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A PERIODIC SOLUTION OF OSCILLATORY COUETTE FLOW THROUGH POROUS MEDIUM IN ROTATING SYSTEM

K. D. SINGH, M. G. GORLA AND HANS RAJ

Department of Mathematics, ICDEOL, Himachal Pradesh University, Summer-Hill, Shimla, India

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An exact solution of oscillatory Ekman boundary layer flow through a porous medium bounded by two horizontal flat plates is obtained. One of the plates is at rest and the other oscillating in its own plane. The entire system rotates about an axis normal to the plates. The effects of coriolis force and the permeability of the porous medium on the flow field are studied. It is found during mathematical analysis that even in the special case of resonance ($w = 2\Omega$) the solution obtained by Mazumder[21] is incorrect as contended by Ganapathy[22].

Key Words : Periodic; Oscillatory Couette Flow; Porous Medium; Rotating System

1. INTRODUCTION

Flows of fluid through Porous media are of principal interest these days and have attracted the attention of a number of scholars due to their applications in the fast growing fields of Science and Technology, viz. in the fields of agricultural engineering to study the underground water resources, seepage of water in riverbeds, in petroleum technology to study the movement of natural gas. oil. and water through the oil reservoirs. In view of the wide ranging applications of flows through porous media, a series of investigation have been made by Raptis *et al.* (1-3) into the steady two-dimensional flows through porous media. Raptis (4) and Raptis and Perdikis (5) further studied the unsteady two-dimensional tree convective flows through highly porous medium.

Apart from these two-dimensional flows a number of three-dimensional steady and unsteady flows through highly porous media along with heat transfer have also been analysed by Singh and Verma (6-7). Singh *et al.* (8-9) and Singh and Sharma (10) for either constant or periodic or variable permeability of the porous medium. The fluid flows through porous medium bounded between channels and duets are important particularly in the fields of chemical engineering, petroleum technology, textile engineering and paper industry for purification and filtration processes. In view of this Singh and Sharma (11) have, very recently analysed three-dimensional Couette flow through a porous medium with heat transfer.

On the other hand the geophysical importance of the flows in the rotating frame of reference has attracted the attention of a number of scholars. There appeared a number of studies in the literature viz. Vidyanidhy and Nigam (12), Gupta (13) and Jana and Datta (14). The effects of uniform transverse magnetic field with or without suction was investigated by Gupta (15), Soundalgekar and Pop (16) and Mazumder *et al.* (17). The similarity solutions of the unsteady Navier-Stokes equations in a rotating frame of reference has been obtained by Gupta (18) Chandran *et al.* (19) and Singh *et al.* (20) studied the unsteady Magnetohydrodynamics Couette flow of electrically conducting fluid in a rotating system. Recently, Mazumder (21) studied an oscillatory Ekman boundary layer flow bounded by two horizontal flat plates, one of which is oscillating about a non-zero constant mean velocity in its own plane and the other at rest. But neither the method adopted by him nor his analysis is valid in the rotating system as has been pointed out by Ganapathy (22). By considering the form of solution as the linear combination (Telionis) (23) of $\exp(i\,\omega t)$ and $\exp(-i\,\omega t)$, Ganapathy (22) presented an alternative solution to the problem. Through the present paper, an attempt has been made to study the effect of the permeability of the porous medium on the oscillatory flow through a porous medium bounded by two parallel flat plates when the entire system rotates about an axis perpendicular to the plates.

2. Mathematical Analysis

Consider an unsteady flow of a viscous, incompressible fluid through a highly porous medium bounded by two horizontal parallel flat plates distance d apart as has been assumed by Mazumder[21]. The lower plate is considered at rest and the upper one is oscillating in its own plane with a velocity $U(t)$ about a non-zero constant mean velocity U_o. Choose the origin on the lower plate and the x-axis parallel to the direction of motion of the upper plate. The z-axis taken perpendicular to the plates is the axis of rotation about which the entire system is rotating with a constant angular velocity Ω. Since the plates are infinite in extent, all the physical quantities, except the pressure. depend on z and t only. Denoting the velocity components u, v, w, in the x, y, z directions, respectively, the equation of continuity $\nabla \vec{\mu} = 0$ gives $w = 0$. Under these assumptions the flow through a highly porous medium in a rotating system is governed by the following equations:

$$\frac{\partial u}{\partial t} = \frac{1}{\rho}\frac{\partial p^*}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} + 2\Omega\, v - \frac{\nu}{K^*}, u, \quad \dots (1)$$

$$\frac{\partial v}{\partial t} = \frac{1}{\rho}\frac{\partial p^*}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} - 2\Omega\, u - \frac{\nu}{K^*}, v. \quad \dots (2)$$

