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BI-ADDITIVE MAPS OF RINGS WITH LEFT IDENTITY

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Let R be a ring with left identity e and suitably-restricted additive torsion, Z its center, and H an additive subgroup of R containing e. Let $G : R \times R \to R$ be a symmetric bi-additive mapping, and let g be the trace of G. The main purpose in this paper is to show that (*i*) if for each $x \in R$, $\langle g(x), x \rangle_n = \langle\langle \cdots \langle g(x), x \rangle, x \rangle, \cdots, x \rangle \in Z$ with $n \geq 1$ fixed, then g is commuting on R; (*ii*) for each $x \in H$, either $\langle g(x), x \rangle_n = 0$ or $\langle\langle g(x), x \rangle_n, x^m \rangle = 0$ with $n \geq 0, m \geq 1$ fixed, then $g = 0$ on H, where $\langle y, x \rangle$ denotes the product $yx + xy$.

Key Words: Rings with Left Identity; Skew-Commuting Mappings' Skew-Centralizing Mappings; Commuting Mappings

1. PRELIMINARIES

Throughout, R will represent an associative ring, and Z will be its center. Let $x, y \in R$. The commutator $yx - xy$ will be denoted by $[y, x]$. We define the $(n + 1)$-tuple $\langle y, x_1, ..., x_n \rangle$ as follows: $\langle y, x_1 \rangle := yx_1 + x_1 y$ and $\langle\langle y, x_1, ..., x_{n-1} x_n \rangle, x_n \rangle$. In particular, in the case $x_1 = x_2 = \ldots = x_n = x$, $\langle y, x \rangle_n$ will stand for the $(n + 1)$-tuple $\langle y, x, ..., x \rangle$ abd ket $\langle y, x \rangle_0 = y$. We will also make extensive use of the following basic properties: for any $x, y, z \in R$, $[xy, z] = x[y, z] + [x, z]y$, $[\langle y, x \rangle, x] = \langle [y, x], x \rangle$.

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Let S be a nonempty subset of R. Then a mapping $f: R \rightarrow R$ is said to be commuting on S if $[f(x), x] = 0$ for all $x \in S$. Similarly f is called skew-commuting (resp. skew-centralizing on S if $\langle f(x), x \rangle = 0$ (resp. $\langle f(x), x \rangle \in Z$) for all $x \in S$. By analogy with the defintion of n-commutativity introduced in[3&5], for $n \geq 2$ we define a mapping $f: R \rightarrow R$ to be n-skew-commuting (resp. n-skew-centralizing) on the subset S if $\langle f(x), x^n \rangle = 0$ (resp. $\langle f(x), x^n \rangle \in Z$) for all $x \in S$. Of course, an 1-skew-commuting mapping (resp. 1-skew-centralizing) is called simply a skew-commuting mapping (resp. skew-centralizing).

A mapping $G: R \times R \rightarrow R$ is said to be symmetric if $G(x, y) = G(y, x)$ for all $x, y \in R$. A mapping $g: R \rightarrow R$ defined by $g(x) = G(x, x)$ for all $x, y \in R$, where $G: R \times R \rightarrow R$ is a symmetric mapping, is called the trace of G. It is obvious that, in case when $G: R \times R \rightarrow R$ is a symmetric mapping which is also bi-additive (i.e., additive in both arguments), the trace g of G satisfies the relation $g(x+y) = g(x) + g(y) + 2G(x, y)$ for all $x, y \in R$.

Recently Bell and Lucier[2] obtained some results for skew-commuting and skew-centralizing additive maps in rings with left identity.

The purpose of this paper is to investigate symmetiric bi-additive maps in rings with left identity, and to improve Bell and Lucier's results by using them.

