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L_p-DUAL MIXED QUERMASSINTEGRALS*

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Associated with the notions of L_p-harmonic radial combination and dual quermassintegrals, we give the "dual" of L_p-mixed quermassintegrals — the notion of L_p-dual mixed quermassintegrals, the L_p-dual mixed volume is its special case. Further, we prove an integral representation and inequalities for L_p-dual mixed quermassintegrals.

Key Words : L_p-harmonic radial combination, dual quermassintegrals, L_p-mixed quermassintegrals, L_p-dual mixed quermassintegrals, L_p-dual mixed volume.

1. INTRODUCTION

In (5) Lutwak posed the notion of dual mixed volume and defined the dual quermassintegrals, he established the dual Brunn-Minkowski theory. Equally, the Brunn-Minkowski-Firey theory are established by Lutwak. In (6) Lutwak showed Fiery L_p-combination (see (3)) lead to the Brunn-Minkowski theory for $p \geq 1$. He introduced the notions of L_p-mixed quermassintegrals and L_p-mixed surface area measure (L_p-mixed volume and L_p-surface area measure are these special cases, respectively) and given an integral representation and inequalities for L_p-mixed quermassintegrals (including L_p-mixed volume).

The L_p-harmonic radial combination were instigated from convex bodies to star bodies by Firey (see (1, 2)) and Lutwak (see (7)) respectively. Associated with L_p-harmonic radial combiantion, Lutwak in (7) showed the notion, integral representation and inequalities for L_p-dual mixed volume.

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The aim of this article is that we extend the notion of L_p-dual mixed volume, and show the dual of L_p-mixed quermassintegrals — the L_p-dual mixed quermassintegrals by combining with the dual quermassintegrals and L_p-harmonic radial combination.

The ideas and techniques of Lutwak play a critical role throughout this paper.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors, write $\mathcal{K}_o^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of body K.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot): \mathbb{R}^n \to (0, \infty)$, is defined by

$$h(K, x) = \max\{x \cdot y : y \in K\} \qquad x \in \mathbb{R}^n.$$

where $x \cdot y$ denotes the standard inner product of x and y.

For $p \geq 1, K, L \in \mathcal{K}_o^n$ and $\lambda . \mu \geq 0$ (not both zero), the Firey L_p-combination $\lambda . K +_p \mu . L \in \mathcal{K}_o^n$ is defined by

$$h(\lambda . K +_p \mu . L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p,$$

where "$.$" denotes the Firey scalar multiplication. Firey L_p-combination of convex bodies were defined and studied by Firey (see (3)).

Associated with the Firey L_p-combination, Lutwak (see (6)) defined the L_p-mixed quermassintegrals (who called mixed p-Quermassintegrals) as follows: For $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$ and real $p \geq 1$, the L_p-mixed quermassintegrals $W_{p,i}(K, L)$ (i = 0.1, ..., n − 1) is defined

$$\frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \to 0+} \frac{W_i(K +_p \varepsilon . L) - W_i(K)}{\varepsilon}. \qquad \text{... (1.1)}$$

Where $W_i(K)$ denotes the Quermassintegrals of $K \in \mathcal{K}^n$ (its definition see (4.8), in particular,

$$W_0(K) = V(K). \qquad \text{... (1.2)}$$

Obviously, for $i = 0$, L_p-mixed quermassintegrals $W_{p,0}(K, L)$ is just the L_p-mixed volume $V_p(K, L)$ by (1.1) and (1.2), namely

$$W_{p,0}(K, L) = V_p(K, L). \qquad \text{... (1.3)}$$

Further, Lutwak in (6) proved the following an analog of the Minkowski inequality for L_p-mixed quermassintegrals:

Theorem A — *If* $K . L \in \mathcal{K}_o^n$, *and* $p > 1$, $0 \le i < n$, *then*

$$W_{p,i}(K, L)^{n-1} \ge W_i(K)^{n-i-p} W_i(L)^p, \qquad \text{... (1.4)}$$

with equality if and only if K and L are dilates.

From (1.4), he established an analog of the Brunn-Minkowski inequality for quermassintegrals.