Where ν is the kinematic viscosity, t is the time, ρ is the density. K^* is the permeability of the porous medium, and p^* is the modified pressure. The boundary conditions for the problem are:

$$u = v = 0 \qquad \text{at} \quad z = 0.$$

$$u = U(t) = U_0(1 + \varepsilon\cos\omega t), \quad v = 0 \qquad \text{at } z = d \quad \dots (3)$$

Where ω is the frequency of oscillations and ε is a small positive constant.

Eliminating the modified pressure gradient, under the usual boundary layer approximations, eqs. (1) and (2) are reduced to

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2} + \frac{\partial U}{\partial t} + 2\Omega\, \nu - \frac{\nu}{K^*}(u - U) \quad \dots (4)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - 2\Omega (u - U) - \frac{\nu}{K^*} v \qquad \text{... (5)}$$

Eqs. (4) and (5) can be combined, in complex form, as

$$\frac{\partial q}{\partial t} = \nu \frac{\partial^2 q}{\partial z^2} + \frac{\partial U}{\partial t} - 2 i \Omega (q - U) - \frac{\nu}{K^*} (q - U), \qquad \text{... (6)}$$

and the corresponding boundary conditions become

$$q = 0 \text{ at } z = 0$$

$$q = U(t) = U_0 \left[1 + \frac{\varepsilon}{2} \left\{ e^{i \omega t} + e^{-i \omega t} \right\} \right] \quad \text{at } z = d, \qquad \text{... (7)}$$

where $q = u + iv$.

The solution sought for eq. (6) by Mazumder (21) is not correct. The correct form of the solution, as given by Ganapathy (22), is

$$q(\eta, t) = U_0 \left[q_0(\eta) + \frac{\varepsilon}{2} \left\{ q_1(\eta) e^{i \omega t} + q_2(\eta) e^{-i \omega t} \right\} \right], \qquad \text{... (8)}$$

where

$$\eta = z/d, \; q_0(\eta) = u_0(\eta) + i v_0(\eta)$$

and

$$\left\{ q_1(\eta) e^{i \omega t} + q_2(\eta) e^{-i \omega t} \right\} = u_1(\eta, t) + i v_1(\eta, t), \qquad \text{... (9)}$$

Substituting (8) into (6) and (7) and comparing the harmonic and non-harmonic terms, we get

$$q_0'' - (2iR + K^{-1}) q_0 = -(2iR + K^{-1}), \qquad \text{... (10)}$$

$$q_1'' - (2iR + I\lambda + K^{-1}) q_1 = -(2iR + i\lambda + K^{-1}), \qquad \text{... (11)}$$

$$q_2'' - (2iR + I\lambda + K^{-1}) q_2 = -(2iR - i\lambda + K^{-1}), \qquad \text{... (12)}$$

with boundary conditions

$$q_0 = q_1 = q_2 = 0 \quad \text{at } \eta = 0,$$

$$q_0 = q_1 = q_2 = 1 \quad \text{at } \eta = 1, \qquad \text{... (13)}$$

where $R = \dfrac{\Omega d^2}{\nu}$ is the rotation parameter, $\lambda = \dfrac{\omega d^2}{\nu}$ is the frequency parameter and $K = \dfrac{k^*}{d^2}$ is the permeability parameter.