2. Symmetric Bi-Additive Mappings in Rings with Left Identity

Let R be a ring with left identity e and n be any positive integer. The resulting tuple after the substitutions $x_1 = \ldots = x_{i-1} = x_{i+1} = \ldots = x_{j-1} = x_{j+1} = \ldots = x_n = e$ and $x_i = x_j = x$ in the $(n+1)$-tuple $\langle y, x_1, \ldots, x_n \rangle$ will be denoted by $T_{i,j}(y, x, e)$ for all $x_i, x_j, y \in R$, where $i, j = 1, 2, \ldots, n$ with $i \neq j$. If $i = j$, then $T_{i,j}(y, x, e)$ stands for the tuple $\langle y, x_1, \ldots, x_n \rangle$ such that $x_i = x$ and $x_l = e$ for all $l \neq i$ and all $x_i y \in R$, where $i, l = 1, 2, \ldots, n$.

We begin our investigation with the following result inspired by Bell and Lucier's theorem [2, Theorem 3] and Brešar's theorem[4].

Theorem 2.1 — *Let $n > 1$. Let R be a $(n+1)!$-tersion-free ring with left identity e. Let $G: R \times R \rightarrow R$ be a symmetric bi-additive mapping, and let g be the trace of G. If $\langle g(x), x \rangle_n \in Z$ for all $x \in R$, then g is commuting on R. In particular, if R is uniquely 2-divisible as it is an additive group, then*

$$g(x) = \lambda(x)\, x + \xi(X) \text{ for all } x \in R,$$

where $\lambda : R \to R$ is an additive commuting mapping and the mapping $\xi : R \to Z$ is the trace of a symmetric bi-additive mapping.

PROOF : In [6, Theorem 6], we showed that the theorem is valid for $n = 1$. We first remark that the relation $[x, e]\, y = 0$ holds for all $x, y \in R$ since e is a left identity. Let $n > 1$. Since our assumption is

$$\langle g(x), x \rangle_n = \langle\langle g(x), x\rangle_{n-1}, x\rangle \in Z \text{ for all } x \in R, \qquad \dots (1)$$

we have

$$\langle g(e), e \rangle_n = \langle g(e), e\rangle_{n-1}, e + g(e), e\rangle_{n-1}\rangle \in Z. \qquad \dots (2)$$

Commuting with e gives $\langle g(e), e \rangle_{n-1} = \langle g(e), e\rangle_{n-1}\, e$. It follows from (2) that 2 $\langle g(e), e\rangle_{n-1} \in Z$, hence $\langle g(e), e\rangle_{n-1} \in Z$. Continuing in the same manner with this expression, we arrive at $\langle g(e), e \rangle = \langle g(e), e\rangle_1 \in Z$, that is,

$$g(e)e + g(e) \in Z. \qquad \dots (3)$$

Commuting with e yields $g(e) = g(e)\, e$; and by (3), $2g(e) \in Z$, and so $g(e) \in Z$.

Let t be any positive integer. Replacing x by $x + te$ in (1) and noting that $g(x + te) = g(x) + t^2\, g(e) + 2tG\,(x, e)$ for all $x \in R$, we obtain

$$tP_1\,(x, e) + t^2\, P_2(x, e) + \dots + t^{\,n+1}\, P_{n+1}\,(x, e) \in Z \text{ for all } x \in R,$$

where $P_k\,(x, e)$ is the sum of terms involving x and e such that $P_k\,(x, te) = t^{\,k}\, P_k\,(x, e)$, $k = 1, 2, \dots, n + 1$.