Theorem B — *If* $K . L \in \mathcal{K}_o^n$, *and* $p > 1$, $0 \le i < n$, *then*

$$W_i(K +_p L)^{p/n-2} \ge W_i(K)^{p/(n-i)} W_i(L)^{p/(n-i)}. \qquad \text{... (1.5)}$$

with equality if and only if K and L are dilates.

Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. For $K \in \mathcal{S}_o^n$ and any real i, $\widetilde{W}_i(K)$ denote the dual quermassintegrals of K. In this paper, we give the definition of the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K, L)$ of $K . L \in \mathcal{S}_o^n$, and mainly obtain the following results.

Theorem 1 — *If* $K . L \in \mathcal{S}_o^n$, $p \ge 1$, *and real* $i < n$ *or* $i > n + p$, *then*

$$\widetilde{W}_{-p,i}(K, L)^{n-i} \ge \widetilde{W}_i(K)^{n+p-i} \widetilde{W}_i(L)^{-p}, \qquad \text{... (1.6)}$$

with equality if and only if K and L are dilates. For $n < i < n+p$, *inequality (1.6) is reversed.*

Theorem 1 is just an analog of the Minkowski inequality for L_p-dual mixed quermassintegrals.

From Theorem 1, the analog of the Brunn-Minkowski inequality for dual quermassintegrals is obtained:

Theorem 2 — *If* $K, L \in \mathcal{S}_o^n$, $\lambda, \mu > 0, p \ge 1$, *and real* $i < n$ *or* $i > n+p$, *then*

$$\widetilde{W}_i(\lambda \cdot K +_p \mu \cdot L)^{-p/(n-i)} \ge \lambda \widetilde{W}_i(K)^{-p/(n-i)} + \mu \widetilde{W}_i(L)^{-p/(n-i)}, \qquad \text{... (1.7)}$$

with equality if and only if K and L are dilates. Inequality (1.7) is reversed for $n < i < n+p$.

In particular, taking $\lambda=\mu=1$ in Theorem 2, we have that

Corollary 3 — If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and real $i < n$ or $i > n+p$, then

$$\widetilde{W}_i(K +_{-p} L)^{-p/(n-i)} \geq \lambda \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)}, \quad \ldots (1.8)$$

with equality and only if K and L are dilates. Inequality (1.8) is reversed for $n<i<n+p$.

Using Theorem 2 and Corollary 3, we prove an isolate form of inequality (1.8).

Theorem 3 — *If* $K, L \in \mathcal{S}_o^n$, $\alpha \in [0, 1]$, $p \geq 1$ *and real number* $i < n$ *or* $i > n+p$, *then*

$$\widetilde{W}_i(K +_{-p} L)^{-p/(n-i)}$$

$$\widetilde{W}_i(\alpha \cdot K +_{-p} (1-\alpha) \cdot L)^{-p/(n-i)} + \widetilde{W}_i((1-\alpha) \cdot K +_{-p} \alpha \cdot L)^{-p/(n-i)}$$

$$\geq \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)}, \quad \ldots (1.9)$$

with equality if and only if K *and* L *are dilates. For* $n<i<n+p$, *inequality (1.9) is reversed.*

2. L_p-Dual Mixed Quermassintegrals

In the section, we shall give the notion and integral representaiton of L_p-dual mixed quermassintegrals.

Let K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its raidal function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$, is defined by

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}. \qquad x \in \mathbb{R}^n \setminus \{0\}$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in \mathcal{S}^{n-1}$.

For $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu > 0$ (not both zero), then for $p \geq 1$, the L_p-harmonic radial combination $\lambda \cdot K +_{-p} \mu \cdot L \in \mathcal{S}_o^n$ is defined by (see (7))

$$\rho(\lambda \cdot K +_{-p} \mu \cdot L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \quad \ldots (2.1)$$

For convex bodies, the L_p-harmonic radial combiantion were investigated by Firey (see (1, 2)). Associated with (2.1), Lutwak (see (7)) defined the L_p-dual mixed volume as follows:

For, $K, L \in S_o^n$ in $\mathbb{R}^n$, $p \geq 1$ and $\varepsilon > 0$, the L_p-dual mixed volume, $V_{-p}(K, L)$, of the K and L is defined by

$$\frac{n}{-p} V_{-p}(K . L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \cdot L) - V(K)}{\varepsilon}. \quad \text{... (2.2)}$$

