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ON THE COMPOSITION OF THE DISTRIBUTIONS x_+^{-r} AND x_+^{μ}

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Let F be a distirbution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta function $\delta(x)$. The distribution $\left(x_+^{\mu}\right)_+^{-r}$ and $\left(|x|^{\mu}\right)_+^{-r}$ are evaluated for $\mu > 0$, $r = 1, 2, \ldots$, and $k\mu \neq 1, 2, \ldots$.

Key Words: Distribution; Delta function; Composition of Distributions; Neutrix; Neutrix Limit

1. INTRODUCTION

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$ and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

$$\ln x_+ = \begin{cases} \ln x, & x > 0, \\ 0, & x < 0, \end{cases} \qquad \ln x_- = \begin{cases} \ln |x|, & x < 0, \\ 0, & x > 0. \end{cases}$$

The distributions x_+^{-r} and x_-^{-r} are then defined by

$$x_+^{-r} = \frac{(-1)^{r-1} (\ln x_+)^{(r)}}{(r-1)!}, \quad x_-^{-r} = -\frac{(\ln x_-)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \ldots$ and not as in Gel'fand and Shilov[7].

We define the locally summable function x_+^λ and x_-^λ for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^\lambda = \begin{cases} |x|^\lambda & x < 0, \\ 0, & x > 0. \end{cases}$$

The distributions x_+^λ and x_-^λ are then defined inductively for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ by

$$\left(x_+^\lambda\right)' = \lambda x_+^{\lambda-1}, \quad \left(x_-^\lambda\right)' = -\lambda x_-^{\lambda-1}.$$

It follows that if r is a positive integer and $-r-1 < \lambda < -r$, then

$$\langle x_+^\lambda, \varphi(x) \rangle = \int_0^\infty x^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right] dx,$$

$$\langle x_-^\lambda, \varphi(x) \rangle = \int_{-\infty}^0 |x|^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right] dx$$

for arbitrary φ in $\mathcal{D}$. The distribution $|x|^\lambda$ is then defined by

$$|x|^\lambda = x_+^\lambda + x_-^\mu.$$

In particular, if φ has its support contained in the interval [–1, 1], then

$$\langle x_+^\lambda, \varphi(x) \rangle = \int_0^1 x^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right] dx + \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!\,(\lambda+k+1)} \quad \ldots (1)$$

if $-r-1 < \lambda < -r$ and

$$\langle |x|^\lambda, \varphi(x) \rangle = \int_{-1}^1 lx^\lambda \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(2k)}(0)}{2k!} x^{2k} \right] dx$$

$$+ \sum_{k=0}^{r-1} \frac{2\,\varphi^{(k)}(0)}{2(2k)!\,(\lambda+2k+1)} \quad \ldots (2)$$

if $-2r-2 < \lambda < -2r$ and $\lambda \neq -2r-1$.

We now let N be the neutrix, see[1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ \ln^r n : \quad \lambda > 0, r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(*i*) $\rho(x) = 0$ for $|x| \geq 1$,

(*ii*) $\rho(x) \geq 0$,

(*iii*) $\rho(x) = \rho(-x)$,

(*iv*) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in[2].

Definition 1 — Let F be a distribution and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$N-\lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x))\, \varphi(x)\, dx = \langle h(x), \varphi(x) \rangle$$

for all test functions φ with compact support contained in (a, b).

The following five theorems were proved in[2-6] respectively.

Theorem 1 — *The distributions* $(x_-^{\mu})_-^{\lambda} = (x_+^{\mu})_-^{\lambda} = 0$

for $\mu > 0$ *and* $\lambda\mu \neq -1, -2, \ldots$ *and*

$$(x_-^{\mu})_-^{\lambda} = (-1)^{\lambda\mu} (x_+^{\mu})_-^{\lambda} = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\mu(-\lambda\mu-1)!}\, \delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0$, $\lambda\mu \neq -1, -2, \ldots$ and $\lambda\mu = -1, -2, \ldots$.

***Theorem* 2** — *The distribution* $(x_+^r)_-^{-s}$ *exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s} c_1(\rho)}{r(rs-1)!} \delta^{(rs-1)}(x)$$

for $r, s = 1, 2, \ldots$.

***Theorem* 3** — *If* $F_s(x)$ *denotes the distribution* $x^{-s} \ln|x|$, *then the distribution* $F_s(x^r)$ *exists and*

$$F_s(x^r) = rF_{rs}(x)$$

for $r, s = 1, 2, \ldots$.

