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MEASURES WITH τ-SMOOTH MARGINALS

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An example is given of two τ-smooth positive measures μ_1, μ_2 on completely regular Hausdorff spaces X_1, X_2, and a net $\{v_\alpha\}$ of τ-smooth positive measures on $X_1 \times X_2$ such that the marginals of $\{v_\alpha\}$ converge to μ_1, μ_2 but $\{v_\alpha\}$ converges to a non-τ-smooth measure. It is proved that if μ_1 is tight then v_α converges to a τ-smooth measure. As a consequence, a Strassen type theorem is proved for τ-smooth measures.

Kew Words: Measure; Finitely Additive Measures; τ-Smooth Measures; Marginal of Measures; Lifting of Measures; Lifting Topology

In this note, all vector spaces are over the field of real numbers R (we will call them scalars). For a completely regular Hausdorff space X, the following notations will be used:

$C(X)$ ($C_b(X)$) will denote the space of all (all bounded) R-valued continuous functions on X,

elements of $\{f^{-1}(0) L f \in C_b(X)\}$ will be called zero-sets of X and their complements the positive-sets of X,

$\mathcal{F}(X)$ will be algebra generated by zero-sets,

$M(X) = (C_b(X), \|\cdot\|)'$ is the lattice of bounded finitely additive measures on $\mathcal{F}(X)$, inner regular by zero-sets and outer regular by positive-sets of X; $M^+(X)$ denotes the positive elements of $M(X)$,

$M_\tau(X)$ are scalar-valued, countably additive Borel measures μ on X, which are inner regular by closed sets and outer regular by open sets of X, and for any increasing net of open sets $\{V_\alpha\} \subset X$ with $\bigcup V_\alpha = V$, we have $\lim \mu(V_\alpha) = \mu(V)$; $M_\tau^+(X)$ denotes the positive elements of these measures. These measures are called τ-smooth measures.

$M_t(X)$ are scalar-valued, countably additive tight Borel measures μ on X; they are inner regular by compact sets and outer regular by open sets of X; $M_t^+(X)$ denotes the positive elements of these measures (7,9,10,1).

If X is compact $M(X) = M_\tau(X) = M_t(X)$.

The topologies on these measures will be the one induced by $\sigma(M(X), C_b(X))$.

For a completely regular Hausdorff space X, $\tilde{X}$ will denote its Stone-Cech compactification; a $\mu \in M(X)$ gives a $\tilde{\mu} \in M(\tilde{X})$ with $\tilde{\mu}(f) = \mu(f_{|X})$, for all $f \in C(\tilde{X})$; this μ will be in $M_\tau(X)$ iff $|\tilde{\mu}|(\tilde{X} \setminus C) = 0$ for all compact $C \subset \tilde{X} \setminus X$ (9,7).

For $i = (1, 2)$, let $\mathcal{A}_i$ be algebras of subsets of the sets X_i and $\mathcal{A}_1 \times \mathcal{A}_2$ the algebra generated by $\{A_1 \times A_2 : A_i \in \mathcal{A}_1 \ (i = 1, 2)\}$. Suppose $\nu : \mathcal{A}_1 \times \mathcal{A}_2 \to [0, 1]$ is a finitely additive measure. The finitely additive measures $\nu^{(1)} : \mathcal{A}_1 \to [0, 1]$, $\nu^{(1)}(A_1) = \nu(A_1 \times A_2)$ and $\nu^{(2)} : \mathcal{A}_2 \to [0, 1]$, $\nu^{(2)}(A_2) = \nu(X_1 \times A_2)$ are called the marginals of ν; they are projections of ν on X_1 and X_2. If X_1 and X_2 are completely regular Hausdorff spaces and $\nu \in M_\tau^+(X_1 \times X_2)$ then its marginals $\nu^{(i)}$, 1, 2, will be in $M_\tau^+(X_i)$ with $\nu(f_1 \otimes 1) = \nu^{(1)}(f_1)$ and $\nu(1 \otimes f_2) = \nu^{(2)}(f_2)$, with $f_i \in C_b(X_i)$, i = 1, 2, (here $f_1 \otimes 1\,(x_1, x_2) = f_1(x_1)$ and $1 \otimes f_2\,(x_1, x_2) = f_2(x_2)$); also it is simple verfication that if $\nu \in M_\tau^+(X_1 \times X_2)$ and $\mu_i \in M^+(X_i)$ such that $\nu(f_1 \otimes 1) = \mu_1(f_1)$ and $\nu(1 \otimes f_2) = \mu_2(f_2)$ with $f_i \in C_b(X_i)$, $i = 1, 2$, then μ_1 and μ_2 are the marginals of ν.

