Indian J. pure appl. Math., 36(3) : 145-150, March 2005

ASYMPTOTIC AND WEAKLY ASYMPTOTIC CONTRACTIONS

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(*Received* 13 *July* 2004; *after final revision* 26 *May* 2005; *accepted* 9 *June* 2005)

Kirk (8) recently introduced asymptotic contractions and proved by an ultraproduct technique that an asymptotic contraction on a complete metric space has a unique fixed point provided the mapping has a bounded orbit. A simple and elementary proof is given in this note. Moreover, weakly asymptotic contractions are introduced and it is proved that a weakly asymptotic contraction on a complete metric space has a unique fixed point should the mapping has a bounded orbit.

Key Words: Asymptotic Contraction, Weakly Asymptotic Contraction, Fixed Point

1. ASYMPTOTIC CONTRACTIONS

Let (M, d) be a complete metric space. Recall a map $T: M \to M$ is called a contraction if

$$d(Tx, Ty) \leq \alpha\, d(x, y), \qquad x, y \in M$$

where $\alpha \in [0, 1)$ is a constant. It is well-known that the Banach contraction principle ensures that every contraction T on a complete metric space M has a unique fixed point u and for each $x \in M, T^n x \to u$.

Due to its wide applications, the Banach contraction principle has been extended in various ways; see, for example, [2], [3] and [4]. For recent applications in inverse problems for differential and integral equations see (9,10 and 11).

A recent extension appeared in (8) in which Kirk introduced asymptotic contractions and proved that an asymptotic contraction T on a complete metric space M has a unique fixed point u and for each $x \in M, T^n x \to u$ provided T has a bounded orbit.

Let Φ denote the collection of all functions φ from $R^+ := [0, \infty) \to R^+$ satisfying the properties:

(*i*) φ is continuous;

(*ii*) $\varphi(t) < t$ for all $t > 0$.

*Supported in part by the National Research Foundation of South Africa.

Definition 1 (8) — Let, (M, d) be a complete metric space. A mapping $T: M \to M$ is said to be an asymptotic contraction if, for each integer $n \geq 1$, there exists a function $\varphi_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(T^n x, T^n y) \leq \varphi_n(d(x, y)) \quad \text{for all } x, y \in M \qquad \text{... (1)}$$

and if $\varphi_n \to \varphi \in \Phi$ uniformly on the range of d.

Then Kirk (8) proves the following theorem.

***Theorem* 1** — *Suppose (M, d) is a complete metric space and suppose $T: M \to M$ is an asymptotic contraction for which the mappings φ_n in (1) are continuous. Assume also some orbit of T is bounded. Then T has a unique fixed point z, and moreover the Picard sequence $\{T^n x\}_{n=1}^{\infty}$ converges to z for each $x \in M$.*

Theorem 1 can be viewed the asymptotic version of the following known result (2,3).

***Theorem* 2** — *Suppose (M, d) is a complete metric space and suppose a mapping $T: M \to M$ satisfies the condition*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \text{for all } x, y \in M \qquad \text{... (2)}$$

where $\varphi \in \Phi$ is a given function. Assume that T has a bounded orbit. Then T has a unique fixed point z, and moreover for each $x \in M$, the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to z.

Theorem 1 was proved in (8) using an ultraproduct technique. In this note we shall give a simple and elementary proof to Theorem 1. Moreover, we will introduce the notion of weakly asymptotic contractions and the main result (Theorem 3) of this note contains Theorem 1. (During the review process of this paper, one of the referees pointed out that an elementary proof of Theorem 1 was also given independently in (1 and 6).

In our argument below we put

$$\overline{\Phi} = \{\overline{\phi} \in \overline{\Phi} : \overline{\phi} \text{ is increasing on } \mathbb{R}^+\}.$$

The key idea of our proof is to replace each function φ_n in (1) by another function $\overline{\varphi}_n$ which is increasing and which is constructed in following manner. For a continuous function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ define $\overline{\varphi}$ by

$$\overline{\varphi}(t) := \max\left\{\varphi(\tau) : \tau \in [0, t] \cap \overline{R(d)}\right\}, \qquad \text{... (3)}$$

where $R(d) = \{d(x, y) : x, y \in M\}$ is the range of d and $\overline{R(d)}$ is the closure of $R(d)$.

Lemma 1 — If $\varphi \in \Phi$, then $\overline{\varphi} \in \overline{\Phi}$.

