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ON THE ASPHERICITY OF CERTAIN 2-COMPLEXES

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The purpose of this paper is to study Whitehead's asphericity question; whether any subcomplex of an aspherical 2-complex is itself aspherical. We obtain criteria for certain 2-complexes to be aspherical in terms of Euler characteristics, and we extend the results obtained by Varadarajan.

Key Words: 2-Complex; Aspherical; Euler Characteristic; Type FP

One of the important topics of low-dimensional homotopy theory is the study of Whitehead's asphericity question that is posed in 1941[(23)].

"Is any subcomplex of a 2-dimensional aspherical complex itself aspherical?"

Even though many mathematicians have tried hard, Whitehead question remains unanswered in general. There is a wide variety of results on this question[(1,5,6,8,9,12,17)]. An excellent survey of the published results and an overview of the methods that have been used in the study of the Whitehead's question until 1993 can be found in[(4)]. In 2002, using Euler characteristics, Varadarajan[(20)] obtained some criteria for certain types of 2-dimensional complexes (in short, 2-complexes) to be aspherical. More precisely, he showed that if the cellular chain complex $C_*(\tilde{X})$ of universal cover $\tilde{X}$ of 2-complex X is chain homotopy equivalent to a finite type *FL*-complex of length 2 and $G = \pi_1(X)$ is of type *FL*, then the asphericity of X is completely determined by the relation between Euler characteristics $\chi(G)$ and $\chi(X)$. This result generalizes several of those results which are based on the Kaplansky-Montgomery theorem[(16,14)] on inverses of matrices over group rings. It is tempting to generalize this result to the more general finiteness properties of groups, say *FP* or *VFP*. If a 2-complex X is aspherical, then $cd\,\pi_1(X) \le 2$ and by a

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result of Eckmann[10], the Bass conjecture over $\mathbb{C}$ and hence the strong Bass conjecture holds for the group $\pi_1(X)$.

The purpose of this paper is to extend Varadarajan's result to the following general situation:

Theorem 1 — *Let X be a 2-complex with* $G = \pi_1(X)$. *Suppose that the strong Bass conjecture is true for G and that there exists a subgroup H of finite index which is of type FP (so G is of type VFP) and that* $C_*(\tilde{X})$ *is chain homotopy equivalent to the following complex over* $\mathbb{Z}G$*:*

$$0 \to C_2 \to C_1 \to C_0 \to 0,$$

where for each $0 \le i \le 2$, C_i is a finitely generated projective $\mathbb{Z}H$-module. Then X is aspherical if and only if $\chi(G) \ge \chi(X)$. When this is valid, automatically we have $\chi(G) = \chi(X)$.

Our proof of Theorem 1 closely proceeds along the same line as that of Varadarajan[20] and Lemma 1 is the crucial ingredient needed to complete the proof.

By the same reason, the following theorem which is the second main result in this paper makes use of Lemma 1. It generalizes the known result which was proved by several authors[4].

Theorem 2 — *Let X be a finitely dominated Cockroft 2-complex. Suppose that the strong Bass conjecture is true for* $\pi_1(X)$. *Then the following are equivalent:*

(1) X is aspherical.

(2) There exists a finitely dominated aspherical 2-complex Y such that $\pi_1(X) \cong \pi_1(Y)$.

(3) $cd\,\pi_1(X) \le 2$ *and* $\pi_1(X)$ *is of type FP.*

(4) $cd\,\pi_1(X) \le 3$, $H_3(\pi_1(X)) = 0$ *and* $\pi_1(X)$ *is of type FP.*

Before we prove Theorem 1 and Theorem 2, let us first recall some basic definitions and preliminaries that will be used. Throughout this paper, by a 2-complex we mean a connected 2-dimensional *CW*-complex and let *G* be a group and $\mathbb{Z}G$ denote the group ring of *G*. All modules considered are $\mathbb{Z}G$-modules which are also called *G*-modules. The group *G* is of *type FP* if the trivial *G*-module $\mathbb{Z}$ admits a resolution of finite length by finitely generated projective $\mathbb{Z}G$-modules, i.e., for some *n* and finitely generated projective modules $P_0, \ldots, P_n$, there is an exact sequence of $\mathbb{Z}G$-modules:

