar{z}^2$. *Then* $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2)$ *is feasible for problem (MD) and the corresponding objective function value are equal. If in addition the hypotheses of Theorem 1 hold, then there exist* $\bar{w}^1, \bar{w}^2$ *such that* $(u^1, u^2, v^1, v^2, w^1, w^2) = (\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{w}^1, \bar{w}^2)$ *is optimal for problem (MD).*

PROOF : Let $q = (x^1, x^2, y^1, y^2, z^1, z^2)$ and

$$F(q) = f(x^1, y^1) + g(x^2, y^2) - (y^1)^T \nabla_2 f(x^1, y^1) - (y^2)^T z^2 +$$

$$s(q \mid \{0\} \times \{0\} \times \{0\} \times \{0\} \times C_1 \times C_2 ,$$

$$G(q) = \nabla_2 f(x^1, y^1) - z^1,$$

$$H(q) = \nabla_2 g(x^2, y^2) - z^2,$$

$$I(q) = -(y^2)^T (\nabla_2 g(x^2, y^2) - z^2),$$

$$J(q) = -x^1,$$

$$K(q) = -x^2,$$

$$D = \mathbb{R}^{|J_1|} \times \mathbb{R}^{|J_2|} \times \mathbb{R}^{|K_1|} \times \mathbb{R}^{|K_2|} \times D_1 \times D_2.$$

Then Problem (MP) canbe restated as follows :

minimize $F(q)$

Subject to :

$$G(q) \le 0,$$

$$H(q) \le 0,$$

$$I(q) \le 0,$$

$$J(q) \le 0$$

$$K(q) \le 0,$$

$$q \in D.$$

Since $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2)$ is optimal for problem (MP) i.e., $\bar{q} = (\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2)$ is optimal for above programming, by the Fritz John conditions[12] and note that $N_C(q) = \{0\} \times \{0\} \times \{0\} \times \{0\} \times N_{D_1}(z^1) \times N_{D_2}(z^2)$ at any $q \in C$, there exist

$\alpha \in \mathbb{R}$, $\alpha_1 \in \mathbb{R}^{|K_1|}$, $\alpha_2 \in \mathbb{R}^{|K_2|}$, $\lambda \in \mathbb{R}$, $\mu_1 \in \mathbb{R}^{|J_1|}$ and $\mu_2 \in \mathbb{R}^{|J_2|}$, such that

$$\alpha[\nabla_1 f(\bar{x}^1, \bar{y}^1) - (\bar{y}^1)^T \nabla_{12}^2 f(\bar{x}^1, \bar{y}^1) + \bar{w}^1] + \alpha_1^T \nabla_{12}^2 f(\bar{x}^1, \bar{y}^1) - \mu_1 = 0, \quad \text{... (17)}$$

$$\alpha[\nabla_1 g(\bar{x}^2, \bar{y}^2) + \bar{w}^2] + (\alpha_2 - \lambda\bar{y}^2)^T \nabla_{12}^2 g(\bar{x}^2, \bar{y}^2) - \mu_2 = 0, \quad \text{... (18)}$$

$$-\alpha \nabla_{22}^2 f(\bar{x}^1, \bar{y}^1)\bar{y}^1 + \alpha_1^T \nabla_{22}^2 g(\bar{x}^1, \bar{y}^1) = 0, \quad \text{... (19)}$$

$$(\alpha - \lambda)[\nabla_2 g(\bar{x}^2, \bar{y}^2) - \bar{z}^2] + (\alpha_2 - \lambda\bar{y}^2)^T g_{22}^2(\bar{x}^2, \bar{y}^2) = 0, \quad \text{... (20)}$$

$$\alpha_1 \in N_{D_1}(\bar{z}^1), \quad \text{... (21)}$$

$$\alpha\bar{y}^2 + (\alpha_2 - \lambda\bar{y}^2) \in N_{D_2}(\bar{z}^2), \quad \text{... (22)}$$

$$\bar{w}^1 \in C_1, (\bar{w}^1)^T \bar{x}^1 = s(\bar{x}^1 \mid C_1), \quad \text{... (23)}$$

$$\bar{w}^2 \in C_2, (\bar{w}^2)^T \bar{x}^2 + s(\bar{x}^2 \mid C_2), \quad \text{... (24)}$$

$$\alpha_1^T[\nabla_2 f(\bar{x}^1, \bar{y}^1) - \bar{z}^1] = 0, \quad \text{... (25)}$$

$$\alpha_2^T [\nabla_2 g(\bar{x}^2, \bar{y}^2) - \bar{z}^2] = 0, \quad \dots (26)$$

$$\lambda (\bar{y}^2)^T [\nabla_2 g(\bar{x}^2, \bar{y}^2) - z^2] = 0, \quad \dots (27)$$

$$\mu_1^T \bar{x}^1 = 0, \quad \dots (28)$$

$$\mu_2^T \bar{x}^2 = 0, \quad \dots (29)$$

$$(\alpha, \alpha_1, \alpha_2, \lambda, \mu_1, \mu_2) \geq 0 \text{ and } (\alpha, \alpha_1, \alpha_2, \lambda, \mu_1, \mu_2) \neq 0. \quad \dots (30)$$