PROOF : It is evident that $\overline{\varphi}$ is increasing. Since $\overline{\varphi}(t) = \varphi(\tau^*)$ for some $\tau^* \in [0, t]$, it follows that $\overline{\varphi}(t) < t$ for all $t > 0$; that is, $\overline{\varphi}$ satisfies condition (*ii*) above. It remains to prove the continuity of $\overline{\varphi}$. Namely, $\overline{\varphi}$ satisfies condition (*i*) above as well; hence $\overline{\varphi} \in \overline{\Phi}$. This follows from the fact that for $0 \le s < t$, $0 \le \overline{\varphi}(t) - \overline{\varphi}(s) \le \max\left\{ \varphi(\tau) - \varphi(s) : \tau \in [s, t] \cap \overline{R(d)} \right\}$ and the uniform continuity of φ over, any closed bounded interval $[a, b]$, where $0 \le a \le b < \infty$. ■

2. WEAKLY ASYMPTOTIC CONTRACTIONS

We now introduce the following notion of weakly asymptotic contractions.

Definition 2 — A continuous mapping T from a complete metric space (M, d) into itself is said to be a weakly asymptotic contraction if for an arbitrary $\varepsilon > 0$, there is an integer $n_\varepsilon \ge 1$ such that

$$d\left(T^{n_\varepsilon} x, T^{n_\varepsilon}\right) \le \varphi(d(x, y)) + \varepsilon \text{ for all } x, y \in M, \quad \ldots (4)$$

where $\varphi \in \Phi$ is given (i.e., independent of ε).

It is easily seen that an asymptotic contraction in the sense of Definition 1 is, due to the requirement that $\varphi_n \to \varphi \in \Phi$ uniformly on the range of d, a weakly asymptotic contraction in the sense of Definition 2.

We next show that a weakly asymptotic contraction on a complete metric space has a unique fixed point provided it has a bounded orbit.

Theorem 3 — *Suppose (M, d) is a complete metric space and suppose $T : M \to M$ is a weakly asymptotic contraction. Assume that T has a bounded orbit at some $x \in M$. Then T has a unique fixed point z. Moreover the Picard sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ converges to z.*

PROOF : Put

$$d_{n,m} = d(T^n x, T^m x) \quad \text{for} \quad n, m \ge 1$$

and

$$d_\infty = \limsup_{n, m \to \infty} d_{n,m} := \lim_{k \to \infty} \sup\{d_{n,m} : n, m \ge k\} < \infty.$$

Observe that for any $\varepsilon > 0$ and for all $n, m > n_\varepsilon$, by the weakly asymptotic contraction condition (4), we have

$$
\begin{aligned}
d_{n,m} &= d\left(T^{n_\varepsilon}(T^{n-n_\varepsilon}x)\, T^{n_\varepsilon}(T^{m-n_\varepsilon}x)\right) \\
&\le \varphi\left(d(T^{n-n_\varepsilon}x, T^{m-n_\varepsilon}x)\right) + \varepsilon \\
&\le \tilde{\varphi}\left(d(T^{n-n_\varepsilon}x, T^{m-n_\varepsilon}x)\right) + \varepsilon \\
&= \tilde{\varphi}(d_{n-n_\varepsilon, m-n_\varepsilon}) + \varepsilon,
\end{aligned}
$$

where $\tilde{\varphi}$ is defined as in (3). Taking the limsup as $n, m \to \infty$ and noting the continuity and increasingness of $\tilde{\varphi}$ we get

$$d_\infty \le \tilde{\varphi}(d_\infty) + \varepsilon.$$

But $\varepsilon > 0$ is arbitrary, it follows that

$$d_\infty \le \tilde{\varphi}(d_\infty).$$

This implies $d_\infty = 0$; hence $\{T^n x\}$ is Cauchy. It is then easily seen that $\{T^n x\}$ converges to the unique fixed point z of T.

Corollary 1 — (Kirk [8]). Suppose (M, d) is a complete metric space and suppose $T: M \to M$ is an asymptotic contraction for which the mappings φ_n in (1) are continuous. Assume also some orbit of T is bounded. Then T has a unique fixed point z, and moreover the Picard sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ converges to z for each $x \in M$.

Note that Theorem 3 also extends a theorem of Browder (3) who assumed that for some integer $N \ge 1$, the map T^N satisfies condition (4).

3. Concluding Remarks

Another way to weaken the notion of asymptotic contractions in Definition 1 is given below.

Definition 3 — A mapping T from a complete metric space (M, d) into itself is said to be a limiting asymptotic contraction provided

$$\limsup_{n\to\infty} d(T^n x, T^n y) \le \varphi(d(x,y)) \quad \text{for all } x, y \in M, \qquad \dots (5)$$

where $\varphi \in \Phi$ is given. Obviously an asymptotic contraction is a limiting asymptotic contraction.

It turns out that the limiting asymptotic contractivity condition (5) is too weak to guarantee the existence of a fixed point, nor the map is necessarily contractive. Indeed, we have the following characterization of limiting asymptotic contractions.

