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GENERALIZATIONS OF ACZÉL'S INEQUALITY AND POPOVICIU'S INEQUALITY

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In this paper, we give new generalizations of Aczél's inequality and Popoviciu's inequality, which has broad applications to mathematical analysis. We improve a number of algebraic inequalities and obtain some new ones. Furthermore, we also point out that a new generalization of Aczél's inequality by Mitrinović (2), Wang (7) and Kuang (8) is not true in general. In the end, we give an analogue of Popoviciu's inequality.

Key Words : Aczél's Inequality; Popoviciu's Inequality; Power Mean Inequality; Generalization; Concave Function

1. INTRODUCTION AND MAIN RESULTS

Aczél (1) proved the following inequality

$$\left(a_1^2 - \sum_{i=2}^{n} a_i^2\right)\left(b_1^2 - \sum_{i=2}^{n} b_i^2\right) \le \left(a_1 b_1 - \sum_{i=2}^{n} a_i b_i\right)^2 \qquad \dots (1)$$

where $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ are two sequences of positive numbers such that $a_1^2 - \sum_{i=2}^{n} a_i^2 > 0$ and $b_1^2 - \sum_{i=2}^{n} b_i^2 > 0$. This inequality is called Aczél's inequality.

Recently, attention has been given to Aczél's inequality and Popoviciu's inequality due to their applications to mathematical analysis. Many authors including Aczél (1), Mitrinović (2), Bellman (3), Popoviciu (4), Dragomir and Mond (5), Vasić and Pečarić (6) considered several inequalities which play an useful role in the theory and applications to matehmatical analysis.

Popoviciu (4) established the following inequality as an extension of Aczél's inequality.

Let $p>0, q>0, \frac{1}{p}+\frac{1}{q} = 1$, $a = (a_1, a_2, ..., a_n)$ and $b=(b_1, b_2, ..., b_n)$ be two sequences of positive numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. Then

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^{n} b_i^q\right)^{\frac{1}{q}} \le a_1 b_1 - \sum_{i=2}^{n} a_i b_i. \qquad ... (2)$$

A further extension was given by Vasić and Pečarić (6).

Let $p_j > 0, a_{ij} > 0, a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j} > 0, i = 1, 2, ..., n, j = 1, 2, ..., m$, and let

$$\frac{1}{p_1}+\frac{1}{p_2} + ... + \frac{1}{p_m} \ge 1.$$

Then,

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}} \le \prod_{j=1}^{m} a_{1j} - \sum_{i=2}^{n}\prod_{j=1}^{m} a_{ij}. \qquad ... (3)$$

Another generalization of Aczel's inequality were given in Mitrinović (2), Wang *et al.* (7) and Kuang (8) as follows:

Let $p \ge 1, a = (a_1, a_2, ..., a_n)$ and $b=(b_1, b_2, ..., b_n)$ be two sequences of positive numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^p - \sum_{i=2}^{n} b_i^p > 0$. Then

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right) \left(b_1^p - \sum_{i=2}^{n} b_i^p\right) \le \left(a_1 b_1 - \sum_{i=2}^{n} a_i b_i\right)^p. \qquad ... (4)$$

We note that inequality (4) cannot be always true for $p > 2$. In fact, if we consider the special case for $n=2, a_1 = b_1 = 1$ and $a_2 = b_2 = t, t \in (0, 1)$. Then

$$\left(a_1^p - a_2^p\right)\left(b_1^p - b_2^p\right) = (1-t^p)^2, \ (a_1 b_1 - a_2 b_2)^p = (1-t^2)^p.$$

It is obvious that $(1-t^p)^2 > (1-t^p)^p > (1-t^2)^p$ for any $p > 2$, which shows that inequality (4) is not true in general.

In this paper, we give a new generalizations of Aczél's inequality and Popoviciu's inequality.