Replacing t by $1, 2, \dots, n + 1$ in turn, and expressing the resulting system of $n + 1$ non-homogeneous equations with the variables $P_1\,(x, e), P_2\,(x, e), \dots, P_{n+1}\,(x, e)$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n+1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ n+1 & (n+1)^2 & \cdots & (n+1)^{n+1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less or equal to $n+1$, and since R is $(n+1)!$-torsion free, it follows immediately that for each $k = 1, 2, \ldots, n+1$,

$$P_k(x, e) \in Z \text{ for all } x \in R,$$

$$Z \ni P_{n+1}(x, e) = \sum_{i=1}^{n} T_{i,i}(g(x), x, e) + 2\langle G(x, e), \overbrace{e, \ldots, e}^{n \text{ times}} \rangle = 0 \quad . \quad \ldots (4)$$

and

$$Z \ni P_n(x, e) = \langle g(x), \overbrace{e, \ldots, e}^{n \text{ times}} \rangle + \sum_{i=1}^{n-1} T_{1,(i+1)}(g(e), x, e)$$

$$+ \sum_{i=2}^{n-1} T_{i,(i+1)}(g(e), x, e) + \sum_{i=2}^{n-2} T_{i,(i+1)}(g(e), x, e)$$

$$+ \sum_{i=1}^{n} 2T_{i,i}(G(x, e), x, e). \quad \ldots (5)$$

Since $g(e) \in Z$, the first sum in(4) becomes $n2^n xg(e)$. A simple calculation shows that the second term in(4) makes $2\{(2^n - 1)\, G(x, e)\, e + G(x, e)\}$.

Hence we conclude that for all $x \in R$,

$$P_{n+1}(x, e) = n2^n xg(e) + 2\{(2^n - 1)\, G(x, e)\, e + G(x, e)\} \in Z. \quad \ldots (6)$$

By using $[x, e]\, y = 0$ for all $x\, y \in R$, commuting with e gives

$$[G(x, e), e] = 0 \text{ for all } x \in R; \quad \ldots (7)$$

that is,

$$G(x, e) = G(x, e)\, e \text{ for all } x \in R.$$

Now it follows from (6) that

$$n2^n xg(e) + 2^{n+1} G(x, e) \in Z \text{ for all } x \in R; \quad \ldots (8)$$

and commuting with x in (8) yields

$$2^{n+1} [G(x, e), x] = 0 = [G(x, e), x] \text{ for all } x \in R. \quad \ldots (9)$$

On the other hand, it follows from an easy calculation that the first term in (5) becomes $(2^n - 1)\, g(x)e + g(x)$. Note that the total number of all the terms in $P_n(x, e)$ is $\frac{n^2+n+2}{2}$, where $n > 1$. Since the number of terms of the second sum in (5) in $n - 1$, and the total number of terms of the third sum and the fourth sum in (5) is $\frac{(n-2)(n-1)}{2}$, we see that the total sum of terms of the second sum, the third sum and the fourth sum in (5), by considering $g(e) \in Z$, amounts to $\frac{(n-1)\,n}{2} 2^n x^2 g(e)$.

Finally, the number of terms of the fifth sum in (5) is n, and hence it follows from (7) and (9) that the sum of the terms comes to $n2^{n+1} xG(x, e)$.

Therefore we conclude that for all $x \in R$,

$$P_n(x, e) = (2^n - 1)\, g(x)\, e + g(x) + \frac{(n-1)n}{2} 2^n x^2 g(x) + n2^{n+1} xG(x, e) \in Z \qquad \text{... (10)}$$

By again using $[x, e]\, y = 0$ for all $x\, y, \in R$, commuting with e gives $g(x) = g(x)e$ for all $x \in R$.

We can now rewrite (10) in the form

$$P_n(x, e) = 2^n g(x) + \frac{(n-1)n}{2} 2^n x^2 g(e) + n2^{n+1} xG(x, e) \in Z$$

for all $x \in R$, that is,

$$g(x) + \frac{(n-1)n}{2} x^2 g(x) + 2nxG(x, e) \in Z \text{ for all } x \in R; \qquad \text{... (11)}$$

thus commuting with x and using (9) gives $[g(x), x] = 0$ for all $x \in R$ which means that g is commuting on R.

In particular, let R be uniquely 2-divisible. In (11), setting

$$\mu(x) = \frac{(n-1)n}{2} g(e)x + 2nG(x, e) \text{ for all } x \in R,$$

we define a mapping $\mu : R \to R$ which is clearly additive and commuting.