In order to define the L_p-dual mixed quermassintegrals, we shall require the notion of dual quermassintegrals. The dual quermassintegrals are defined base on the dual mixed volumes. Lutwak in (5) defined the dual mixed volumes as follows :

For $K_1, K_2, ..., K_n \in S_o^n$. $1 \leq i \leq n$, the dual mixed volumes $\tilde{V}(K_1, K_2, ..., K_n)$ are defined by

$$\tilde{V}(K_1, K_2, ..., K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \rho_{K_2}(u) \ldots \rho_{K_2}(u)\, du.$$

Let $K_1 = \ldots = K_i = K . K_{i=1} = \ldots = K_1 = B$, where B denote the standard unit ball B in $\mathbb{R}^n$ and allow i is any real number, then the definition of dual quermassintegrals are stated that (see (4, 8)).

For $K \in S_o^n$ and any real i, the dual quermassintegrals. $\tilde{W}_i(K)$, of K are defined by

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i}\, dS(u). \quad \text{... (2.3)}$$

Obviously,

$$\tilde{W}_0(K) = V(K). \quad \text{... (2.4)}$$

For $K, L \in S_o^n$, $\varepsilon > 0$, $p \geq 1$ and real $i = n$, the L_p-dual mixed quermassintegrals, $\tilde{W}_{-p,i}(K, L)$, of K and L are defined by

$$\frac{n-i}{-p} \tilde{W}_{-p,i}(K . L) = \lim_{\varepsilon \to 0^+} \frac{\tilde{W}_i(K +_{-p} \varepsilon \cdot L) - \tilde{W}_i(K)}{\varepsilon}. \quad \text{... (2.5)}$$

If $i = 0$, and using (2.3), we easily see that definition (2.5) is just definition (2.2) of L_p-dual mixed volume. Namely

$$\widetilde{W}_{-p,0}(K, L) = V_{-p}(K, L). \qquad \text{... (2.6)}$$

From this, the L_p-dual mixed quermasintegrals is the extension of L_p-dual mixed volume.

From definition (2.5), we give the integral representationof the L_p-dual mixed quermassintegrals as follows:

Proposition 1 — If $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and real $i \neq n . n+p$, then

$$\widetilde{W}_{-p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_L^{n+p-i}(u)\, \rho_L^{-p}(u)\, dS(u). \qquad \text{... (2.7)}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

PROOF : According to definition (2.3) and (2.1), for $i \neq n$, we have that

$$\lim_{\varepsilon \to 0^+} \frac{\widetilde{W}_i(K +_{-p} \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{\rho(K +_{-p} \varepsilon \cdot L, u)^{n-i} - \rho(K, u)^{n-i}}{\varepsilon} dS(u)$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{n} \int_{S^{n-1}} \frac{[\rho(K, u)^{-p} + \varepsilon \rho(L, u)^{-p}]^{-n-i/p} - \rho(K, u)^{n-i}}{\varepsilon} dS(u)$$

since

$$\lim_{\varepsilon \to 0^+} \frac{[\rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}]^{-n-1/p} - \rho(K, \cdot)^{n-i}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \rho_K^{n-i} \frac{[1 + \varepsilon(\rho_K/\rho_L)^p]^{-n-i/p} - 1}{\varepsilon}$$

$$= -\frac{n-i}{p} \rho_K^{n+p-i} \rho_L^{-p} .$$

thus using definition (2.5), we get formula (2.7).

For $i = n+p$, above the process of proof is right, but according to (2.7), we know

$$\widetilde{W}_{-p . n+p}(K . L) = \widetilde{W}_{n+p}(L). \qquad \text{... (2.8)}$$

for any $K \in \mathcal{S}_o^n$. Therefore, $i \neq n+p$ in formula (2.7). ■

Together with (2.3), (2.7) and (2.8), we get that for $K \in \mathcal{S}_o^n$, $p \geq 1$, and $i \neq n$, then

$$\widetilde{W}_{-p,i}(K.K) = \widetilde{W}_i(K). \qquad \text{... (2.9)}$$

Taking $i = 0$ in Proposition 1, we have that

Corollary 1 — If $K, L \in \mathcal{S}_o^n$, then for $p \geq 1$

$$V_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u)\, \rho_L^{-p}(u)\, dS(u).$$

It is just integral representation of the L_p-dual mixed volume (see Lutwak (7)).