***Theorem* 4** — *LET* $F_s(x)$ *denote the distribution* $x_-^{-s} \ln x_-$. *Then the distribution* $F_{ms}\left(x_+^{r-p/m}\right)$ *exists and*

$$F_{ms}\left(x_+^{r-p/m}\right) = \frac{(-1)^{ms+msr+sp} \phi(ms-1) c_1(\rho)}{(r-p/m)(msr-sp-1)!} \delta^{(msr-sp-1)}(x)$$

$$+ \frac{(-1)^{ms+msr+sp+1} [\phi_1(ms) + \phi_2(ms) - \phi^2(ms) + c_2(\rho)]}{(r-p/m)(msr-sp-1)!} \delta^{(msr-sp-1)}(x)$$

for $r, s = 1, 2, \ldots$ *and* $m = 2, 3, \ldots$, *where*

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} 1/i, & r \geq 1, \\ 0, & r = 0 \end{cases}, \qquad \phi_1(r) = \begin{cases} \sum_{i=1}^{r-1} \frac{\phi(i)}{i+1}, & r = 2, 3, \ldots, \\ 0, & r = 0, 1. \end{cases}$$

$$\phi_2(r) = \sum_{i=1}^{r+1} \frac{\phi(i)}{i}, \quad r = 0, 1, 2, \ldots,$$

$$c_1(\rho) = \int_0^1 \ln t \, \rho(t) \, dt, \quad c_2(\rho) = \int_0^1 \ln^2 t \, \rho(t) \, dt,$$

$1 \leq p < m$ *and* p *and* m *are coprime.*

***Theorem* 5** — *The distribution* $(x_+^r)^{-1}$ *exists and*

$$(x_+^r)^{-1} = x_+^{-r} + (-1)^r \frac{2c_1(\rho) - r\phi(r-1)}{r!} \delta^{(r-1)}(x),$$

for $r = 1, 2, \ldots$.

We now prove

Theorem 6 — *The distribution $(x_+^{\mu})_+^{-r}$ exists and*

$$(x_+^{\mu})_+^{-r} = x_+^{-\mu r} \qquad \ldots (3)$$

for $\mu > 0$, $r = 1, 2, \ldots$ and $\mu k \neq 1, 2, \ldots$ for $k = 1, 2, \ldots$.

PROOF: We put

$$(x_+^{-r})_n = x_+^{-r} * \delta_n(x) = \frac{(-1)^{r-1}}{(r-1)!} \ln x_+ * \delta_n^{(r)}(x)$$

and so

$$(-1)^{r-1}(r-1)!\,(x_+^{-r})_n = \begin{cases} \int_{-1/n}^{1/n} \ln(x-t)\,\delta_n^{(r)}(t)\,dt, & 1/n < x, \\ \int_{-1/n}^{x} \ln(x-t)\,\delta_n^{(r)}(t)\,dt, & -1/n \le x \le 1/n, \\ 0, & x < -1/n \end{cases}$$

Then

$$(-1)^{r-1}(r-1)!\,[(x_+^{\mu})_+^{-r}]_n = \begin{cases} \int_{-1/n}^{1/n} \ln(x^{\mu}-t)\,\delta_n^{(r)}(t)\,dt, & 1/n < x^{\mu} \\ \int_{-1/n}^{x^{\mu}} \ln(x^{\mu}-t)\,\delta_n^{(r)}(t)\,dt, & 0 \le x^{\mu} \le 1/n, \\ \int_{-1/n}^{0} \ln(-t)\,\delta_n^{(r)}(t)\,dt, & x \le 0. \end{cases} \qquad \ldots (4)$$

It follows that

$$(-1)^{r-1}(r-1)! \int_{-1}^{1} x^k \left[\left(x_+^{\mu}\right)_+^{-r}\right]_n dx = \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^{\mu}} \ln(x^{\mu}-t)\,\delta_n^{(r)}(t)\,dt\,dx$$

$$+\int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} \ln(x^\mu - t)\, \delta_n^{(r)}(t)\, dt\, dx$$

$$+\int_{n^{-1}}^{0} x^k \int_{-1/n}^{0} \ln(-t)\, \delta_n^{(r)}(t)\, dt\, dx$$

$$= \frac{n^{r-(k+1)/\mu}}{\mu} \int_0^1 v^{(k+1)/\mu - 1} \int_{-1}^{v} [\ln(v-u) - \ln n]\, \rho^{(r)}(u)\, du\, dv$$

$$+ \frac{n^{r-(k+1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(r)}(u) \int_1^n v^{(k+1)/\mu - 1} \ln(v-u)\, dv\, du$$

$$- \frac{n^{r-(k+1)/\mu} \ln n}{\mu} \int_{-1}^{1} \rho^{(r)}(u) \int_1^n v^{(k+1)/\mu - 1}\, dv\, du$$

$$+ n^r \int_{-1}^{0} x^k \int_{-1}^{0} [\ln(-u) - \ln n]\, \rho^{(r)}(u)\, du\, dx$$

$$= I_1 + I_2 + I_3 + I_4, \qquad \text{... (5)}$$

where the substitutions $u = nt$ and $v = nx^\mu$ have been made.