Solving eqs. (10) to (12) under the boundary conditions (13), we obtain

$$q_0(\eta) = 1 - \frac{\sinh \ell(1-\eta)}{\sinh \ell} \quad \text{... (14)}$$

$$q_1(\eta) = 1 - \frac{\sinh m(1-\eta)}{\sinh m} \quad \text{... (15)}$$

$$q_2(\eta) = 1 - \frac{\sinh n(1-\eta)}{\sinh n} \quad \text{... (16)}$$

where

$$\ell = (2iR + K^{-1})^{1/2}, m = (2iR + i\lambda + K^{-1})^{1/2}, n = (2iR - i\lambda + K^{-1})^{1/2}.$$

As $K \to \infty$, the solutions for $q_0(\eta)$, $q_1(\eta)$ and $q_2(\eta)$ are in agreement with those obtained by Ganapathy (22).

3. Shear Stress

The amplitude and the phase differences of shear stresses al the plate $\eta = 0$ for the steady flow can be obtained as:

$$\tau_{or} = \left(\tau_{ox}^2 + \tau_{oy}^2\right)^{1/2}, \quad \theta_{or} = \tan^{-1}\left(\frac{\tau_{oy}}{\tau_{ox}}\right) \quad \text{... (17)}$$

where τ_{ox} and τ_{oy} are, respectively, the shear stresses at the plate due to the primary and secondary velocity components.

The numerical values for the resultant shear stress and the phase angle due to the shear stresses are listed in Table 1.

TABLE 1 : Values of τ_{0r} and θ_{0r} for various R and K

R	K	τ_{0r}	θ_{0r}
1		1.265549	0.539750
1	1	1.750183	0.549237
5	1	3.228127	0.734216
25	1	7.072347	0.775385
50	1	10.000263	0.780398
100	1	14.1422240	0.782898
1000	1	44.721362	0.785148
5	5	232222	0.773713
25	5	7.071716	0.783383
50	5	10.000024	0.784398
100	5	14.142139	0.784898

4. Results and Discussion

The solution (14) corresponds to the steady part, which gives u_0, as the primary and v_0 as the secondary velocity components. The amplitude and phase difference due to these primary and secondary velocities for the steady flow are given by:

$$|A_0| = \left(u_0^2 + v_0^2\right), \quad \theta_0 = \tan^{-1}\left(\frac{v_0}{u_0}\right). \qquad \text{... (18)}$$

For large values of R, the expressions for u_0 and v_0 can be approximated from eq. (14) as

$$u_0(\eta) \cong 1 - \exp(-\zeta_r \eta)\cos \zeta_i \eta \qquad \text{... (19)}$$

$$v_0(\eta) \cong \exp(-\zeta_r \eta)\cos \zeta_i \eta \qquad \text{... (20)}$$

where

$$\zeta_r = \left[\left\{\frac{(K^{-2} + 4R^2)^{1/2} + K^{-1}}{2}\right\}\right]^{1/2}, \quad \zeta_i = \left[\left\{\frac{(K^{-2} + 4R^2)^{1/2} - K^{-1}}{2}\right\}\right]^{1/2}$$

These approximations for u_0 and v_0 represent the spiral distribution of velocity and show clearly the existence of a thin boundary layer of order $0\left(\zeta_r^{-1}\right)$ in the neighbourhood of the plates and is known as Ekman layer which decreases with the increase of the rotation parameter R hut increases with the increase of permeability parameter.

The amplitude or the resultant velocity $|A_0|$ and the phase angle θ_0 for the steady part are shown graphically in Fig. 1(a, b) for various values of the rotation parameter R and the permeability parameter K. It is observed from Fig. 1(a) that the amplitude $|A_0|$ increases with the increase of the permeability parameter K for all values of the rotation parameter R large or small. An increase in $|A_0|$ also noticed with the increasing rotation parameter R and it becomes approximately one for large rotation in the upper hall of the channel width. Fig. l(b) shows that the phase angle θ_0 for the steady flow increases with increasing permeability K of the porous medium for any value of rotation large or small. It is also clear from this figure that an increase in the small rotation parameter R leads to an increase in θ_0 , but an increase in large values of R results in the decrease of phase angle θ_0. In the upper half of the channel the phase angle becomes approximately zero.