***Theorem* 1** — *Let* $p_j > 0, a_{ij} > 0,\ a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j} > 0,\ i = 1, 2, ..., n,\ j = 1, 2, \ldots m$

and let $\rho_m = \min\left\{\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m}, 1\right\}$. *Then*

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}} \leq n^{1-\rho_m} \prod_{j=1}^{m} a_{1j} - \sum_{i=2}^{n}\prod_{j=1}^{m} a_{ij}. \qquad \ldots (5)$$

and equality holds if and only if $a_{1j} = n^{\frac{1}{p_j}} a_{2j} = \ldots = n^{\frac{1}{p_j}} a_{nj}, j = 1, 2, ..., m$ *for* $\rho_m < 1$, *or*

$$\frac{a_{11}^{p_1}}{a_{ij}^{pj}} = \frac{a_{21}^{p_1}}{a_{2j}^{pj}} = \ldots = \frac{a_{n1}^{p_1}}{a_{nj}^{pj}},\ j = 2, 3, \ldots m \text{ for } \rho_m = 1.$$

In order to prove this Theorem, we will use following Lemmas.

2. Lemma

Lemma 1 — If $x_i > 0, \lambda_i > 0, i = 1, 2, ..., n$. Then

$$\prod_{i=1}^{n}\left(\frac{x_i}{\lambda_i}\right)^{\lambda_i} \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}\right)^{\lambda_1 + \lambda_2 + \ldots + \lambda_n}, \qquad \ldots (6)$$

and equality holds if and only if

$$\frac{x_1}{\lambda_1} = \frac{x_2}{\lambda_2} = \ldots = \frac{x_n}{\lambda_n}.$$

Lemma 1 is equivalent to theorem of arithmetic and geometric means (see Hardy *et al.*[9], pp. 17).

Lemma 2 — If $0 < p \leq 1, x_i > 0, i = 1, 2, ..., n$. Then

$$\sum_{i=1}^{n} x_i^p \leq n^{1-p}\left(\sum_{i=1}^{n} x_i\right)^p, \qquad \ldots (7)$$

and equality holds if and only if

$x_1 = x_2 = \ldots = x_n$ for $p < 1$.

The inequality (7) is the well-known power mean inequality (see Hardy *et al.* (9), pp. 28).

Lemma 3 — Let $p_j > 0, a_{ij} > 0, i = 1, 2, \ldots n, j = 1, 2, \ldots m$, and let $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} \le 1$. Then

$$\prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{pj} \right)^{\frac{1}{p_j}} \ge n^{\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - 1} \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \qquad \ldots (8)$$

when $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} < 1$, equality of inequality (8) holds if and only if $a_{1j} = a_{2j} = \ldots = a_{nj}, j = 1, 2, \ldots, m$, when $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1$, equality of inequality (8) holds if and only if

$\frac{a_{11}^{p_1}}{a_{1j}^{pj}} = \frac{a_{21}^{p_1}}{a_{2j}^{pj}} = \ldots = \frac{a_{n1}^{p_1}}{a_{nj}^{pj}}$, $j = 2, 3, \ldots, m$, $a_{1j} = a_{2j} = \ldots = a_{nj}$, $j = 1, 2, \ldots, m$.

PROOF : Using inequality (6), we have

$$\prod_{j=1}^{m} a_{ij} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{-\frac{1}{p_j}} = \prod_{j=1}^{m} \left[\frac{\frac{1}{p_j} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{-1} a_{ij}^{p_j}}{\frac{1}{p_j}} \right]^{\frac{1}{p_j}}$$

$$\le \left[\frac{\sum_{j=1}^{m} \frac{1}{p_j} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{-1} a_{ij}^{p_j}}{\sum_{j=1}^{m} \frac{1}{p_j}} \right]^{\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m}}$$

that is

$$\prod_{j=1}^{m} a_{ij} \le \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \right]$$

$$\left[\frac{\sum_{j=1}^{m} \frac{1}{p_j}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{-1} a_{ij}^{p_j}}{\sum_{j=1}^{m} \frac{1}{p_j}}\right]^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}}$$

Therefore,

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \leq \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}\right]$$

$$\sum_{i=1}^{n}\left[\frac{\sum_{j=1}^{m} \frac{1}{p_j}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{-1} a_{ij}^{p_j}}{\sum_{j=1}^{m} \frac{1}{p_j}}\right]^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}} \quad \ldots (9)$$

