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NOTE ON 'OPTIMALITY CONDITIONS FOR LINEAR FRACTIONAL BILEVEL PROGRAMS'

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N. Malhotra and S. R. Arora [Indian Journal of Pure and Applied Mathematics 30(4) : 373-384, 1999] present some results on necessary and sufficient optimality conditions for linear fractional bilevel programs based on the duality theory. This note shows that Malhotra and Arora's proofs contain a flaw that makes their conclusions invalid and provides alternative optimality conditions.

Key Words: Bilevel Programming; Fractional Programming; Duality

1. INTRODUCTION

Bilevel programming involves two optimization problems where the constraint region of the first level problem is implicitly determined by another optimization problem. It has been applied to decentralized planning problems involving a decision process with a hierarchical structure. The bilevel programming problem is a nonconvex optimization problem that has received increasing attention in the literature (see[1,7,9,10] and references therein).

Malhotra and Arora[8] consider the linear fractional bilevel programming (LFBP) problem in which both objective functions are linear fractional and the common constraint region is a polyhedron. This problem has also been considered in[3,4,5]. The LFBP problem is formulated as:

$$\max_{x} \quad F(x, y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}$$

where y solves

$$\max_{y} \quad f(x, y) = \frac{r^T x + p^T y + \gamma}{s^T x + q^T y + \delta}$$

subject

$$Ax + By \leq t$$

where $x \in R^{n_1}$ and $y \in R^{n_2}$ are the variables controlled by the first level (leader) and the second level (follower) decision maker; $a, c, r, s \in R^{n_1}$; $b, d, p, q \in R^{n_2}$; $\alpha, \beta, \gamma, \delta$ are scalars; $t \in R^m$; and A, B are $m \times n_1$ and $m \times n_2$ matrices, respectively.

Let $S = \{(x, y) : Ax + By \leq t\}$. It is assumed that:

(i) S is nonempty and compact.

(ii) $c^T x + d^T y + \beta > 0, s^T x + q^T y + \delta > 0$ for all $(x, y) \in S$.

Finally, for each value of x, let (P_x) be the second level problem:

$$(P_x): \qquad \max_y \quad f_1(x, y) = \frac{p^T y + \gamma_1}{q^T y + \delta_1} \qquad \dots (2)$$

subject to

$$By \leq t - Ax$$

where $\gamma_1 = r^T x + \gamma$ and $\delta_1 = s^T x + \delta$. To ensure that the LFBP problem is posed well, it is also assumed that for each value of x, the optimal solution to (P_x) is a singleton. See[1,7,10] for problems caused by multiple optima in the second level problem.

Let $\bar{S}$ be the set of feasible solutions to the LFBP problem, i.e.,

$$\bar{S} = \{(x, y) \in S : y \text{ solves } (P_x)\}$$

Then, the LFBP problem can be written as $\max_{(x, y) \in \bar{S}} F(x, y)$.

By considering duality theory applied to the second level problem, Malhotra and Arora[8] developed optimality conditions for the LFBP problem. In this note, we show that these optimality conditions are incorrect because the theoretical analysis contains a flaw that makes its conclusions invalid and derive alternative optimality conditions.

2. Examining the Theorems and Proofs in [8]

Taking into account (2), Malhotra and Arora state that the LFBP problem becomes equivalent to:

$$\max \quad F(x, y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}$$

where y solves

$$\max \quad f_1(x,y) = \frac{p^T y + \gamma_1}{q^T y + \delta_1} \qquad \dots (3)$$

subject to

$$By \leq t - Ax.$$

Notice that this formulation of the problem is misleading as it hides the dependence of γ_1 and δ_1 on x.