It follows immediately that

$$\underset{n\to\infty}{N-\lim}\, I_1 = \underset{n\to\infty}{N-\lim}\, I_3 = \underset{n\to\infty}{N-\lim}\, I_4 = 0 \qquad \text{... (6)}$$

for $k = 0, 1, 2, \ldots$.

Further,

$$\int_1^n v^{(k+1)/\mu - 1} \ln(v-u)\, dv = \int_1^0 v^{(k+1)/\mu - 1} \ln v\, dv$$

$$- \sum_{i=1}^{\infty} u^i \int_1^0 \frac{v^{(k+1)/\mu - i - 1}}{i}\, dv$$

$$= \frac{\mu\, n^{(k+1)/\mu} \ln n}{k+1} - \frac{\mu^2 (n^{(k+1)/\mu} - 1)}{(k+1)^2}$$

$$- \sum_{i=1}^{\infty} \frac{u^i [n^{(k+1)/\mu - i} - 1]}{i[(k+1)/\mu - i]}$$

and so

$$N-\lim_{n\to\infty} I_2 = \frac{1}{r(\mu r-k-1)} \int_{-1}^{1} u^r \rho^{(r)}(u)\,du$$

$$= \frac{(-1)^r (r-1)!}{(\mu r-k-1)} \qquad \dots (7)$$

for $k = 0, 1, 2, \dots$.

It now follows from eqs. (5), (6) and (7) that

$$N-\lim_{n\to\infty} \int_{-1}^{1} x^k \left[\left(x_+^{\mu}\right)_+^{-1}\right]_n dx = -(\mu r-k-1)^{-1}, \qquad \dots (8)$$

for $k = 0, 1, 2, \dots$.

We now consider the case $k = s$, where s is chosen so that $0 < -\mu r+s+1<1$, and let ψ be an arbitrary continuous function. Then

$$(-1)^{r-1}(r-1)! \int_0^{n^{-1/\mu}} x^s \psi(x) \left[\left(x_+^{\mu}\right)_+^{-1}\right]_n dx$$

$$= \frac{n^{(\mu r-s-1)/\mu}}{\mu} \int_0^1 v^{(r+1)/\mu-1} \int_{-1}^{v} (v-u)^{s-r} \rho^{(s)}(u)\,du\,dv$$

and it follows that

$$\lim_{n\to\infty} \int_0^{n^{-1/\mu}} x^s \psi(x) \left[\left(x_+^{\mu}\right)_+^{-1}\right]_n dx = 0. \qquad \dots (9)$$

When $x \le 0$, we have

$$(-1)^{r-1}(r-1)! \int_{-1}^{0} x^s \psi(x) \left[\left(x_+^{\mu}\right)_+^{-r}\right]_n dx$$

$$= n^r \int_{-1}^{0} x^s \psi(x) \int_{-1}^{0} (-u)^{s-r} \rho^{(r)}(u)\,du\,dx$$

and it follows that

$$N-\lim_{n\to\infty} \int_{-1}^{0} x^s \psi(x) \left[\left(x_+^{\mu}\right)_+^{-r}\right]_n dx = 0. \qquad \dots (10)$$

When $x^{\mu} \geq 1/n$, we have

$$(-1)^{r-1}(r-1)!\left[\left(x_{+}^{\mu}\right)_{+}^{-r}\right]_{n} = \int_{-1/n}^{1/n} \ln(x^{\mu}-t)\,\delta_{n}^{(r)}(t)\,dt$$

$$= n^{r}\int_{-1}^{1} \ln(x^{\mu}-u/n)\;\rho^{(r)}(u)\,du$$

$$= n^{r}\mu\ln x\int_{-1}^{1} \rho^{(r)}(u)\,du - n^{r}\sum_{i=1}^{\infty}\int_{-1}^{1}\frac{u^{i}\rho^{(r)}(u)}{in^{i}x^{\mu i}}\,du$$