By $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m} \leq 1$ and inequality (7), we obtain

$$\sum_{i=1}^{n}\left[\frac{\sum_{j=1}^{m} \frac{1}{p_j}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{-1} a_{ij}^{p_j}}{\sum_{j=1}^{m} \frac{1}{p_j}}\right]^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}}$$

$$\leq n^{1-\frac{1}{p_1}-\frac{1}{p_2}-\ldots-\frac{1}{p_m}}\left[\frac{\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{p_j}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{-1} a_{ij}^{p_j}}{\sum_{j=1}^{m} \frac{1}{p_j}}\right]^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}}$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{p_j}\left(\sum_{i=1}^{n}a_{ij}^{p_j}\right)^{-1}a_{ij}^{p_j}=\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{p_j}\left(\sum_{i=1}^{n}a_{ij}^{p_j}\right)^{-1}a_{ij}^{p_j}$$

$$=\sum_{j=1}^{m}\frac{1}{p_j}\left(\sum_{i=1}^{n}a_{ij}^{p_j}\right)^{-1}\left(\sum_{i=1}^{n}a_{ij}^{p_j}\right)=\sum_{j=1}^{m}\frac{1}{p_j}, \qquad \ldots (10)$$

therefore,

$$\sum_{i=1}^{n}\left[\frac{\sum_{j=1}^{m}\frac{1}{p_j}\left(\sum_{i=1}^{n}a_{ij}^{p_j}\right)^{-1}a_{ij}^{p_j}}{\sum_{j=1}^{m}\frac{1}{p_j}}\right]^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}}$$

$$\leq n^{1-\frac{1}{p_1}-\frac{1}{p_2}-\ldots-\frac{1}{p_m}}. \qquad \ldots (11)$$

Combining inequalities (9) and (11), we get

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n}a_{ij}^{p_j}\right)^{\frac{1}{p_j}}\geq n^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1}\sum_{i=1}^{n}\prod_{j=1}^{m}a_{ij}.$$

Lemma 1 and Lemma 2 shows that inequality (8) becomes an equality if and only if $\frac{a_{11}^{p1}}{a_{1j}^{p_j}}=\frac{a_{21}^{p1}}{a_{2j}^{p_j}}=\ldots=\frac{a_{n1}^{p1}}{a_{nj}^{p_j}}, j = 2, 3, \ldots, m$, for the case of $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}=1$, from which is easy to see equality of inequality (8) holds if and only if $a_{1j}=a_{2j}=\ldots=a_{nj}, j = 1, 2, \ldots, m$,. for the case of $\left(\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}<1\right)$.

The proof of Lemma 3 is complete.

An analogue of inequality (8) was given by Vasić and Pečarić (10), as follows:

Lemma 4 — Let $p_j>0, a_{ij}>0, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$ and let $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}>1$. Then

$$\prod_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}} > \sum_{i=1}^{n}\prod_{j=1}^{m} a_{ij}. \qquad \text{... (12)}$$

3. Proof of Theorem 1

When $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}>1$, then $\rho_m = 1$. Clearly, inequality (5) of Theorem 1 is equivalent to inequality (3). Let us now prove that inequality (5) for the case of $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}\le 1$, i.e.

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}$$

$$\le n^{1-\frac{1}{p_1}-\frac{1}{p_2}-\ldots-\frac{1}{p_m}} \prod_{j=1}^{m} a_{1j} - \sum_{i=2}^{n'}\prod_{j=1}^{m} a_{ij}. \qquad \text{... (13)}$$

Since $p_j>0$, $a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}>0$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, we can assume that

$$a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j} = x_j^{p_j} \qquad (x_j > 0,\ j = 1, 2, ..., m - 1),$$

and

$$\prod_{j=1}^{m} a_{1j} - n^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1} \sum_{i=2}^{n}\prod_{j=1}^{m} a_{ij}$$

$$= n^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1} \prod_{j=1}^{m} x_j .$$