In their first theorem (Theorem 1 in[8]), by considering the dual problem of the second level problem, Malhotra and Arora prove that (3) is equivalent to the following nonlinear programming problem:

$$(P_1): \qquad \max_{(x,y,w)} \quad F(x,y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}$$

subject to

$$Ax + By \leq t \qquad \dots (4)$$

$$w^T (Ax + By - t) = 0 \qquad \dots (5)$$

$$B^T w + \frac{\gamma_1 + (t - Ax)^T w}{\delta_1} q = p \qquad \dots (6)$$

$$w \geq 0 \qquad \dots (7)$$

Notice that this is again a misleading formulation of the problem by the same reasons previously mentioned. Moreover, it is implicitly assumed that $\delta_1 \neq 0$. The authors in[8] continue by reformulating (P_1) as:

$$\max_{w \geq 0} \quad \max_{(x,y) \in S[w]} \quad F(x,y) \qquad \dots (8)$$

where, for a given $w \geq 0$,

$$S[w] = \{(x,y) : (x,y) \text{ satisfies (4)-(7)}\}$$

and $\max \{F(x,y) : (x,y) \in S[w]\} = -\infty$ when $S[w] = 1$.

For a given $w \geq 0$, they state that the inner problem in (8) is a linear fractional programming problem and write it as:

$$(P_w): \qquad \max_{(x,y)} \quad F(x,y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}$$

subject to

$$Ax + By \leq t$$

$$- w^T Ax - w^T By \leq - w^T t$$

$$q \frac{w^T Ax}{\delta_1} = B^T w + \frac{\gamma_1 + w^T t}{\delta_1} q - p \qquad \dots (9)$$

Then they continue by writing its dual problem as:

$$(D_w): \quad \min_{(u_1, u_2, u_3)} \quad h_w(u_1, u_2, u_3)$$

$$= \frac{(u_1 - u_2 w)^T t + u_3^T \left\{ \left(B + \frac{tq^T}{\delta_1} \right)^T w + \left(\frac{\gamma_1 q}{\delta_1} - p \right) \right\} + \alpha}{\beta}$$

subject to

$$(u_1 - u_2 w)^T A + u_3^T \left(\frac{q w^T A}{\delta_1} \right)$$

$$+ \frac{(u_1 - u_2 w)^T t + u_3^T \left\{ \left(B + \frac{tq^T}{\delta_1} \right)^T w + \left(\frac{\gamma_1 q}{\delta_1} - p \right) \right\} + \alpha}{\beta} c^T = a^T$$

$$(u_1 - u_2 w)^T B$$

$$+ \frac{(u_1 - u_2 w)^T t + u_3^T \left\{ \left(B + \frac{tq^T}{\delta_1} \right)^T w + \left(\frac{\gamma_1 q}{\delta_1} - p \right) \right\} + \alpha}{\beta} d^T = b^T$$

$$u_1, u_2 \geq 0$$

where $u_1 \in R^m$, $u_2 \in R$ and $u_3 \in R^{n_2}$.

This is the key point to establish the remaining theorems on paper[8]. However, the problem (D_w) is not the dual problem of (P_w). Note that (P_w) contains constraint (9) whose right-hand side depends on x through $\gamma_1 = r^T x + \gamma$ and $\delta_1 = s^T x + \delta$. But, it has been considered as a constant when writing the dual problem. In other words, Malhotra and Arora consider a dual problem, (D_w), which depends on the variables of its primal problem (P_w). This fact invalidates the remaining theorems in[8]. Besides, it is implicitly assumed that $\beta \neq 0$.

In the next section, we derive necessary and sufficient optimality conditions for the LFBP problem in such a way that it preserves the general philosophy of Malhotra and Arora's paper, but overcomes its difficulties.

3. Optimality Conditions for the LFBP Problem

Before proving the next theorem it is convenient to mention that, for a given x, by using the Charnes and Cooper transformation[6], we obtain that the dual problem of (P_x) is [2]:

$$(D_x) : \quad \min_{(w_1, w_2)} \quad w_2$$

subject to

$$B^T w_1 + q w_2 = p$$

$$w_1^T (Ax - t) + w_2 \delta_1 = \gamma_1$$

$$w_1 \geq 0$$

where $w_1 \in R^m$ and $w_2 \in R$. As a consequence of well known results of duality theory in linear programs, given feasible solutions to the primal and dual problems, they are respectively optimal if and only if