Let $SO(n)$ denote the group of rotation transformations about the origin. If $\phi \in SO(n)$, then obviously $\rho_{\phi K}(x) = \rho_K(\phi^{-1} x)$. From this and definition (2.1), it follows immediately that for $K.L \in \mathcal{S}_o^n$, $p \geq 1$, $\phi \in SO(n)$ and $\varepsilon > 0$.

$$\phi(K +_{-p} \varepsilon \cdot L) = \phi K +_{-p} \varepsilon \cdot \phi L.$$

This, together with the facts that $\phi B = B$, $\widetilde{W}_i(\phi Q) = \widetilde{W}_i(Q)$, for all $Q \in \mathcal{S}_o^n$, and using the definition (2.5) of $\widetilde{W}_{-p,i}$, immediately yields.

Proposition 2 — If $K, L \in \mathcal{S}_o^n$, $\phi \in SO(n)$, $p \geq 1$, and real $i \neq n + p$, then

$$\widetilde{W}_{-p,i}(\phi K, \phi L) = \widetilde{W}_{-p,i}(K, L).$$

3. The Proof of Theorems

Proof of Theorem 1.

For real $i < n$, from (2.7) and Hölder inequality, we have that

$$\widetilde{W}_{-p,i}(K, L)^{\frac{n-i}{n+p-i}}\, \widetilde{W}_i(L)^{\frac{p}{n+p-1}}$$

$$= \left[\frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p-i}(u)\, \rho_L^{-p}(u)\, dS(u) \right]^{\frac{n-i}{n-p-i}}$$

$$\left[\frac{1}{n}\int_{S^{n-1}}\rho_L^{n-i}(u)\,dS(u)\right]^{\frac{p}{n+p-i}}$$

$$=\left[\frac{1}{n}\int_{S^{n-1}}\left(\rho_K^{n-i}(u)\,\rho_L^{\frac{-p(n-i)}{n+p-i}}(u)\right)dS(u)\right]^{\frac{n-i}{n-p-i}}$$

$$\left[\frac{1}{n}\int_{S^{n-1}}\left(\rho_L^{\frac{p(n-i)}{n+p-i}}(u)\right)^{\frac{n+p-i}{p}}dS(u)\right]^{\frac{p}{n-p-i}}$$

$$\geq\frac{1}{n}\int_{S^{n-1}}\rho_K^{n-i}(u)\,dS(u)$$

$$=\widetilde{W}_i(K),$$

from this, we obtain inequality (1.6) when $i<n$. According to the condition of equality holds for Hòlder inequality, we know the equality holds if and only if K and L are dilaties in inequality (1.6).

Similarly, we can prove for $i>n+p$, inequality (1.6) is true; for $n<i<n+p$, inequality (1.6) is reversed. ■

For $i = 0$, we have that by Theorem 1:

Corollary 2 — If $K, L \in S_o^n$, then for $p\geq 1$

$$V_{-p}(K,L)^n \geq V(K)^{n+p}\,V(L)^{-p},$$

with equality if and only if K and L are dilates.

Corollary 2 is just an analog of Minkowski inequality for L_p-dual mixed volume (see (7)).

The proof of Theorem 2.

For $i < n$ or $i > n + p$, according to definition (2.1), formula (2.7), and using inequality (1.6), we know that for any $Q \in S_o^n$

$$\widetilde{W}_{-p,i}(Q,\lambda\cdot K -_{-p}\mu\cdot L)$$

$$=\lambda\,\widetilde{W}_{-p,i}(Q,K)+\mu\,\widetilde{W}_{-p,i}(Q,L)$$

$$\geq \widetilde{W}_i(Q)^{(n+p-i)/(n-i)}\left[\lambda\,\widetilde{W}_i(K)^{-p/(n-i)}+(\mu\,\widetilde{W}_i L)^{-p/(n-i)}\right],$$

with equality if and only if K and L are dilates. Taking $Q=\lambda\cdot K+_{-p}\mu\cdot L$ and using (2.9), the result is desired inequality (1.7).