Hence, by rewriting (11) in the form $g(x) = -\mu(x)\, x + \xi(x)$, where $\xi(x) \in Z$, we define a mapping $\xi : R \to Z$ which is the trace of a symmetric bi-additive mapping $H : R \times R \to R$ defined by

$$H(x, y) = \frac{1}{2} \{2G(x, y) + \mu(x)y + \mu(y)x\}$$

for all $x, y \in R$. By putting $\lambda(x) = -\mu(x)$ for all $x \in R$, we have the desired structure $g(x) = \lambda(x)x + \xi(x)$ for all $x \in R$ and the proof is complete. □

***Theorem* 2.2** — *Let $n > 1$. Let R be a $(n+1)!$-torsion-free ring with left identity e, and let H be an additive subgroup of R containing e. Let $G : R \times R \to R$ be a symmetric bi-additive mapping, and let g be the trace of G. If $\langle g(x), x \rangle_n = 0$ for all $x \in H$, then $g = 0$ on H. In particular, if $\langle g(x), x \rangle_n = 0$ for all $x \in R$, then $G = 0$.*

PROOF : In [6, Theorem 3], we proved that the theorem is true for $n = 1$. Let $n > 1$. We follow the same argument as in the proof of Theorem 2.1.

In the proof of Theorem 2.1, letting $Z = \{0\}$ and then using (1), (2) and (3), we obtain $g(e) = 0$. Then (6), in conjunction with (4), shows that

$$2\langle G(x,e), \overbrace{e, \ldots, e}^{n \text{ times}} \rangle = 2\{(2^n - 1)\, G(x,e)\, e + G(x,e)\} = 0 \qquad \ldots (12)$$

for all $x \in H$. Right-multiplying by e, obtaining $2^n G(x,e)\, e = 0 = G(x,e)e$ for all $x \in H$, and so, by (12), $G(x,e) = 0$ for all $x \in H$. Consequently, (5) becomes

$$\langle g(x), \overbrace{e, \ldots, e}^{n \text{ times}} \rangle = (2^n - 1)\, g(x)\, e + g(x)\} = 0 \qquad \ldots (13)$$

for all $x \in H$. Right-multiplying by e gives $2^n g(x)e = 0 = g(x)e = 0$ which means that, in view of (13), $g(x) = 0$ for all $x \in H$. □

We here present a result for m-skew-commuting mappings as in [2].

***Theorem* 2.3** — *Let $n \geq 0$ and $m \geq 1$. Let R be a $(n+m+1)!$-torsion-free ring with left identity e, and let H be an additive subgroup of R containing e. Let $G : R \times R \to R$ be a symmetric bi-additive mapping, and let g be the trace of G. If the mapping $x \mapsto \langle g(x), x \rangle_n$ is m-skew-commuting on H, then $g = 0$ on H. In particular, if the mapping $x \mapsto \langle g(x), x \rangle_n$ is m-skew-commuting on R, then $G = 0$.*

PROOF : Suppose that

$$\langle\langle g(x), x \rangle_n, x^m \rangle = 0 \text{ for all } x \in H. \qquad \ldots (14)$$

Then we get

$$\langle\langle g(e), e \rangle_n, e^m \rangle = \langle g(e), e \rangle_n\, e + \langle g(e), e \rangle_n = 0; \qquad \ldots (15)$$

and right-multiplying by e gives $2\langle g(e), e \rangle_n e = 0 = \langle g(e), e \rangle_n e$. Hence, (15) yields $\langle g(e), e \rangle_n = 0$. Using similar approach as in the proof of Theorem 2.1, we obtain $g(e) = 0$.