$$0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to \mathbb{Z} \to 0.$$

If the P_i can be chosen to be finitely generated free $\mathbb{Z}G$-modules, then *G* is of *type FL*. A virtually torsion-free group G is of *type VFP* (resp. *VFL*) if it contains a finite index subgroup of type *FP* (resp. *FL*). Now recall a notion of rank for finitely generated projective modules and the Euler characteristics of groups. For more details, see[7]. For an arbitrary ring R, let $T(R)$ be the quotient $R/[R, R]$, where $[R, R]$ is the additive subgroup of R generated by all commutators $rs - sr$, $(r, s \in R)$. Let G be an arbitrary group. Define a $\mathbb{Z}$-valued rank $\rho_G(P)$ of a finitely generated projective $\mathbb{Z}G$-module P by $\rho_G(P) = R_G(P)(1)$, where $R_G(P) = tr(id_P)$ is the Hattori-Stallings rank of P[13,18,7]. On the other hand, there is another rank $\varepsilon_G(P)$. If M is a G-module, we denote by M_G the abelian group $H_0(G, M) = \mathbb{Z} \otimes_{\mathbb{Z}G} M$. If P is a finitely generated projective $\mathbb{Z}G$-module, then P_G is a finite generated free $\mathbb{Z}$-module. Thus, we can define a rank of P as follows:

$$\varepsilon_G(P) = \mathrm{rank}_{\mathbb{Z}}(P_G).$$

This rank $\varepsilon_G(P)$ is also expressed by $\sum_{g \in C} R_G(P)(g)$, where C is a complete set of representatives of the conjugacy classes of G[7]. By Swan[19], if G is finite, then $\varepsilon_G(P) = \rho_G(P)$. For an infinite group G, it is still open whether $\varepsilon_G(P) = \rho_G(P)$ holds or not. This is what is called the strong Bass conjecture[2]. The Bass conjecture over $\mathbb{C}$ asserts that for a finitely generated projective $\mathbb{C}G$-module and $g \in G$ of infinite order, $R_G(P)(g) = 0$. It follows from Linnell's work[15] that the Bass conjecture over $\mathbb{C}$ is indeed stronger than the strong Bass conjecture. The strong Bass conjecture has been established for various classes of groups[2,3,10,11,15] such that linear groups, groups with $cd_{\mathbb{C}} \leq 2$, subgroups of semihyperbolic groups, and mapping class groups Γ_g of closed surfaces of genus g.

A group G is said to be of *finite homological type* if vcd $(G) < \infty$ and for every G-module M which is finitely generated as an abelian group, $H_i(G, M)$ is finitely generated for all i. The main examples of groups of finite homological type are the groups of type *VFP*. If G is finite homological type and torsion-free, then we define the *Euler characteristic* $\chi(G)$ by

$$\chi(G) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}}(H_i(G)).$$

When G is of type *FP*, $\chi(G) = \sum_i (-1)^i \varepsilon_G(P_i)$, where $P_* \to \mathbb{Z} \to 0$ is a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

By definition, a connected complex X is *aspherical* if its universal cover $\tilde{X}$ is contractible. For a 2-complex X, this is equivalent to $\pi_2(X) = 0$, by Whitehead's theorem. A 2-complex X is said to be *Cockroft* if the Hurewicz homomorphism $\rho : \pi_2(X) \to H_2(X)$ is trivial. The Cockcroft property has been extensively studied in[8,5,6,4].

Now, let us start the proof of main theorems. First, we introduce the following lemma which will play an important role in proving our main theorems.

Lemma 1 — Let G be an arbitrary group and P be a finitely generated projective $\mathbb{Z}G$-module. Suppose that the strong Bass concjecture is true for G. If $\varepsilon_G(P) = 0$, then $P = 0$.

PROOF : By Exercise 8 in Chapter IX, 2,[7], if $\rho_G(P) = 0$, then $P = 0$. This is an immediate consequence of the result of Montogomery and Kaplansky[16,14]. By the assumption, $\varepsilon_G(P) = \rho_G(P)$. This completes the proof. □

Remark 1 : For any free $\mathbb{Z}G$-module F of finite rank, $\varepsilon_G(F)$ and $\rho_G(F)$ are simply equal to the free rank of F. It follows easily that if $\varepsilon_G(F) = 0$, then $F = 0$. Therefore the results of this paper indeed generalize the works in[20] and in[4].