$$w_1^T (Ax + By - t) = 0.$$

Hence, for any fixed $(x, y) \in \overline{S}$, since y is an optimal solution to (P_x), there exist [2] $w_1 \in R^m$ and $w_2 \in R$ such that (x, y, w_1, w_2) satisfies

$$Ax + By \leq t \quad \text{... (10)}$$

$$w_1^T (Ax + By - t) = 0 \quad \text{... (11)}$$

$$w_1^T (Ax - t) + w_2 (s^T x + \delta) - r^T x = \gamma \quad \text{... (12)}$$

$$B^T w_1 + q w_2 = p \quad \text{... (13)}$$

$$w_1 \geq 0. \quad \text{... (14)}$$

Similarly, if (x, y, w_1, w_2) satisfies (10)-(14) then $(x, y) \in \overline{S}$.

***Theorem* 3.1** — *(x^*, y^*) is an optimal solution to the LFBP problem if and only if there exist $w_1^* \in R^m$ and $w_2^* \in R$ such that (x^*, y^*, w_1^*, w_2^*) is an optimal solution to the following nonlinear programming problem*

$$(NP): \qquad \max_{(x, y, w_1, w_2)} \quad F(x, y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}$$

subject to (10) - (14).

PROOF : Similar to Theorem 1 in[8]. □

Given $w_1 \in R^m$ and $w_2 \in R$, let

$$W = \{(w_1, w_2) : (w_1, w_2) \text{ satisfies (13) - (14)}\}$$

For a given $(w_1, w_2) \in W$, let

$$S[w_1, w_2] = \left\{ (x, y) \in R^{n_1 + n_2} : (x, y) \text{ satisfies (10)–(12)} \right\}.$$

If $S[w_1, w_2] = \phi$ we define $\max\{F(x, y) : (x, y) \in S[w_1, w_2]\} = -\infty$. Hence, if the problem (NP) has an optimal solution it can be reformulated as:

$$\max_{(w_1, w_2) \in W} \quad \max_{(x, y) \in S[w_1, w_2]} \quad F(x, y) \qquad \text{... (15)}$$

For a given $(w_1, w_2) \in W$, since $w_1^T (Ax + By - t) \leq 0$, the inner problem in (15) can be written as:

$$(P_{w_1, w_2}) : \qquad \max_{(x, y)} \quad F(x, y) = \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}$$

subject to

$$Ax + By \leq t$$

$$-w_1^T Ax - w_1^T By \leq -w_1^T t$$

$$\left(w_1^T A + w_2 s^T - r^T \right) x = \gamma + w_1^T t - w_2 \delta.$$

Since w_1 and w_2 are fixed, this is a linear fractional programming problem whose dual problem is [2]:

$$(D_{w_1, w_2}) : \quad \min_{(u_1, u_2, u_3, u_4)} \quad h(u_1, u_2, u_3, u_4) = u_4$$

subject to

$$(u_1 - u_2 w_1)^T A + \left(w_1^T A + w_2 s^T - r^T \right) u_3 + u_4 c^T = a^T \qquad \dots (16)$$

$$(u_1 - u_2 w_1)^T B + u_4 d^T = b^T \qquad \dots (17)$$

$$-(u_1 - u_2 w_1)^T t - \left(\gamma + w_1^T t - w_2 \delta \right) u_3 + \beta u_4 = \alpha \qquad \dots (18)$$

$$u_1 \geq 0, \ u_2 \geq 0 \qquad \dots (19)$$

where $u_1 \in R^m$ and $u_2, u_3, u_4 \in R$.