From

$$p_j>0,\ a_{ij}>0,\ a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}>0,\ i = 1, 2, ..., n,\ j = 1, 2, ..., m, \text{ we get}$$

$$\prod_{j=1}^{m} a_{1j} > \prod_{j=1}^{m} \left(x_j^{p_j} + \sum_{i=2}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}.$$

Further, by Lemma 3 with $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} \le 1$, we obtain

$$\prod_{j=1}^{m} a_{1j} > (n-1)^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1} \sum_{i=2}^{n} \prod_{j=1}^{m} a_{ij}$$

$$> n^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1} \sum_{i=2}^{n} \prod_{j=1}^{m} a_{ij}.$$

It follows from the above inequality that $\prod_{j=1}^{m} x_j > 0$, it also follows that $x_j > 0$ for $j = 1, 2, \ldots, m$.

Using inequality (8), we obtain

$$\prod_{j=1}^{m} \left(x_j^{p_j} + \sum_{i=2}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}} \ge n^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1} \left(\prod_{j=1}^{m} x_j + \sum_{i=2}^{n} \prod_{j=1}^{m} a_{ij} \right),$$

that is,

$$\left(x_m^{p_m} + \sum_{i=2}^{n} a_{im}^{p_m} \right)^{\frac{1}{p_m}} \prod_{j=1}^{m-1} \left(x_j^{p_j} + \sum_{i=2}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}$$

$$\ge n^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1} \left(\prod_{j=1}^{m} x_j + \sum_{i=2}^{n} \prod_{j=1}^{m} a_{ij} \right). \quad \ldots (14)$$

We note that

$$x_j^{p_j} + \sum_{i=2}^{n} a_{ij}^{p_j} = a_{1j}^{p_j},\ j = 1, 2, \ldots, m-1,$$

and

$$\prod_{j=1}^{m} a_{1j} = n^{\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m}-1} \left(\prod_{j=1}^{m} x_j + \sum_{i=2}^{n} \prod_{j=1}^{m} a_{ij} \right).$$

Hence, it follows from inequality (14) that

$$\left(x_m^{p_m} + \sum_{i=2}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}} \prod_{j=1}^{m-1} a_{1j} \geq \prod_{j=1}^{m} a_{1j},$$

that is,

$$\left(x_m^{p_m} + \sum_{i=2}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}} \geq a_{1m},$$

hence, we get

$$x_m^{p_m} \geq a_{1m}^{p_m} - \sum_{i=2}^{n} a_{im}^{p_m}.$$

We also deduce that

$$x_m \geq \left(a_{1m}^{p_m} - \sum_{i=2}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}} > 0.$$

Therefore,

$$\begin{aligned}\prod_{j=1}^{m} x_j &\geq \left(a_{1m}^{p_m} - \sum_{i=2}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}} \prod_{j=1}^{m-1} x_j = \left(a_{1m}^{p_m} - \sum_{i=2}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}} \prod_{j=1}^{m-1} \left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}} \\ &= \prod_{j=1}^{m} \left(a_{1j}^{p_j} - \sum_{i=2}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}.\end{aligned}$$

From

$$\prod_{j=1}^{m} x_j = n^{1-\frac{1}{p_1}-\frac{1}{p_2}-\dots-\frac{1}{p_m}} \prod_{j=1}^{m} a_{1j} - \sum_{i=2}^{n} \prod_{j=1}^{m} a_{ij},$$

we immediately obtain

$$\prod_{j=1}^{m}\left(a_{1j}^{p_j}-\sum_{i=2}^{n}a_{ij}^{p_j}\right)^{\frac{1}{p_j}}\le n^{1-\frac{1}{p_1}-\frac{1}{p_2}-\ldots-\frac{1}{p_m}}\prod_{j=1}^{m}a_{1j}-\sum_{i=2}^{n}\prod_{j=1}^{m}a_{ij}.$$

The condition of equality for inequality (5) can easily be obtained by Lemma 3. This completes the proof of Theorem 1.