Proof of Theorem 1 — Let X_H be the covering space associated to the subgroup H. By the well known fact on Euler characteristic, we get

$$\chi(X_H) = |G : H|\chi(X) \text{ and } \chi(H) = |G : H|\chi(G).$$

Since $C_*(\tilde{X}) \simeq 0 \to C_2 \to C_1 \to C_0 \to 0$, we get the following exact sequence of $\mathbb{Z}G$-modules:

$$0 \to H_2(\tilde{X}) \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0,$$

because $H_1(\tilde{X}) = 0$ and $H_0(\tilde{X}) = \mathbb{Z}$. Since $\mathbb{Z}G$ is $\mathbb{Z}H$-free, we have the following exact sequence of $\mathbb{Z}G$-modules:

$$0 \to \mathbb{Z}G \otimes_{\mathbb{Z}H} H_2(\tilde{X}) \to D_2 \to D_1 \to D_0 \to \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z} \to 0,$$

where $D_j = \mathbb{Z}G \otimes_{\mathbb{Z}H} C_j$ for $0 \le j \le 2$. Notice that

$$C_*(X_H) = C_*(\tilde{X}/H) \simeq \mathbb{Z} \otimes_{\mathbb{Z}H} C_*(\tilde{X}).$$

Thus, $C_*(X_H)$ is chain homotopy equivalent to the following chain complex:

$$E_* : 0 \to \mathbb{Z} \otimes_{\mathbb{Z}H} C_2 \to \mathbb{Z} \otimes_{\mathbb{Z}H} C_1 \to \mathbb{Z} \otimes_{\mathbb{Z}H} C_0 \to 0$$

and the homology of E_* is the homology of X_H. Since H is of type FP, we get an exact sequence of finitely generated projective $\mathbb{Z}H$-modules:

$$0 \to Q_n \to Q_{n-1} \to \dots \to Q_1 \to Q_0 \to \mathbb{Z} \to 0.$$

Since $\mathbb{Z}G$ is $\mathbb{Z}H$-free, we have the following exact sequence in $\mathbb{Z}G$:

$$0 \to F_n \to F_{n-1} \to \dots \to F_1 \to F_0 \to \mathbb{Z}G \otimes_{\mathbb{Z}H} Z \to 0,$$

where $F_j = \mathbb{Z}G \otimes_{\mathbb{Z}H} Q_j$ for $0 \le j \le n$. Notice that F_i is finitely generated $\mathbb{Z}G$-projective and $\varepsilon_G(F_j) = \varepsilon_H(Q_j)$ for $0 \le j \le n$. By Schanuel's lemma, we get

$$D_0 \oplus F_1 \oplus D_2 \oplus F_3 \oplus F_5 \oplus \dots$$

$$\cong F_0 \oplus D_1 \oplus F_2 \oplus (\mathbb{Z}G \otimes_{\mathbb{Z}H} H_2(\tilde{X})) \oplus F_4 \oplus F_6 \oplus \dots .$$

Notice that $\varepsilon_G(D_j) = \varepsilon_H(C_j)$ for each j. Let $\alpha_j = \varepsilon_G(D_j)$ for $0 \le j \le 2$ and $\beta_j = \varepsilon_G(F_j)$ for $0 \le j \le n$. Notice also that

$$\chi(H) = \sum_{j \ge 0} (-1)^j \varepsilon_H(Q_j) = \sum_{j \ge 0} (-1)^j \varepsilon_G(F_j) = \sum_{j \ge 0} (-1)^j \beta_j.$$

Since the homology of E_* is the homology of X_H, $\chi(X_H) = \alpha_0 - \alpha_1 + \alpha_2$.

Let

$$A = D_0 \oplus F_1 \oplus D_2 \oplus F_3 \oplus F_5 \oplus \dots,$$

$$B = F_0 \oplus D_1 \oplus F_2 \oplus F_4 \oplus F_6 \oplus \dots .$$

Then A and B are finitely generated projective G-modules and we have

$$\varepsilon_G(A) = \alpha_0 + \alpha_2 + \sum_{j \ge 0} \beta_{2j+1}$$

and

$$\varepsilon_G(B) = \alpha_1 + \sum_{j \ge 0} \beta_{2j}.$$

Since $\chi(G) \ge \chi(X)$, we have $\chi(H) \ge \chi(X_H)$. Thus,

$$\sum_{j\geq 0} (-1)^j \beta_j \geq \alpha_0 - \alpha_1 + \alpha_2.$$

Hence $\varepsilon_G(A) \leq \varepsilon_G(B)$. Since $B \oplus (\mathbb{Z}G \otimes_{\mathbf{Z}H} H_2(\tilde{X})) \cong A$, we have $\varepsilon_G(A) = \varepsilon_G(B)$. Notice that $\mathbb{Z}G \otimes_{\mathbf{Z}H} H_2(\tilde{X})$ is finitely generated projective and $\varepsilon_G(A) = \varepsilon_G(B) + \varepsilon_G(\mathbb{Z}G \otimes_{\mathbf{Z}H} H_2(\tilde{X}))$. Thus, $\varepsilon_G(\mathbb{Z}G \otimes_{\mathbf{Z}H} H_2(\tilde{X})) = 0$. By Lemma 1, we have $\mathbb{Z}G \otimes_{\mathbf{Z}H} H_2(\tilde{X}) = 0$ and then $H_2(\tilde{X}) = 0$. Since $\tilde{X}$ is simply connected and 2-dimensional, it follows that $\tilde{X}$ is contractible. Hence X is aspherical. □