These values clearly show that the shear stress τ_{0r} increases with increasing rotation parameter R. However, the increase in the permeability parameter K leads to an increase in the shear stress for small rotation but to decrease thereafter for large rotation. It is also clear from this table that the phase angle θ_{0r} increases with increasing rotation and permeability of the porous medium.

The solutions (15) and (16) together give the unsteady part of the flow. The primary and secondary velocity components u_1 and v_1 respectively, for the fluctuating flow can be obtained from the asymptotic expansion of q for large rotation R as

$$u_1(\eta, t) \cong 2\cos\omega t - e^{-m_r\eta}\cos(m_i\eta - \omega t) - e^{-n_i\eta}\cos(n_i\eta + \omega t) \qquad \dots (21)$$

$$v_1(\eta, t) \cong e^{-m_r\eta}\sin(m_i\eta - \omega t) + e^{-n_r\eta}\sin(n_i\eta + \omega t), \qquad \dots (22)$$

where

$$m_r = \left[\left\{\frac{(K^{-2} + \overline{2R + \lambda^2})^{1/2} + K^{-1}}{2}\right\}\right]^{1/2},$$

$$m_i = \left[\left\{\frac{(K^{-2} + \overline{2R + \lambda^2})^{1/2} - K^{-1}}{2}\right\}\right]^{1/2}$$

$$n_r = \left[\left\{\frac{(K^{-2} + \overline{2R - \lambda^2})^{1/2} + K^{-1}}{2}\right\}\right]^{1/2},$$

$$n_i = \left[\left\{\frac{(K^{-2} + \overline{2R - \lambda^2})^{1/2} - K^{-1}}{2}\right\}\right]^{1/2}$$

These expressions for $u_1(\eta, t)$ and $v_1(\eta, t)$ show the emergence of a boundary layer of thickness of order $O\left(m_r^{-1}\right)$ superimposed with a boundary layer of thickness of order $O\left(n_r^{-1}\right)$. These boundary layers which are due to the cyclonic and anticyclonic components of the impressed harmonic oscillations increase with the increase of permeability parameter and decrease with the combined effect of rotation and frequency. The combination of Stokes layer and Ekman layer is exhibited in these boundary layers which appear in the neighbourhood of both the plates. In the case of resonance when the natural frequency 2Ω of the rotating fluid is equal to the forcing frequency ω, that is for $2\Omega - \omega = 0$ or $2R - \lambda = 0$, the differential eq. (12) for q_2 reduces to