From (19) and nonsingularity of the Hessian matrix $\nabla_{22}^2 f(\bar{x}^1, \bar{y}^1)$, we have

$$\alpha_1 = \alpha \bar{y}^1. \quad \dots (31)$$

Multiplying (20) by $\alpha_2 - \lambda \bar{y}^2$, and from (26) and (27), we obtain

$$(\alpha_2 - \lambda \bar{y}^2)^T \nabla_{22}^2 g(\bar{x}^2, \bar{y}^2)(\alpha_2 - \lambda \bar{y}^2) = 0. \quad \dots (32)$$

Since $\nabla_{22}^2 g(\bar{x}^2, \bar{y}^2)$ is positive or negative definite, we have

$$\alpha_2 = \lambda \bar{y}^2. \quad \dots (33)$$

From (20), (33) and the hypothesis $\nabla_2 g(\bar{x}^2, \bar{y}^2) \neq \bar{z}^2$, we have

$$\alpha = \lambda. \quad \dots (34)$$

If $\alpha = 0$, then $\lambda = 0$ and from (33), $\alpha_1 = 0$, and from (17), $\mu_1 = 0$ and from (18), $\mu_2 = 0$. This contradicts (30). Hence $\alpha > 0$ and $\lambda > 0$. From (33) and (30), we have

$$\bar{y}^2 \geq 0. \quad \dots (35)$$

From (31) and (30), we have

$$\bar{y}^1 \geq 0. \quad \dots (36)$$

From (17), (31) and (30), we have

$$\nabla_1 f(\bar{x}^1, \bar{y}^1) + \bar{w}^1 \geq 0. \quad \dots (37)$$

From (18), (33), (30) and $\alpha > 0$, we have

$$\nabla_1 g(\bar{x}^2, \bar{y}^2) + \bar{w}^2 \geq 0. \quad \dots (38)$$

From (18), (33), (29) and $\alpha > 0$, we have

$$\mu_2^T [\nabla_1 g(\bar{x}^2, \bar{y}^2) = \bar{w}^2] \leq 0. \quad \dots (39)$$

Hence from (23), (24), (35), (36, (37), (38) and (39), $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{w}^1, \bar{w}^2)$ is feasible for (MD). Now from (17), (23), (28), (31) and $\alpha > 0$,, we have

$$s(\bar{x}^1 \mid C_1) = -(\bar{x}^1)^T \nabla_1 f(\bar{x}^1, \bar{y}^1). \qquad \text{... (40)}$$

From (21) and (31), we know that $\bar{y}^1 \in N_{D_1}(\bar{z}^1)$, i.e.,

$$(\bar{y}^1)^T z^1 = s(\bar{y}^1 \mid D_1). \qquad \text{... (41)}$$

From (22), (33) and $\alpha > 0$, we have

$$\bar{y} \in N_{D_2}(\bar{z}^2).$$

That is, $$(\bar{y}^2)^T \bar{z}^2 = s(\bar{y}^2 \mid D_2). \qquad \text{... (42)}$$

Finally, from (24), (40), (41) and (42), we give

$$f(\bar{x}^1, \bar{y}^1) + g(\bar{x}^2, \bar{y}^2) + s(\bar{x}^1 \mid C_1) + s(\bar{x}^2 \mid C_2) - (\bar{y}^1)^T \nabla_2 f(\bar{x}^1, \bar{y}^1) - (\bar{y}^2)^T \bar{z}^2$$

$$= f(\bar{x}^1, \bar{y}^1) + g(\bar{x}^2, \bar{y}^2) - s(\bar{y}^1 \mid D_1) - s(\bar{y}^2 \mid D_2) - (\bar{x}^1)^T \nabla_1 f(\bar{x}^1, \bar{y}^1) + (x^2)^T \bar{w}^2. \qquad \text{... (43)}$$

By the weak duality and (43), $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{w}^1, \bar{w}^2)$ is an optimal solution of (MD). □

By the similar method of Theorem 2, we can prove the following converse duality theorems

Theorem 3 — *(Converse duality) Suppose that $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{w}^1, \bar{w}^2)$ is pptimal for problem (MD) and that the Hessian matrix $\nabla_1^2 f(\bar{x}^1, \bar{y}^1)$ is nonsingular, that $\nabla_1^2 g(\bar{x}^2, \bar{y}^2)$ is positive definite or negative definite and that $\nabla_1 g(\bar{x}^2, \bar{y}^2) \neq \bar{w}^2$. Then $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{w}^1, \bar{w}^2)$ is feasible for problem (MP) and the corresponding objective function value are equal. If in addition the hypotheses of Theorem 1 hold, then there exist $\bar{z}^1, \bar{z}^2$ such that $(x^1, x^2, y^1, y^2, z^1, z^2) = (\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{z}^1, \bar{z}^2)$ is opimal for problem(MP).*

4. Special Cases

In this section we consider some special cases of problem (MP) and problem (MD) by choosing particular forms of sublinear functionals F_1, F_2, G_1 and G_2 and the compact convex sets C_1, C_2, D_1 and D_2.