PROOF OF THEOREM 2 — The implication (2) ⇒ (3) follows from the fact that if there exists a finitely dominated $K(G, 1)$-complex, then G is of type FP. Since the implications (1) ⇒ (2) and (3) ⇒ (4) are trivial, it remains to show the implication (4) ⇒ (1). For this implication, we need to use the assumption that the strong Bass conjecture is true for $\pi_1(X)$. Let $G = \pi_1(X)$. Since cd $G \leq 3$ and G is of type FP, there exists a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$0 \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$

where each P_i is a finitely generated projective G-module. The augmented cellular chain complex of the universal cover $\tilde{X}$ of X gives the following exact sequence of $\mathbb{Z}G$-modules

$$0 \to \pi_2(X) \to \tilde{C}_2 \to \tilde{C}_1 \to \tilde{C}_0 \to \mathbb{Z} \to 0,$$

where $\tilde{C}_i = C_i(\tilde{X})$ for each i. By Schanuel's lemma, there is an isomorphism of $\mathbb{Z}G$-modules

$$\pi_2(X) \oplus P_2 \oplus \tilde{C}_1 \oplus P_0 \cong P_3 \oplus \tilde{C}_2 \oplus P_1 \oplus \tilde{C}_0.$$

Notice that

$$\chi(G) = \sum_{i=0}^{3} (-1)^i \varepsilon_G(P_i)$$

$$= \sum_{i=0}^{3} (-1)^i \operatorname{rank}_{\mathbf{Z}}(\mathbb{Z} \otimes_{\mathbf{Z}G} P_i)$$

$$= \sum_{i=0}^{3} (-1)^i \operatorname{rank}_{\mathbf{Z}}(H_i(G)).$$

Since X is Cockroft, we get $H_2(G) \cong H_2(X)$ by Hopf's theorem (cf. 7). Notice also that

$H_1(G) \cong G/[G, G] \cong H_1(X)$. Thus,

$$\sum_{i=0}^{3} (-1)^i \operatorname{rank}_{\mathbf{Z}}(H_i(G)) = \sum_{i=0}^{2} (-1)^i \operatorname{rank}_{\mathbf{Z}}(H_i(X))$$
$$= \sum_{i=0}^{2} (-1)^i \operatorname{rank}_{\mathbf{Z}}(C_i)$$
$$= \sum_{i=0}^{2} (-1)^i \operatorname{rank}_{\mathbf{Z}G}(\tilde{C}_i),$$

where $C_i = C_i(X)$. Now the fact $\sum_{i=0}^{3} (-1)^i \varepsilon_G(P_i) = \sum_{i=0}^{2} (-1)^i \varepsilon_G(\tilde{C}_i)$ and

$$\pi_2(X) \oplus P_2 \oplus \tilde{C}_1 \oplus P_0 \cong P_3 \oplus \tilde{C}_2 \oplus P_1 \oplus \tilde{C}_0$$

imply that $\varepsilon_G(\pi_2(X)) = 0$. Therefore, $\pi_2(X) = 0$ by Lemma 1 and hence X is aspherical. □

We end this paper with a question. Let X be a connected CW-complex whose fundamental group $G = \pi_1(X)$ is finitely presented, and let C be the chain complex of the universal cover of X, regarded as a complex of $\mathbb{Z}G$-modules. It is well known that (a) X is homotopy equivalent to a finite CW-complex if and only if C is chain homotopy equivalent to a finite free chain complex; and (b) X is finitely dominated if and only if C is chain homotopy equivalent to a finite projective chain complex[21,22]. In this view point, we can naturally ask the following question which is an analogue of [20, Theorem 3.2].

Question — Let X be a finitely dominated Cockroft 2-complex satisfying the condition that $G - \pi_1(X)$ is of type FP. Suppose that the strong Bass conjecture is true for G. Does the condition

$$\sum_{i \geq 3} (-1)^i \operatorname{rank}_{\mathbf{Z}} H_i(G) \geq 0$$

implies that X is aspherical?

Notice that the positive answer to this qeustion implies Theorem 2.

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