Let t be any positive integer. Replacing x by $x + te$ in (14) and considering $g(x + te) = g(x) + t^2 g(e) + 2tG(x, e) = g(x) + 2tG(x, e)$ for all $x \in R$, we obtain

$$tP_1(x, e) + t^2 P_2(x, e) + \ldots + t^{n+m+1} P_{n+m+1}(x, e) = 0 \text{ for all } x \in H,$$

where $P_k(x, e)$ is the sum of terms involving x and e such that $P_k(x, te) = t^k P_k(x, e), k = 1, 2, \ldots, n + m + 1$.

Replacing t by $1, 2, \ldots, n + m + 1$ in turn, and expressing the resulting system of $n + m + 1$ homogeneous equations with the variables $P_1(x, e), P_2(x, e), \ldots, P_{n+m+1}(x, e)$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n+m+1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ n+m+1 & (n+m+1)^2 & \cdots & (n+m+1)^{n+m+1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less or equal to $n + m + 1$, and since R is $(n + m + 1)!$-torsion free, it follows immediately that for each $k = 1, 2, \ldots, n + m + 1$,

$$P_k(x, e) = 0 \text{ for all } x \in H.$$

In particular, we have, by utilizing $e^m = e$, for all $x \in H$,

$$0 = P_{n+m+1}(x, e) = 2T_{(n+1),(n+1)}(G(x, e), e, e) \qquad \ldots (16)$$

and

$$0 = P_{n+m}(x, e) \qquad \ldots (17)$$

$$= T_{(n+1),(n+1)}(g(x), e, e) + \sum_{i=1}^{n} 2T_{i,i}(G(x, e), x, e)$$

$$+ 2(m-1)\, T_{(n+1),(n+1)}(G(x, e), xe, e) + 2T_{(n+1),(n+1)}(G(x, e), x, e).$$

By inspecting (16), we now obtain that

$$2\{(2^{n+1}-1)\,G(x, e)e + G(x, e)\} = 0 \text{ for all } x \in H; \qquad \dots (18)$$

and right-multiplying by e gives $2^{n+2}\,G(x, e)\,e = 0 = G(x, e)\,e$ for all $x \in H$, therefore, by (18), $G(x, e) = 0$ for all $x \in H$. This forces (17) to

$$T_{(n+1),(n+1)}(g(x), e, e) = \langle g(x), \overbrace{e, \dots, e}^{n \text{ times}} \rangle = 0 \text{ for all } x \in H. \qquad \dots (19)$$

Calculating (19), we get

$$(2^{n+1}-1)\,g(x)\,e + g(x) = 0 \text{ for all } x \in H; \qquad \dots (20)$$

and right-multiplying by e gives $2^{n+1}\,g(x)\,e = 0 = g(x)e$ for all $x \in H$, hence, by (2), $g(x) = 0$ for all $x \in H$ which completes the proof of the theorem. □

3. Some Results for Additive Mappings in Rings with Left Identity

In this section, we use the results in Section 2 to obtain some results concerning additive mappings in rings with left identity which are to improve Bell and Lucier's ones in[2]. We have need of the following remark as a basic tool.

Remark : Let R be a ring and let $f: R \to R$ be an additive mapping. We define a mapping $G: R \times R \to R$ by

$$G(x, y) = \langle f(x), y \rangle + \langle f(y), x \rangle \text{ for all } x, y \in R.$$

or

$$G(x, y) = [f(x), y] + [f(y), x] \text{ for all } x, y \in R.$$

Then it is clear that G is symmetric and bi-additive, and that the mapping $g: R \to R$ defined by $g(x) = G(x, x)$ for all $x \in R$ is the trace of G.