For a given $(w_1, w_2) \in W$, let

$$S\left\{ w_1, w_2 \right\} = \left\{ (u_1, u_2, u_3, u_4) \in R^{m+3} : (u_1, u_2, u_3, u_4) \text{ satisfies (16)–(19)} \right\}.$$

If $S\left\{ w_1, w_2 \right\} = \phi$ we define $\min\left\{ h(u_1, u_2, u_3, u_4) : (u_1, u_2, u_3, u_4) \in S\left\{ w_1, w_2 \right\} \right\} = -\infty$. Let (DP) be the following max-min problem:

$$(DP) : \quad \max_{(w_1, w_2) \in W} \quad \min_{(u_1, u_2, u_3, u_4) \in S\left\{ w_1, w_2 \right\}} \quad h(u_1, u_2, u_3, u_4)$$

***Theorem* 3.2** — *If (x, y, w_1, w_2) and $(u_1, u_2, u_3, u_4, w_1, w_2)$ are feasible solutions to the problems (NP) and (DP), respectively, then*

$$u_4 \geq \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}.$$

PROOF: Multiplying (16) by x, (17) by y and then adding them together with (18),

$$(u_1 - u_2 w_1)^T (Ax + By - t) + \left(w_1^T Ax + w_2 s^T x - r^T x - \gamma - w_1^T t + w_2 \delta \right) u_3$$
$$+ u_4 (c^T x + d^T y + \beta) = a^T x + b^T y + \alpha.$$

Taking into account (11) and (12), the above equality can be rewritten as

$$u_1^T (Ax + By - t) + u_4 (c^T x + d^T y + \beta) = a^T x + b^T y + \alpha$$

Since $c^T x + d^T y + \beta > 0$, $u_1 \geq 0$ and $Ax + By - t \leq 0$,

$$u_4 \geq \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta}.$$

and the proof is completed. □

Theorem 3.3 — *Let* $\left(x^*, y^*, w_1^*, w_2^*\right)$ *and* $\left(u_1^*, u_2^*, u_3^*, u_4^*, w_1^*, w_2^*\right)$ *be feasible solutions to the problems (NP) and (DP), respectively. Then they are respectively optimal if and only if*

$$u_4^* \geq \frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta} \qquad \text{... (20)}$$

and, for all $(x, y) \in \overline{S}$,

$$\left(u_1^* - u_2^* w_1^*\right)^T (Ax + By - t) + \left(w_1^{*T} Ax + w_2^* s^T x - r^T x - \gamma - w_1^{*T} t + w_2^* \delta\right) u_3^* \leq 0 \qquad \text{... (21)}$$

PROOF: For a given $(w_1, w_2) \in W$, if $S[w_1, w_2] \neq \phi$ then the problem $\left(P_{w_1, w_2}\right)$ reaches an optimal solution. Bearing in mind duality theory in linear fractional programs[2], its dual problem $\left(D_{w_1, w_2}\right)$ also reaches an optimal solution and their optimal objective values are equal, i.e.,

$$\max_{(x, y) \in S[w_1, w_2]} F(x, y) = \min_{(u_1, u_2, u_3, u_4) \in S\{w_1, w_2\}} h(u_1, u_2, u_3, u_4)$$

Moreover, if $S[w_1, w_2] = \phi$, by convention

$$\max_{(x, y) \in S[w_1, w_2]} F(x, y) = \min_{(u_1, u_2, u_3, u_4) \in S\{w_1, w_2\}} h(u_1, u_2, u_3, u_4) = -\infty$$

Hence,

$$\max_{(w_1, w_2) \in W} \max_{(x, y) \in S[w_1, w_2]} F(x, y) = \max_{(w_1, w_2) \in W} \min_{(u_1, u_2, u_3, u_4) \in S\{w_1, w_2\}} h(u_1, u_2, u_3, u_4) \qquad \text{... (22)}$$

In other words, if $\left(x^*, y^*, w_1^*, w_2^*\right)$ and $\left(u_1^*, u_2^*, u_3^*, u_4^*, w_1^*, w_2^*\right)$ are optimal solutions to the problems (NP) and (DP), respectively, then (20) holds.

Let $(x, y) \in \overline{S}$, then there exist $w_1 \in R^m$ and $w_2 \in R$ such that (x, y, w_1, w_2) is a feasible solution to the problem (NP).