(*i*) If $C_1 = C_2 = \{0\}$, $D_1 = D_2 = \{0\}$, then (MP) and (MD) reduce to a pair of primal problem and dual problem programs studied in Chandra, *et al.*[5]

Primal Problem $(MP)_1$:

Minimize

$$f(x^1, y^1) + g(x^2, y^2) - (y^1)^T \nabla_2 f(x^1, y^1)$$

subject to:

$$\nabla_2 f(x^1, y^1) \leq 0,$$

$$\nabla_2 g(x^2, y^2) \leq 0,$$

$$(y^2)^T \nabla_2 g(x^2, y^2) \geq 0,$$

$$x^1 \geq 0, x^2 \geq 0.$$

Dual Problem $(MD)_1$:

Maximize

$$f(u^1, v^1) + g(u^2, v^2) - (u^1)^T \nabla_1 f(u^1, v^1)$$

subject to:

$$\nabla_1 f(u^1, v^1) \geq 0,$$

$$\nabla_1 g(u^2, v^2) \geq 0,$$

$$(u^2)^T \nabla_1 f(u^2, v^2) \leq 0,$$

$$v^1 \geq 0, v^2 \geq 0.$$

(*ii*) If $J_2 = 0$ and $K_2 = 0$ the symmetric dual pair (MP) and (MD) reduces tot he pair (P) and (D) of Mond and Schechter[11]

Primal Problem $(MP)_2$:

Minimize

$$f(x^1, y^1) + s(x^1 \mid c_1) - (y^1)^T \nabla_2 f(x^1, y^1)$$

subject to :

$$\nabla_2 f(x^1, y^1) - z^1 \leq 0,$$

$$x^1 \geq 0,$$

$$z^1 \in D_1.$$

Dual Problem $(MD)_2$:

Maximize

$$f(u^1, v^1) - s(v^1 \mid D_1) - (u^1)^T \nabla_1 f(u^1, v^1)$$

subject to :

$$\nabla_1 f(u^1, v^1) - z^1 \leq 0,$$

$$v^1 \geq 0,$$

$$w^1 \in C_1.$$

where C_1 and D_1 are compact and convex sets of $I\!R^n$ and $I\!R^m$, respectively.

If $J_1 = 0$ and $K_1 = 0$ the symmetric dual pair (MP) and (MD) reduces to the pair (P_1) and (D_1) of Mond and Schechter[11]

Primal Problem $(MP)_3$

Minimize

$$g(x^2, y^2) + s(x^2 \mid C_2) - (y^2)^T z^2$$

subject to:

$$\nabla_2 g(x^2, y^2) - z^2 \leq 0,$$

$$(y^2)^T (\nabla_{2g}(x^2, y^2) - z^2) \geq 0,$$

$$x^2 \geq 0,$$

$$z^2 \in D_2.$$

Dual problem $(MD)_3$:

Maximize

$$g(u^2, v^2) - s(v^2 \mid D_2) + (u^2)^T w^2$$

subject to :

$$\nabla_1 g(u^2, v^2) + w^2 \geq 0,$$

$$(u^2)^T (\nabla_1 f(u^2, v^2) + w^2) \leq 0,$$

$$v^2 \geq 0,$$

$$w^2 \in C_2,$$

where C_2 and D_2 are compact and convex sets of $I\!R^n$ and $I\!R^m$ respectively.

Chandra, Husain and Abha (see [5]) proved the weak and strong duality theorems for $(MP)_1$ and $(MD)_1$ amd Mond and Schechter (see [11]) proved the weak and strong duality theorems for $(MP)_2$ and $(MD)_2$ or $(MP)_3$ and $(MD)_3$ under convex-concave functions. Here we prove our weak, strong and converse duality theorem under F-convex and F-concave functions. So our theorem 1 and 2 generalize the main results in [5], and improve, extend and unified Mond and Schechter's work in[11].

(*iii*) From the symmetric dual models (MP) and (MD), we can construct other symmetric dual pairs. For example, if we take $C_i = \left\{ A_i y : y^T A_{iy} \leq 1 \right\}$ $(i = 1, 2)$ and $D_i = \left\{ B_{ix} : x^T B_{ix} \leq 1 \right\}$ $(i = 1, 2)$ where A_i and B_i are positive semi-definite, then it can be readily verified that $(x^T A_i x)^{1/2} -= s\,(x \mid C_i)$ and $(y^T B_i y)^{1/2} = s\,(y \mid D_i)$, and thus a number of symmetric dual pairs and duality results are obtained. In particular, $(MP)_2$ and $(MD)_2$ reduce to the symmetric dual pair S. Chandra and I. Husain[4].

(*iv*) These results in this paper can also be extended to multiobjective programming, and integer programming on the line of [2, 3] under various types of generalized convexity assumption.

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