***Theorem* 3.1** — *Let $n \geq 0$ and $m \geq 1$. Let R be a $(n + m + 1)!$-torsion-free ring with left identity e, and let H be an additive subgroup of R containing e. If f is an additive map on R such that mapping $x \mapsto \langle g(x), x \rangle_n$ is m-skew-centralizing on H, then f is commuting H.*

Proof : Since $\langle \langle f(x), x \rangle_n, x^m \rangle \in Z$ for all $x \in H$, we have, by recalling $[\langle y, x \rangle, x] = \langle [y, x], x \rangle$,

$$[\langle f(x), x \rangle_n\, x^m + x^m \langle f(x), x \rangle_n, x] = 0 \text{ for all } x \in H$$

which implies that $[\langle f(x), x\rangle_n, x]\, x^m + x^m\, [\langle f(x), x\rangle_n, x] = 0$ for all $x \in H$. This reduces to $[\langle f(x), x\rangle_n, x_n\, x^m + x^m \langle [f(x), x]\rangle_n = 0$ for all $x \in H$, and so it follows from Remark and Theorem 2.3 that f is commuting on H. □

Theorem 3.2 — *Let $n \geq 1$. Let R be a $(n + 1)!$-torsion-free ring with left identity e, and let H be an additive subgroup of R containing e. If f is an additive map on R such that the mapping $x \mapsto \langle g(x), x\rangle_n$ is commuting H, then f is commuting on H*

PROOF : By hypothesis, we have $[\langle f(x), x\rangle_n, x] = 0$ for all $x \in H$, and so we get $\langle [f(x), x]\, x\rangle_n = [\langle f(x), x\rangle_n, x] = 0$ for all $x \in H$. Hence Theorem 2.2 and Remark guarantee the conclusion. □

Theorem 3.3 — *Let $n \geq 1$. Let R be a $(n + 1)!$-torsion-free ring with left identity e, and let H be an additive subgroup of R containing e. If f is an additive map on R such that the mapping $x \mapsto \langle f(x), x\rangle_n$ is skew-commuting on H, then $f = 0$ on H*

PROOF : By hypothesis, we have $\langle\langle f(x), x\rangle, x\rangle_n = \langle\langle f(x), x\rangle_n, x\rangle = 0$ for all $x \in H$; therefore Remark and Theorem 2.2 yield $\langle f(x), x\rangle = 0$ for all $x \in H$, i.e., f is skew-commuting on H. This implies that the mapping $x \mapsto \langle f(x), x\rangle_n$ is commuting on H, and hence f is commuting on H according to Theorem 3.2.

From the fact that f is both skew-commuting and commuting, we conclude that

$$xf(x) = f(x)x = 0 \text{ for all } x \in H.$$

Since $ef(x) = f(e)e = 0$, we have $f(e) = 0$. Substituting $x + e$ for x in the relation $xf(x) = 0$, we obtain

$$xf(x) + xf(e) + ef(x) = ef(e) = 0 \text{ for all } x \in H,$$

whence we see that $f(x) = 0$ for all $x \in H$ which gives the conclusion. □

Let R be a semiprime ring. If $\langle x, y\rangle = 0$ for all $x, y \in R$, then R is commutitative. This result can be obtained as a corollary to the main result of Awtar[1] which is not mentioned specially there.

Finally, as a special case one obtains an condition which implies commutativity in rings with identity as motivated by the above result.

Corollary 3.4 — Let $n \geq 1$. Let R be a $(n+1)!$-torsion free ring with identity such that $\langle x, y \rangle_{n+1} \in Z$ for all $x, y \in R$. Then R is commutative.

PROOF : For any fixed $y \in R$, putting $f(x) = [y, x]$ for all $x \in R$, we define a mapping $f : R \to R$ which is additive. Since our hypothesis implies that $[\langle y, x \rangle_{n+1}, x] = 0$ for all $x \in R$, the relation $\langle [y, x], x \rangle_{n+1} = [\langle y, x, \rangle_{n+1}, x]$ yields $\langle f(x), x \rangle_{n+1} = 0$ for all $x \in R$. According to Theorem 3.3, we obtain $f(x) = 0$ for all $x \in R$, that is, $[y, x] = 0$ for all $x \in R$. Since y is arbitrary, therefore R is commutative. This completes the proof. □

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