Since $\left(u_1^*, u_2^*, u_3^*, u_4^*, w_1^*, w_2^*\right)$ is a feasible solution to the problem (DP), multiplying (16) by x, (17) by y and then adding them together with (18) we get

$$\left(u_1^* - u_2^* w_1^*\right)^T (Ax + By - t) + \left(w_1^{*T} Ax + w_2^* s^T x - r^T x - \gamma - w_1^{*T} t + w_2^* \delta\right) u_3^*$$
$$+ u_4^* (c^T x + d^T y + \beta) = a^T x + b^T y + \alpha.$$

Bearing in mind that $c^T x + d^T y + \beta \neq 0$ and the fact that $\left(x^*, y^*, w_1^*, w_2^*\right)$ is an optimal solution to the problem (NP), we can write

$$u_4^* + \frac{1}{c^T x + d^T y + \beta} \left\{ \left(u_1^* - u_2^* w_1^*\right)^T (Ax + By - t) \right.$$
$$\left. + \left(w_1^{*T} Ax + w_2^* s^T x - r^T x - \gamma - w_1^{*T} t + w_2^*\right) u_3^* \right\}$$
$$= \frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta} \leq \frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta} = u_4^*.$$

Since $c^T + d^T y + \beta > 0$, we can conclude that

$$\left(u_1^* - u_2^* w_1^*\right)^T (Ax + By - t) + \left(w_1^{*T} Ax + w_2^* s^T x - r^T x - \gamma - w_1^{*T} t + w_2^* \delta\right) u_3^* \leq 0$$

which proves (21).

Conversely, let (x, y, w_1, w_2) be a feasible solution to the problem (NP). Hence $(x, y) \in \overline{S}$. Since $\left(u_1^*, u_2^*, u_3^*, u_4^*, w_1^*, w_2^*\right)$ is a feasible solution to the problem (DP), in the same way as we have done previously, multiplying (16) by x, (17) by y and then adding them together with (18) we get

$$\frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta} = u_4^* + \frac{1}{c^T x + d^T y + \beta} \left\{ \left(u_1^* - u_2^* w_1^*\right)^T (Ax + By - t) \right.$$
$$\left. + \left(w_1^{*T} Ax + w_2^* s^T x - r^T x - \gamma - w_1^{*T} t + w_2^*\right) u_3^* \right\}.$$

Applying conditions (20) and (21) we get

$$\frac{a^T x + b^T y + \alpha}{c^T x + d^T y + \beta} \leq \frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta}$$

and so we conclude that $\left(x^*, y^*, w_1^*, w_2^*\right)$ is an optimal solution to the problem (NP).

Finally, since $\left(x^*, y^*, w_1^*, w_2^*\right)$ solves the problem (NP), from (20) and (22) directly follows that $\left(u_1^*, u_2^*, u_3^*, u_4^*, w_1^*, w_2^*\right)$ is an optimal solution to the problem (DP). □

***Theorem* 3.4** — *$(x^*, y^*) \in S$ is an optimal solution to the LFBP problem if and only if there exist $w_1^*, u_1^* \in R^m$ and $w_2^*, u_2^*, u_3^*, u_4^* \in R$ satisfying $w_1^* \geq 0$, $u_1^* \geq 0$, $u_2^* \geq 0$, such that*

$$B^T w_1^* + q w_2^* = p \qquad \text{... (23)}$$

$$w_1^{*T} (Ax^* + By^* - t) = 0 \qquad \text{... (24)}$$

$$w_1^{*T} (Ax^* - t) + w_2^* (s^T x^* + \delta) - r^T x^* = \gamma \qquad \text{... (25)}$$

$$\left(u_1^* - u_2^* w_1^*\right)^T A + \left(w_1^{*T} A + w_2^* s^T - r^T\right) u_3^* + u_4^* c^T = a^T \qquad \text{... (26)}$$

$$\left(u_1^* - u_2^* w_1^*\right)^T B + u_4^* d^T = b^T \qquad \text{... (27)}$$

$$-\left(u_1^* - u_2^* w_1^*\right)^T t - \left(\gamma + w_1^{*T} t - w_2^{*T} \delta\right) u_3^* + \beta u_4^* = \alpha \qquad \text{... (28)}$$

$$u_1^{*T} (Ax^* + By^* - t) = 0 \qquad \text{... (29)}$$

$$\left(u_1^* - u_2^* w_1^*\right)^T (Ax + By - t) + \left(w_1^{*T} Ax + w_2^* s^T x - r^T x - \gamma - w_1^{*T} t + w_2^* \delta\right) u_3^* \leq 0,$$
$$\forall (x, y) \in \overline{S} \qquad \text{... (30)}$$