In the celebrated Strassen's paper (8), conditions are obtained for the existence of measures with given marginals. In (3, Theorem 1, p. 160), a result is proved about the limit of a net of τ-smooth measures, having τ-smooth marginals; unfortunately there is a gap in the proof. The following example proves that.

Example 1 — For $i = 1, 2$, there exist completely regular Hausdorff spaces X_i, $\mu_i \in M_\tau^+(X_i)$ and a net $\{\nu_\alpha\} \subset M_\tau^+(X_1 \times X_2)$ having the following properties:

(*i*) $\nu_\alpha^{(i)} \to \mu_i$ in $\sigma(M_\tau(X_i), C_b(X_i))$

(*ii*) $\nu_\alpha \to \nu \in M^+(X_1 \times X_2) \setminus M_\tau^+(X_1 \times X_2)$.

PROOF : The following construction is done in (4, p. 167):

I = [0,1], m the Lebesgue measure on I, $\mathcal{B}$ all Borel subsets of I, Z_1, Z_2 a decomposition of I such that the Lebesgue outer measures of Z_1, Z_2 are 1 each;

for i = 1, 2, $\mu_i : Z_i \cap \mathcal{B} \to I$, $\mu_i(Z_i \cap B) = m(B)$, for all $B \in \mathcal{B}$ are countably additive probability measures;

there is a finitely additive measure ν_0 on the algebra generated, in $(Z_1 \times Z_2)$, by $\{(Z_1 \times Z_2) \cap (B_1 \times B_2) : B_1$ and B_2 Borel subsets of $I\}$, $\nu_0((Z_1 \times Z_2) \cap (B_1 \times B_2)) = m(\{x : (x, x) \in B_1 \times B_2\})$; this ν_0 has total mass 1, is not countably additive and has μ_1, μ_2 as its marginals.

The completions of the measures μ_i (i = 1, 2) are also denoted by μ_i. We fix some liftings ρ_i for these finite measures μ_i (2), (5). Putting $\mathcal{A}_i = \{\rho_i(B \cap Z_i) : B \in \mathcal{B}\}$ (i = 1, 2), $\mathcal{A}_1 \mathcal{A}_2$ are algebras of subsets of Z_1 and Z_2 and forms a base for the lifting topology $\mathcal{T}_{\rho\, i}$ on Z_i (these are clopen sets; note these topologies are extremely disconnected). We denote by $\left(X_i, \mathcal{T}_{\rho_i}\right)$ the associated completely regular Hausdorff space. The canonical mapping $Z_i \to X_i$ is denoted by ϕ_i (note for an $A \in \mathcal{A}_i$, $\phi_i^{-1}(\phi_i(A)) = A$). The space X_i is extremely disconnected and the algebra of clopen sets, $\mathcal{S}_i = \{\phi_i(A) : A \in \mathcal{A}_i\}$, forms a base of the topology of X_i for i = 1, 2. As is well-known (2), the measures μ_i can be considered as elements of $M_\tau^+(X_i)$.

Let $\mathcal{S}$ be the algebra generated by $\{S_1 \times S_2 : S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2\}$. Take $S_i \in \mathcal{S}_i$, i = 1, 2 and put $\phi_i^{-1}(S_i) = S_i^z$. There are $B_i \in \mathcal{B}$ such that $S_i^z = B_i \cap Z_i$, a.e. $[\mu_i]$. Define $\nu(S_1 \times S_2) = \nu_0((B_1 \times B_2) \cap (Z_1 \times Z_2))$; ν can be extended to a finitely additive measure on $\mathcal{S}$. Since ν_0 is not countably additive, it is a routine verification that this measure in not countably additive. This means there is sequence $\{W_n\} \subset \mathcal{S}$ such that $W_n \downarrow 0$ but $\nu(W_n)$ does not converge to 0.