$$q_2^* - K^{-1} q_2 = -K^{-1} \qquad \dots (23)$$

which under the boundary conditions (13) is integrated and we get

$$q_2(\eta) = 1 - \frac{\sinh K^{-1}(1-\eta)}{\sinh K^{-1}} \qquad \dots (24)$$

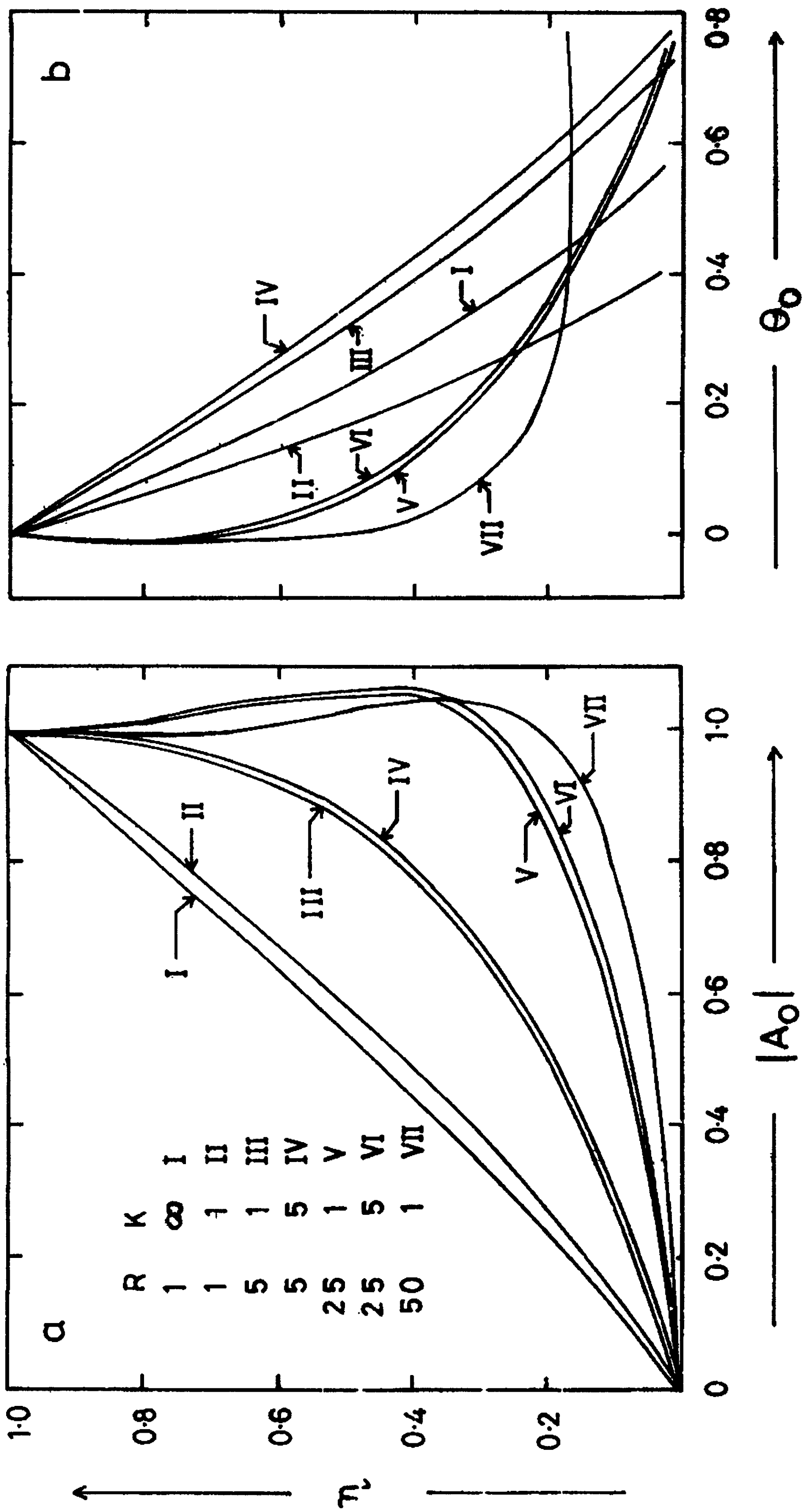

FIG. 1.(a,b). The amplitude | Ao | and phase angle $_0\theta$ due to $_0U$ and $_0V$ for steady flow.

Further, for ordinary medium, i.e. as $K \to \infty$ the solution (24) reduces to

$$q_2(\eta) = \eta. \qquad \text{... (25)}$$

Even otherwise, the differential eq. (5) for q_2 of Ganapathy (22) in the case of resonance gives $q_2(\eta) = \eta$ as the solution. This means that $q_2(\eta)$ does not vanish in the case of resonance which is a contradiction to the claim of Ganapathy (22) that $q_2(\eta)$ becomes zero and the solution

$$q(\eta, t) = U_0 \left[q_0(\eta) + \frac{\varepsilon}{2} q_1(\eta) \exp(i \omega t) \right], \qquad \text{... (26)}$$

of Mazumder (21) is valid for this special case. Therefore, the solution obtained by Mazumder (21) does not agree with the correct solution even in the special case of resonance as contended by Ganapathy (22).

For this situation of resonance in the ordinary medium when the rotation R is very large the velocity components for the unsteady flow are given by

$$u_1(\eta, t) = (1 + \eta) \cos \omega t - e^{-\sigma_1 \eta} \cos(\sigma_1 \eta - \omega t), \qquad \text{... (27)}$$

$$v_1(\eta, t) = (1 - \eta) \sin \omega t + e^{-\sigma_1 \eta} \sin(\sigma_1 \eta - \omega t), \qquad \text{... (28)}$$

where $\sigma_1 = \left\{ R + \left(\frac{\lambda}{2} \right) \right\}^{1/2}$. These equations are different from those obtained by Mazumder (21).