$$= -(r-1)!\,x^{-\mu r} + O(n^{-1}).$$

and so

$$\left[\left(x_{+}^{\mu}\right)_{+}^{-r}\right]_{n} = (-1)^{r}x^{-\mu r} + O(n^{-1}). \qquad \dots (11)$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{s-1}\frac{x^{k}}{k!}\varphi^{(k)}(0) + \frac{x^{s}}{s!}\varphi^{(s)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\langle [(x_{+}^{\mu})_{+}^{-r}]_{n}, \varphi(x)\rangle = \int_{-1}^{1}[(x_{+}^{\mu})_{+}^{-r}]_{n}, \varphi(x)\,dx$$

$$= \sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}[(x_{+}^{\mu})_{+}^{-r}]_{n}, dx + \int_{n^{-1/\mu}}^{1}\frac{x^{s}}{s!}\int_{-1}^{1}[(x_{+}^{\mu})_{+}^{-r}]_{n}, \varphi^{(s)}(\xi x)\,dx$$

$$+ \int_{0}^{n^{-1/\mu}}\frac{x^{s}}{s!}\int_{-1}^{1}[(x_{+}^{\mu})_{+}^{-r}]_{n}, \varphi^{(s)}(\xi x)\,dx + \int_{-1}^{0}\frac{x^{s}}{s!}\int_{-1}^{1}[(x_{+}^{\mu})_{+}^{-r}]_{n}, \varphi^{(s)}(\xi x)\,dx.$$

Using eqs. (8) to (11), it follows that

$$N-\lim_{n\to\infty}\langle [(x_{+}^{\mu})_{+}^{-r}]_{n}, \varphi(x)\rangle = \sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{(-\mu r+k+1)\,k!} + \int_{0}^{1}\frac{x^{-r\mu+s}}{s!}\varphi^{(r)}(\xi x)\,dx$$

$$= \int_0^1 x^{-\mu r}\left[\varphi(x) - \sum_{k=0}^{s-1} \frac{x^k}{k!}\varphi^{(k)}(0)\right]dx + \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{(-\mu r + k + 1)\,k!}$$

$$= \langle x_+^{-\mu r}, \varphi(x)\rangle,$$

on using eq. (1). This proves eq (3) on the interval [–1, 1]. However, eq. (3) clearly holds on any interval not containing the origin, and the proof is complete.

Corollary 6.1 — The distribution $(x_-^{\mu})_-^{-r}$ exists and

$$(x_-^{\mu})_-^{-r} = x_-^{-\mu r} \qquad \ldots (12)$$

for $\mu > 0$, $r = 1, 2, \ldots$ and $k\mu \neq -1, -2, \ldots$ for $k = 1, 2, \ldots$.

PROOF: Equation (12) follows on replacing x by $-x$ in eq. (3).

Theorem 7 — *The distribution* $(|x|^{\mu})_+^{-r}$ *exists and*

$$(|x|^{\mu})_+^{-r} = |x|^{-\mu r} \qquad \ldots (13)$$

for $\mu > 0$, $r = 1, 2, \ldots$ *and* $k\mu \neq -1, -2, \ldots$ *for* $k = 1, 2, \ldots$.

PROOF: It follows from eq. (4) that

$$(-1)^{r-1}(r-1)!\,[(|x|^{\mu})_+^{-r}]_n = \begin{cases} \displaystyle\int_{-1/n}^{1/n} \ln(|x|^{\mu} - t)\,\delta_n^{(r)}(t)\,dt, & 1/n < |x|^{\mu}, \\ \displaystyle\int_{-1/n}^{|x|^{\mu}} \ln(|x|^{\mu} - t)\,\delta_n^{(r)}(t)\,dt, & 0 \le |x|^{\mu} \le 1/n. \end{cases} \qquad \ldots (14)$$

Since $[(|x|^{\mu})_+^{-r}]_n$ is an even function, it follows that

$$\int_{-1}^{1} x^k [(|x|^{\mu})_+^{-r}]_n\,dx = 0 \qquad \ldots (15)$$

for $k = 1, 3, \ldots$

If k is even, we have

$$(-1)^{r-1}(r-1)! \int_{-1}^{1} x^k\,[(|x|^{\mu})_+^{-r}]_n\,dx$$

$$= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{|x|^\mu} \ln(|x|^\mu - t)\ \delta_n^{(r)}(t)\,dt\,dx$$

$$+ \int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} \ln(|x|^\mu - t)\ \delta_n^{(r)}(t)\,dt\,dx$$

$$= I_1 + I_2$$

and it follows as above that

$$N\!-\!\lim_{n\to\infty} \int_{-1}^{1} x^k\ [(|x|^\mu)_+^{-r}]_n\,dx = 2(-\mu r + k + 1)^{-1} \qquad \text{... (16)}$$

for $k = 0, 2, 4, \ldots$.