For any $S_i \in \mathcal{S}_i$, $i = 1, 2$, $\chi_{s_1} \chi_{s_2}$ can be considered an element of $C_b(X_1 \times X_2)$. Let F be the subspace of $C_b(X_1 \times X_2)$, generated by $\{\chi_{S_1} \chi_{S_2} : S_i \in \mathcal{S}_i, i = 1, 2\}$; F contains constant functions and $\nu : F \to R$, $\nu(f) = \int f d\nu$, is linear, positive and $|\nu(f)| \le \|f\|$, $\forall f \in F$ ($\|f\|$ is the sup-norm). By Hahn-Banach theorem, ν extends to $\nu : C_b(X_1 \times X_2) \to R$ which is also linear, continuous (in sup-norm topology), and $|\nu(f)| \le \|f\|$, $\forall f \in C_b(X_1 \times X_2)$. We claim this is also positive: to prove this, take an $f \in C_b(X_1 \times X_2)$, $0 \le f \le 1$ and suppose $\nu(f) = -c\ (c > 0)$; this means $\nu(1 - f) = 1 + c > 1$; but $\|1 - f\| \le 1$ and so we have a contradiction. Thus $\nu \in M^+(X_1 \times X_2)$. For any $\mathcal{S}_i$-simple function $f \in C_b(X_1)$, we have $\nu(f \otimes 1) = \mu_1(f)$. Now the elements of $C_b(X_1)$ are, individually, bounded μ_1-measurable functions on X_1; this implies that for an $f \in C_b(X_1)$, there is a sequence $\{f_n\}$, of μ_1 measurable simple functions, such that $f_n \to f$ uniformly on X_1. Taking liftings, it immediately follows that f is the uniform limit of a sequence of $\mathcal{S}_i$-simple functions. Since ν is continuous in sup-norm, we get $\nu(f \otimes 1) = \mu_1(f)$ for every $f \in C_b(X_1)$, Similarly for every $f \in C_b(X_2)$, we have $\nu(f \otimes 1) = \mu_2(f)$.

Take a net $\{\nu_\alpha\} \subset M_\tau^+(X_1 \times X_2)$, of discrete probability measures, such that $\nu_\alpha \to \nu$. This implies that, for an $f \in C_b(X_1)$, $\nu_\alpha(f \otimes 1) \to \nu(f \otimes 1) = \mu_1(f)$. Similar result holds for an $f \in C_b(X_2)$. From this it follows that $\lim \nu_\alpha^{(i)} = \mu_i$, ($i = 1, 2$). Were (3, Theorem 1, p. 160) true, ν must be in $M_\tau^+(X_1 \times X_2)$, which is not the case.

However, the following theorem holds for τ-smooth measures:

***Theorem* 2** — *For $i = 1, 2$, let X_1, X_2 be completely regular Hausdorff spaces, $\mu_1 \in M_t^+(X_1), \mu_2 \in M_\tau^+(X_2)$, and a $\nu \in M^+(X_1, X_2)$ such that for every $f \in C_b(X_1)$, $\nu(f \otimes 1) = \mu_1(f)$ and for every $f \in C_b(X_2)$, $\nu(1 \otimes f) = \mu_2(f)$. Then $\gamma \in M_\tau^+(X_1 \times X_2)$.*