PROOF: If (x^*, y^*) is an optimal solution to the LFBP problem, by Theorem 3.1, there exist $w_1^* \in R^m$ and $w_2^* \in R^m$ such that $\left(x^*, y^*, w_1^*, w_2^*\right)$ is an optimal solution to the problem (NP). Hence, it is clear that (23), (24) and (25) are satisfied.

Since (x^*, y^*) is an optimal solution to $\left(P_{w_1^*, w_2^*}\right)$, by applying duality theory in linear fractional programs [2] there exist $\overset{*}{u}_1 \in R^m$ and $\overset{*}{u}_2, \overset{*}{u}_3, \overset{*}{u}_4 \in R$ such that $\overset{*}{u}_1 \geq 0, \overset{*}{u}_2 \geq 0$ and $\left(\overset{*}{u}_1, \overset{*}{u}_2, \overset{*}{u}_3, \overset{*}{u}_4\right)$ is an optimal solution to $\left(D_{\overset{*}{w}_1, \overset{*}{w}_2}\right)$. Hence, it is clear that (26), (27) and (28) are satisfied. In addition, their optimal objective values are equal

$$\overset{*}{u}_4 = \frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta} \qquad \dots (31)$$

Moreover, multiplying (26) by x^*, (27) by y^* and then adding them together with (28), while applying (24) and (25), we get

$$\overset{*T}{u}_1 (Ax^* + By^* - t) + \overset{*}{u}_4 (c^T x^* + d^T y^* + \beta) = a^T x^* + b^T y^* + \alpha$$

Taking into account (31), we conclude that $\overset{*T}{u}_1 (Ax^* + By^* - t) = 0$, which proves (29).

Finally, bearing in mind that $\left(\overset{*}{u}_1, \overset{*}{u}_2, \overset{*}{u}_3, \overset{*}{u}_4, \overset{*}{w}_1, \overset{*}{w}_2\right)$ is a feasible solution to the problem (DP) which satisfies (31) and that $\left(x^*, y^*, \overset{*}{w}_1, \overset{*}{w}_2\right)$ is an optimal solution to the problem (NP), we conclude that $\left(\overset{*}{u}_1, \overset{*}{u}_2, \overset{*}{u}_3, \overset{*}{u}_4, \overset{*}{w}_1, \overset{*}{w}_2\right)$ is an optimal solution to the problem (DP). As a result of Theorem 3.3, (30) is satisfied.

Conversely, let $(x^*, y^*) \in S$. As a consequence of (23), (24) and (25), $\left(x^*, y^*, \overset{*}{w}_1, \overset{*}{w}_2\right)$ is a feasible solution to the problem (NP). Similarly, since (26), (27) and (28) hold $\left(\overset{*}{u}_1, \overset{*}{u}_2, \overset{*}{u}_3, \overset{*}{u}_4, \overset{*}{w}_1, \overset{*}{w}_2\right)$ is a feasible solution to the problem (DP).

Moreover, multiplying (26) by x^*, (27) by y^* and then adding them together with (28), while applying (24), (25) and (29), we get

$$\overset{*}{u}_4 = \frac{a^T x^* + b^T y^* + \alpha}{c^T x^* + d^T y^* + \beta}.$$

From this condition and (30), by applying Theorem 3.3, we conclude that $\left(x^*, y^*, \overset{*}{w}_1, \overset{*}{w}_2\right)$ and $\left(\overset{*}{u}_1, \overset{*}{u}_2, \overset{*}{u}_3, \overset{*}{u}_4, \overset{*}{w}_1, \overset{*}{w}_2\right)$ are respectively optimal. From Theorem 3.1 we get that (x^*, y^*) is an optimal solution to the LFBP problem. □

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