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REFERENCES

1. A. Raptis, G. Perdikis and G. Tzivanidis, Free convection flow through a porous medium bounded by a vertical surface, *J. Phys. D. Appl. Phys.*, **14** (1981), 99-102.
2. A. Raptis, G. Tzivanidis and N. Kafousias, Free convection and mass transfer flow through a porous medium bounded by an infinite vertical limiting surface with constant suction, *Letters Heat Mass Transfer*, **8** (1981), 417-24.
3. A. Raptis, N. Kafousias and C. Massalas, Free convection and mass transfer flow through a porous medium bounded by an infinite vertical porous plate with constant heat flux, *ZAMM*, **62** (1982), 489-91.
4. A. A. Raptis, Unsteady free convective flow through a porous medium, *Int. J. Engng. Sci.*, **21** (1983), 345-48.
5. A. A. Raptis and C. P. Perdikis, Oscillatory flow through a porous medium by the presence of free convective flow, *Int. J. Engng. Sci.*, **23** (1985), 51-55.
6. K. D. Singh and G. N. Verma, Three-dimensional oscillatory flow through a porous medium with periodic permeability, *ZAMM*, **75** (1995), 599-604.
7. K. D. Singh and G. N. Verma, Three-dimensional fluctuating flow and heat transfer though a porous medium, *Indian Natn. Sci. Acad.*, **62** (1996), 33-44.
8. K. D. Singh, Khem Chand and G. N. Verma, Heat transfer in a three-dimensional flow through a porous medium with periodic permeability, *ZAMM*, **75** (1995), 950-52.

9. K. D. Singh, Rakesh Sharma and Khem Chand, Three-dimensional fluctuating flow and heat transfer through a porous medium with variable permeability, *ZAMM*, **80** (2000), 473-80.

10. K. D. Singh and Rakesh Sharma, Three-dimensional free convective flow and heat transfer through a porous medium with periodic permeability, *Indian J. Pure Appl. Math.*, **33** (2002), 941-49.

11. K. D. Singh and Rakesh Sharma, Three-dimensional couette flow through a porous medium with heat transfer, *J. Pure Appl. Math.*, **32** (2001), 1819-29.

12. V. Vidyanidhu and S. D. Nigam, Secondary flow in a rotating channel, *J. Math. and Phys. Sci.*, **1** (1967), 85.

13. A. S. Gupta, Ekman layer on a porous plate, *Phys. Fluid*, **15** (1972), 930-31.

14. R. N. Jana and N. Datta, Couette flow and heat transfer in a rotating system, *Acta Mechanica*, **26** (1977), 301-06.

15. A. S. Gupta, Magnetohydrodynamic Ekman layer, *Acta Mechanica*, **13** (1972), 155-60.

16. V. M. Soundalgekar and I. Pop, On hydromagnetic flow in a rotating fluid past an infinite porous wall, *ZAMM*, **53** (1973), 718-19.

17. B. S. Mazumder, A. S. Gupta and N. Datta, Flow and heat transfer in a hydromagnetic Ekman layer on a porous plate with hall effects, *Int. J. Heat Mass Transfer*, **19** (1976), 523-27.

18. A. S. Gupta, *Mechanics Research Communication*, **27** (2000), 485-92.

19. P. Chandran, N. C. Sacheli and A. K. Singh, *Int. Jr. Engg. Sci.*, **202** (1993), 1-10.

20. A. K. Singh, N. C. Sacheli and P. Chandran, *Int. Jr. Engng. Sci.*, **32** (1994), 133-39.

21. B. S. Mazumder, An exact solution of oscillatary couette flow in a rotating system, *ASME J. Appl. Mech.*, **58** (1991), 1104-07.

22. R. Ganapathy, A notion oscillatary couette flow in a rotating system, *ASME J. Appl. Mech.*, **61** (1994), 208-09.

23. D. P. Telionis, *Unsteady Viscous Flows*, Springer-Verlag, New York, 1981.