PROOF : Let $\tilde{X}_1$ and $\tilde{X}_2$ be the Stone-Cech compactifications of X_1 and X_2. For any $f \in C(\tilde{X}_1 \times \tilde{X}_2)$, define $\overline{\nu}(f) = \nu(f_{|(X_1 \times X_2)})$. We get $\overline{\nu} \in M^+(\tilde{X}_1 \times \tilde{X}_2)$. For any $\tilde{f}_i \in C(\tilde{X}_i)$, i, $= 1$, 2, put $f_i = \tilde{f}_{i\,|X_i}$; we have $\overline{\nu}(\tilde{f}_1 \otimes 1) = \mu_1(f_1)$ and $\overline{\nu}(1 \otimes \tilde{f}_2) = \mu_2(f_2)$. This means for any Borel B_i in $\tilde{X}_i$, $i = 1, 2$, $\overline{\nu}(B_1 \times \tilde{X}_2) = \mu_1(B_1 \cap X_1)$ and $\overline{\nu}(\tilde{X}_1 \times B_2) = \mu_2(B_2 \cap X_2)$ (note μ_1 and μ_2 are τ-smooth). Take an increasing sequence $\{C_n\}$ of compact subsets of X_1 such that $\overline{\nu}((\tilde{X}_1 \setminus C_n) \times X_2) \leq \frac{1}{n}$, for all n (here we are very much using that $\mu_1 \in M_t^+(X_1)$). First we prove that for any Borel K of $(\tilde{X}_1 \times \tilde{X}_2)$, $K \subset (\tilde{X}_1 \times (\tilde{X}_2 \setminus X_2))$, $\overline{\nu}(K) = 0$. Because of the regularity of $\overline{\nu}$. It is enough to prove the result when K is compact. Let $\psi_1 : \tilde{X}_1 \times \tilde{X}_2 \to \tilde{X}_1$ and $\psi_2 : \tilde{X}_1 \times \tilde{X}_2 \to \tilde{X}_2$ be the canonical mappings; they are continuous. $K_i = \psi_i(K)$ are compact subsets of $\tilde{X}_i$, $i = 1, 2$, $K \subset K_1 \times K_2$ and $K_2 \subset (\tilde{X}_2 \setminus X_2)$. Since μ_2 is τ-smooth, $\overline{\nu}(\tilde{X}_1 \times K_2) = 0$. This means $\overline{\nu}(K_1 \times K_2) = 0$ and so $\overline{\nu}(K) = 0$, proving the result. Also if a Borel B in $(\tilde{X}_1 \times \tilde{X}_2)$ has the property that, for some n, $B \cap (C_n \times X_2) = \phi$, then $\overline{\nu}(B \cap (C_n \times \tilde{X}_2) = 0$; to prove this, one has only to note that the Borel set $B \cap (C_n \times \tilde{X}_2)$ is a subset of $(\tilde{X}_1 \times (\tilde{X}_2 \setminus X_2))$. Now take a Borel B of $(\tilde{X}_1 \times \tilde{X}_2)$ such that $B \subset (\tilde{X}_1 \times \tilde{X}_2) \setminus (X_1 \times X_2)$. This means $B \cap (C_n \times X_2) = \phi$, for all n and so $\overline{\nu}(B \cap (C_n \times \tilde{X}_2) = 0$, for all n. This implies that $\overline{\nu}(B) = 0$.

Let X be the Stone-Cech compactification of $(X_1 \times X_2)$. This means $\tilde{\nu} \in M^+(X)$. Let $\phi : X \to (\tilde{X}_1 \times \tilde{X}_2)$ be the unique continuous extension of the identity mapping $(X_1 \times X_2) \to (\tilde{X}_1 \times \tilde{X}_2)$; ϕ maps $X \setminus (X_1 \times X_2)$ onto $(\tilde{X}_1 \times \tilde{X}_2) \setminus (X_1 \times X_2)$. It is easily verified that for any $f \in C(\tilde{X}_1 \times \tilde{X}_2)$, $\overline{\nu}(f) = \tilde{\nu}(f \circ \phi)$. By regularity, we get $\overline{\nu}(K) = \tilde{\nu}(\phi^{-1}(K))$, for any compact $K \subset (\tilde{X}_1 \times \tilde{X}_2)$. Take a compact $C \subset X \setminus (X_1 \times X_2)$. Then $C_1 = \phi^{-1}(\phi(C))$ is compact and contains C, and $\phi(C)$ is disjoint from $(X_1 \times X_2)$. Now $\tilde{\nu}(C) \leq \tilde{\nu}(C_1) = \overline{\nu}(\phi(C)) = 0$. This means $\nu \in M_\tau^+(X_1 \times X_2)$. This proves the result.