4. APPLICATIONS OF THEOREM 1

Let $p_1=p_2=\ldots=p_m=p$. Then, from Theorem 1, we obtain the following results:

Corollary 1 — Assume $a_{ij}>0, a_{1j}^{p}-\sum_{i=2}^{n}a_{ij}^{p}>0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $p > 0$, $\rho_m = min\left\{\frac{m}{p}, 1\right\}$. Then the following inequalities hold

$$\prod_{j=1}^{m}\left(a_{1j}^{p}-\sum_{i=2}^{n}a_{ij}^{p}\right)$$

$$\le\left(n^{1-\rho_m}\prod_{j=1}^{m}a_{1j}-\sum_{i=2}^{n}\prod_{j=1}^{m}a_{ij}\right)^{p}. \quad \ldots (15)$$

Let $m=2, a_{i1}=a_i, a_{i2}=b_i, i=1,2,\ldots,n$, then from Corollary 1, we get

Corollary 2 — Assume $a_i>0, b_i>0, i = 1, 2, \ldots, n$ $a_1^p-\sum_{i=2}^{n}a_i^p>0$, $b_1^p-\sum_{i=2}^{n}b_i^p>0$, $p>0$, $\rho = \min\left\{\frac{2}{p}, 1\right\}$. Then the following inequalities hold

$$\left(a_1^p-\sum_{i=2}^{n}a_i^p\right)\left(b_1^p-\sum_{i=2}^{n}b_i^p\right)\le\left(n^{1-\rho}a_1b_1-\sum_{i=2}^{n}a_ib_i\right)^{p}. \quad \ldots (16)$$

Let $m = 2$, $p_1=p, p_2=q, a_{i1}=a_i, a_{i2}=b_i, i = 1, 2, \ldots, n$. Then from Theorem 1, we obtain

Corollary 3 — Assume $a_i>0, b_i>0, i = 1, 2, \ldots, n$, $a_1^p-\sum_{i=2}^{n}a_i^p>0$, $b_1^q-\sum_{i=2}^{n}b_i^q>0$, $p>0, q>0$, $\rho = \min\left\{\frac{1}{p}+\frac{1}{q} 1\right\}$. Then the following inequalities hold

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{\frac{1}{p}} \left(b_{1q}^p - \sum_{i=2}^{n} b_i^q\right)^{\frac{1}{q}}$$

$$\leq n^{1-\rho} a_1 b_1 - \sum_{i=2}^{n} a_i b_i. \qquad \dots (17)$$

Using $\frac{1}{p} + \frac{1}{q} = 1$ and Corollary 3, we obtain Popoviciu's inequality. It is clear that Aczél's inequality as well as Popoviciu's inequality is a simple consequence of the Theorem 1 presented in this article.

5. The Analogue of Popoviciu's Inequality

In this section, we shall give an analogue of Popoviciu's inequality, which is interesting and useful.

***Theorem* 2** — *Let* $p_j > 0, \lambda_j > 0, a_{ij} > 0,$ $\left(\sum_{i=1}^{n} a_{ij}\right)^{p_j} - \lambda_j^{p_j} \sum_{i=1}^{n} a_{ij}^{p_j} > 0,$ $i = 1, 2, \dots, n,$ $j = 1, 2, \dots, m.$ If $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} < 1,$ *then*

$$\prod_{j=1}^{m} \left(\left(\sum_{i=1}^{n} a_{ij}\right)^{p_j} - \lambda_j^{p_j} \sum_{i=1}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}$$

$$\leq (n+1)^{1 - \frac{1}{p_1} - \frac{1}{p_2} - \dots - \frac{1}{p_m}} \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij}\right) - \lambda_1 \lambda_2 \dots \lambda_m \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}, \qquad \dots (18)$$

and equality holds if and only if $a_{1j} = a_{2j} = \dots = a_{nj}$ *and* $\lambda_j = n/(n+1)^{1/p_j}$, $j = 1, 2, \dots, m.$