We now consider the case $k = 2s$, where s is chosen so that $0 < -\mu r + s + 2 < 1$, and let ψ be an arbitrary continuous function. Then it follows as above that

$$\lim_{n\to\infty} \int_0^{n^{-1/\mu}} x^{2s}\ \psi(x)\ [(|x|^\mu)_+^{-r}]_n\,dx$$

$$= \lim_{n\to\infty} \int_{-n^{-1/\mu}}^{0} x^{2s}\ \psi(x)\ [(|x|^\mu)_+^{-r}]_n\,dx = 0 \qquad \text{... (17)}$$

and

$$[(|x|^\mu)_+^{-r}]_n = |x|^{-\mu r} + O(n^{-1}) \qquad \text{... (18)}$$

if $|x|^\mu \geq 1/n$.

Again let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. Then

$$\varphi(x) = \sum_{k=0}^{2s-1} \frac{x^k}{(k)!}\ \varphi^{(k)}(0) + \frac{x^{2s}}{(2s)!}\ \varphi^{(2s)}(\xi x)$$

where $0 < \xi < 1$. Then

$$\langle [(|x|^\mu)_+^{-r}]_n, \varphi(x) \rangle = \int_{-1}^{1} [(|x|^\mu)_+^{-r}]_n, \varphi(x)\,dx$$

$$= \sum_{k=0}^{r-1} \frac{\varphi^{(2k+1)}(0)}{(2k+1)!} \int_{-1}^{1} x^{2k} [(|x|^{\mu})_+^{-r}]_n, dx$$

$$+ \sum_{k=0}^{s-1} \frac{\varphi^{(2k)}(0)}{(2k)!} \int_{-1}^{1} x^{2k} [(|x|^{\mu})_+^{-r}]_n, \varphi(x)\, dx$$

$$+ \int_{n^{-1/\mu}}^{1} \frac{x^{2s}}{(2s)!} [(|x|^{\mu})_+^{-r}]_n, \varphi^{(2s)}(\xi x)\, dx$$

$$+ \int_{0}^{-n^{-1/\mu}} \frac{x^{2s}}{(2s)!} [(|x|^{\mu})_+^{-r}]_n, \varphi^{(2s)}(\xi x)\, dx$$

$$+ \int_{0}^{n^{-1/\mu}} \frac{x^{2s}}{(2s)!} [(|x|^{\mu})_+^{-r}]_n, \varphi^{(2s)}(\xi x)\, dx$$

$$+ \int_{-n^{-1/\mu}}^{0} \frac{x^{2s}}{(2s)!} [(|x|^{\mu})_+^{-r}]_n, \varphi^{(2s)}(\xi x)\, dx.$$

Using eqs. (15) to (18), it follows that

$$N - \lim_{n \to \infty} \langle [(|x|^{\mu})_+^{-r}]_n, \varphi(x) \rangle$$

$$= \sum_{k=0}^{s-1} \frac{\varphi^{(2k)}(0)}{(-\mu r + 2k + 1)(2k)!} + \int_{-1}^{1} \frac{|x|^{-\mu r + 2s}}{(2s)!} \varphi^{(2s)}(\xi x)\, dx$$

$$= \int_{-1}^{1} |x|^{\mu r} \left[\varphi(x) - \sum_{k=0}^{r-1} \frac{x^{2k}}{(2k)!} \varphi^{(2k)}(0) \right] dx$$

$$+ \sum_{k=0}^{s-1} \frac{\varphi^{(2k)}(0)}{(-\mu r + 2k + 1)(2k)!}$$

$$= \langle |x|^{\mu^{-\mu r}}, \varphi(x) \rangle,$$

on using eq. (2). This proves eq. (13) on the interval [–1, 1]. However, eq. (13) clearly holds on any interval not containing the origin, and the proof is complete.

Corollary 7.1 — The distribution $(|x|^{\mu})_{-}^{-r}$ exists and

$$(|x|^{\mu})_{-}^{-r} = |x|^{\mu - r} \qquad \text{... (19)}$$

for $\mu > 0$, $r = 1, 2, \ldots$ and $k\mu \neq -1, -2, \ldots$ for $k = 1, 2, \ldots$.

PROOF: Eq. (19) follows on replacing x by $-x$ in eq. (13).

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