Remark 3 : ([3], Theorem 1, p. 160) holds in the following form:

Corollary 4 — Let X_1, X_2 be completely regular Hausdorff spaces, $\mu_1 \in M_t^+(X_1)$, $\mu_2 \in M_\tau^+(X_2)$ and a net $\{\nu_\alpha\} \subset M_\tau^+(X_1 \times X_2)$ such that $\nu_\alpha^{(i)} \to \mu_i$, for $i = 1, 2$. Then there exist a subnet of $\{\nu_\alpha\}$ which converges to some $\nu \in M_\tau^+(X_1 \times X_2)$ with $\nu^{(i)} = \mu_i$, for $i = 1, 2$.

PROOF : A subnet of $\{v_\alpha\}$ converges to some $v \in M^+(X_1 \times X_2)$ such that for every $f \in C_b(X_1)$, $v(f \otimes 1) = \mu_1(f)$ and for every $f \in C_b(X_2)$, $v(1 \otimes f) = \mu_2(f)$. By Theorem 2, the result follows.

Now the Strassen's theorem for τ-smooth measures can be put in the form:

***Theorem* 5** — *Let X_1, X_2 be completely regular Hausdorff spaces, $\mu_1 \in M_t^+(X_1)$, $\mu_2 \in M_t^+(X_2)$ and Q a uniformly bounded, convex and closed subset of $M_\tau^+(X_1 \times X_2)$. Then there exists a $\lambda \in Q$ such that $\lambda^{(i)} = \mu_i$ $(i = 1, 2)$, iff for any $\{f_i\} \subset C_b(X_i)$ $(i = 1, 2)$, $\mu_1(f_1) + \mu_2(f_2) \leq sup\ \{v(f_1 \otimes 1 + \otimes f_2) : v \in Q\}$.*

PROOF : The condition is trivially necessary. We take the topology $\sigma(M(X_1 \times X_2), C_b(X_1 \times X_2))$ on $M(X_1 \times X_2)$ and the topology $\sigma(M(X_i, C_b(X_i))$ on $M(X_i)$ $(i = 1, 2)$. $\overline{Q}$, the closure of Q, is a compact and convex subset of $M^+(X_1 \times X_2)$. For a $v \in \overline{Q}$, define $v^{(1)} \in M^+(X_1)$, $v^{(1)}(f) = v(f \otimes 1)$ for every $f \in C_b(X_1)$; $v^{(2)} \in M^+(X_2)$ is defined in a similar way. Now $Q_0 = \{(v^{(1)}, v^{(2)}) : v \in \overline{Q}\}$ is a compact convex subset of $M(X_1) \times M(X_2)$, with product topology. We claim that $(\mu_1, \mu_2) \in Q_0$; if not, by separation theorem ([6], 9.2, p. 65), there are, for $i = 1, 2$, $f_i \in C_b(X_i)$ such that

$$\mu_1(f_1) + \mu_2(f_2) > \sup\left\{ v^{(1)}(f_1) + v^{(2)}(f_2) : (v^{(1)}, v^{(2)}) \in Q_0 \right\}.$$

Now the right hand side of this inequality is $\geq \sup\{v(f_1 \otimes 1 + \otimes f_2) : v \in Q\}$. This contradicts the given hypothesis. Thus $(\mu_1, \mu_2) \in Q_0$. So there is a $\lambda \in \overline{Q}$ such that $(\mu_1, \mu_2) = (\lambda^{(1)}, \lambda^{(2)})$. By Theorem 2, $\lambda \in M_\tau^+(X_1 \times X_2)$. Since Q is closed in $M_\tau^+(X_1 \times X_2)$, $\lambda \in Q$.

Remark 6 : In ([3], Theorem 2, p. 164), one of the marginals should be tight.

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