If $\frac{1}{p_1} + \frac{1}{p_2} + \dots + \frac{1}{p_m} \geq 1$, then

$$\prod_{j=1}^{m} \left(\left(\sum_{i=1}^{n} a_{ij}\right)^{p_j} - \lambda_j^{p_j} \sum_{i=1}^{n} a_{ij}^{p_j}\right)^{\frac{1}{p_j}}$$

$$\leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} \right) - \lambda_1 \lambda_2 \ldots \lambda_m \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}, \qquad \ldots (19)$$

and equality holds if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = 1$$

and

$$\frac{a_{11}^{p_1}}{a_{1j}^{p_j}} = \frac{a_{21}^{p_1}}{a_{2j}^{p_j}} = \ldots = \frac{a_{n1}^{p_1}}{a_{nj}^{p_j}} = \left(\frac{1}{\lambda_1} \sum_{i=1}^{n} a_{i1} \right)^{p_1} \Bigg/ \left(\frac{1}{\lambda_j} \sum_{i=1}^{n} a_{ij} \right)^{p_j}, j = 2, 3, \ldots, m.$$

PROOF : Let $\lambda_j^{p_j} a_{n+1j}^{p_j} = \left(\sum_{i=1}^{n} a_{ij} \right)^{p_j} - \lambda_j^{p_j} \sum_{i=1}^{n} a_{ij}^{p_j}, j = 2, 3, \ldots, m.$

From the above condition with $\left(\sum_{i=1}^{n} a_{ij} \right)^{p_j} - \lambda_j^{p_j} \sum_{i=1}^{n} a_{ij}^{p_j} > 0$, we obtain $a_{n+1j} > 0$, $j = 2, 3, \ldots, m.$

When $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} \leq 1$, by Lemma 3, we have

$$\prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} \right) = \lambda_1 \lambda_2 \ldots \lambda_m \prod_{j=1}^{m} \left(\sum_{i=1}^{n+1} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}$$

$$\geq \lambda_1 \lambda_2 \ldots \lambda_m (n+1)^{\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - 1} \sum_{i=1}^{n+1} \prod_{j=1}^{m} a_{ij}$$

$$= \lambda_1 \lambda_2 \ldots \lambda_m (n+1)^{\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} - 1} \left(\prod_{j=1}^{m} a_{n+1j} + \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \right),$$

therefore,

$$\lambda_1 \lambda_2 \ldots \lambda_m \prod_{j=1}^{m} a_{n+1j}$$

$$\leq (n+1)^{1-\frac{1}{p_1}-\frac{1}{p_2}-\cdots-\frac{1}{p_m}} \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} \right) - \lambda_1 \lambda_2 \cdots \lambda_m \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}.$$

Substituting $\lambda_j a_{n+1j} = \left(\left(\sum_{i=1}^{n} a_{ij} \right)^{p_j} - \lambda_j^{p_j} \sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}$ into above inequality, we get inequality (18).

When $\frac{1}{p_1}+\frac{1}{p_2}+\ldots+\frac{1}{p_m} > 1$, by Lemma 4, we have

$$\prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} \right) = \lambda_1 \lambda_2 \cdots \lambda_m \prod_{j=1}^{m} \left(\sum_{i=1}^{n+1} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}$$

$$\geq \lambda_1 \lambda_2 \cdots \lambda_m \sum_{i=1}^{n+1} \prod_{j=1}^{m} a_{ij}$$

$$= \lambda_1 \lambda_2 \cdots \lambda_m \left(\prod_{j=1}^{m} a_{n+1j} + \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij} \right),$$

therefore,

$$\lambda_1 \lambda_2 \cdots \lambda_m \prod_{j=1}^{m} a_{n+1j} \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} \right) - \lambda_1 \lambda_2 \cdots \lambda_m \sum_{i=1}^{n} \prod_{j=1}^{m} a_{ij}.$$

Substituting $\lambda_j a_{n+1j} = \left(\left(\sum_{i=1}^{n} a_{ij} \right)^{p_j} - \lambda_j^{p_j} \sum_{i=1}^{n} a_{ij}^{p_j} \right)^{\frac{1}{p_j}}$ into the above inequality gives inequality (19).

The proof is complete. Identity condition of (18) and (19) follows immediately from the proof and Lemma 3, Lemma 4.

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