Theorem 4 — *A map T on a complete metric space (M, d) is a limiting asymptotic contraction if and only if*

$$\limsup_{n\to\infty} d(T^n x, T^n y) = 0 \quad \text{for all } x, y \in M. \qquad \dots (6)$$

PROOF : Indeed, that (6) implies (5) is trivial. To show that (5) implies (6), replacing the x and y in (5) by $T^m x$ and $T^m y$, respectively, where $m \ge 1$ is a fixed integer, we get (noticing that

$$\limsup_{n\to\infty} d(T^n x, T^n y) = \limsup_{n\to\infty} d(T^{n+m} x, T^{n+m} y))$$

$$\limsup_{n\to\infty} d(T^n x, T^n y) \le \tilde{\varphi}(d(T^m x, T^m y)) \text{ for all } x, y \in M. \qquad \dots (7)$$

Taking the lim sup as $m \to \infty$ in (7) yields

$$\limsup_{n\to\infty} d(T^n x, T^n y) \le \tilde{\varphi}\left(\limsup_{m\to\infty} d(T^m x, T^m y)\right).$$

It follows that $\limsup_{n\to\infty} d(T^n x, T^n y) = 0$.

Tingley (12) constructed a limiting asymptotic contraction T on a bounded closed convex subset K of a Hilbert space which is fixed point free. Note that his map is not contractive (i.e., $\| Tx - Ty \| \ge \| x - y \|$ for some $x \ne y, x, y \in K$). However even a contractive limiting· asymptotic contraction may fail to have a fixed point, as shown in the following example.

Example 1 — (cf. [7]) Let $M = \{f \in C[0,1] : f(0) = f(1) = 1, 0 \le f(x) \le 1\}$ which is equipped with the usual sup norm. Define an operator $T : M \to M$ by

$$Tf(x) = xf(x) \text{ for all } x \in [0, 1].$$

It is easily seen that T is contractive and does not have a fixed point. We now show that T is a limiting asymptotic contraction. As a matter of fact, it is not hard to see that given $f \in M$ and $x \in [0, 1]$, we have

$$T^n f(x) = x^n f(x).$$

Thus, if given another $g \in M$, we have

$$|T^n f(x) - T^n g(x)| = |x^n| f(x) - g(x)|.$$

For $\varepsilon > 0$, since $f(1) - g(1) = 0$, by continuity of $f - g$, we can find, $\delta \in (0,1)$ such that

$$|f(x) - g(x)| < \varepsilon \quad \text{for all } x \in [1 - \delta, 1]. \qquad \dots (8)$$

Now take an integer $N > 1$ big enough so that

$$\gamma(1 - \delta)^n < \varepsilon \quad \text{for all } n \geq N \qquad \dots (9)$$

where $\gamma = \max \{|f(x) - g(x)| : x \in [0, 1 - \delta]\}$. Combining (8) and (9) we find that

$$d(T^n f, T^n g) = \max \{x^n |f(x) - g(x)| : x \in [0, 1]\} < \varepsilon \text{ for all } n \geq N.$$

Namely, $\limsup_{n \to 0} d(T^n f, T^n g) = 0$, i.e., T is a limiting asymptotic contraction.

ACKNOWLEDGEMENT

The author is grateful to the referees for their helpful comments and suggestions which improved the presentation of this manuscript.

REFERENCES

1. I. D. Arandelovic, On a fixed point theorem of Kirk, *J. Math. Anal. Appl.*, **301** (2005), 384-85.
2. D. W. Boyd and J. S. W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.*, **20** (1969), 458-64.
3. F. E. Browder, On the Convergence of successive approximations for nonlinear functional equations, *Idag. Math.*, **30** (1968), 27-35.
4. A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
5. S. Heinrich, Ultraproducts in Banach space theory, *J. Reine Angew. Math.*, **313** (1980), 72-104.
6. J. Jachymski and J. Jozwik, On Kirk's asymptotic contractions, *J. Math. Anal. Appl.*, **300** (2004), 645-50.
7 W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly*, **72** (1965), 1004-1006.
8. W. A. Kirk, Fixed points of asymptotic contractions, *J. Math. Anal. Appl.*, **277** (2003), 645-50.
9. H. E. Kunze and E. R. Vrscay, Solving inverse problems for ordinary differential equations using the Picard contraction mappings, *Inverse Problems*, **15** (1999), 745-70.
10. H. E. Kunze and S. Gomes, Solving an inverse problem for Urison-type integral equations using Banach's fixed point theorem, *Inverse Problems*, **19** (2003), 411-18.
11. H. E. Kunze, J. E. Hichen and and E. R. Vrscay, Inverse problems for ODEs using contraction maps and suboptimality of the 'collage methods', *Inverse Problems*, **20** (2004), 977-91.
12. D. Tingley, An asymptotically nonexpansive commutative semigroup with no fixed points, *Proc. Amer. Math. Soc.*, **97** (1986), 107-13.