Similar the above way, we easily prove inequality (1.7) is reversed when $n < i < n + p$. ■

Taking $\lambda=\mu=1$ in Theorem 2, we immediately yield Corollary 3.

The case $i = 0$ of inequality (1.7) is established by Lutwak (see (7)); namely

Corollary 4 — If $K, L \in \mathcal{S}_o^n$, $\lambda, \mu>0$ and $p\geq 1$, then

$$V(\lambda\cdot K+_{-p}\mu\cdot L)^{-p/n} \geq \lambda V(K)^{-p/n} + \mu V(L)^{-p/n},$$

with equality if and only if K and L are dilates.

The proof of Theorem 3.

Let $M=\alpha\cdot K+_{-p}(1-\alpha)\cdot L, N=(1-\alpha)\cdot K+_{-p}\alpha\cdot L$. Since $K, L \in \mathcal{S}_o^n$, then $M, N \in \mathcal{S}_o^n$ and

$$\widetilde{W}_i(K+_{-p}L) = \frac{1}{n}\int_{S^{n-1}} \rho(K+_{-p}L, u)^{n-i}\, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} [\rho(K+_{-p}L, u)^{-p}]^{-\frac{n-i}{p}}\, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} [\rho(K\,.\,u)^{-p}+\rho(L\,.\,u)^{-p}]^{-\frac{n-i}{p}}\, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} [(\alpha\,\rho(K\,.\,u)^{-p}+(1-\alpha)\,\rho(L, u)^{-p})$$

$$+((1-\alpha)\,\rho(K\,.\,u)^{-p}+\alpha\,\rho L, u)^{-p})]^{-\frac{n-i}{p}}\, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} [\rho(\alpha\cdot K+_{-p}(1-\alpha)\cdot L, u)^{-p}$$

$$+\rho[(1-\alpha)\cdot K+_{-p}\alpha\cdot L\,.\,u)^{-p}]^{-\frac{n-i}{p}}\, dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} [\rho(M\,.\,u)^{-p} + \rho(N\,.\,u)^{-p}]^{-\frac{n-i}{p}}\, dS(u)$$

$$= \frac{1}{n} \int_{S^{n-1}} [\rho(M +_{-p} N . u)^{-p}]^{-\frac{n-i}{p}} dS(u)$$

$$= \frac{1}{n} \int_{S^{n-1}} \rho(M +_{-p} N . u)^{n-i} dS(u)$$

$$= \widetilde{W}_i(M +_{-p} N).$$

Using inequality (1.8) of Corollary 3, we get that for $i < n$ or $i > n + p$.

$$\widetilde{W}_i(K +_{-p} L)^{-p/(n-i)} = \widetilde{W}_i(M +_{-p} N)^{-p/(n-i)}$$

$$\geq \widetilde{W}_i(M)^{-p/(n-i)} + \widetilde{W}_i(N)^{-p/(n-i)}, \quad \text{... (3.1)}$$

which equality if and only if M and N are dilates. Inequality (3.1) is just first inequality of (1.9).

Because M and N are dilates if and only if $\rho_M^{-p}(u) = c\rho_N^{-p}(u)$ $(c > 0)$ for any $u \in S^{n-1}$, together with (2.1), we have

$$\alpha \rho_K^{-p}(u) + (1 - \alpha \rho_L^{-p}(u)) = c\,[(1+\alpha)\,\rho_K^{-p}(u) - \alpha \rho_L^{-p}(u)], \qquad u \in S^{n-1}$$

namely

$$(\alpha + c\alpha - c)\,\rho_K^{-p}(u) = (c\alpha + \alpha - 1)\,\rho_L^{-p}(u), \qquad u \in S^{n-1}$$

Hence M and N are dilates if and only if K and L are dilates. From this, we know the equality holds of first inequality of (1.9) if and only if K and L are dilates.

But according to inequality (1.7) of Theorem 2 for real $i < n$ or $i > n + p$

$$\widetilde{W}_i(M)^{-p/(n-i)} + \widetilde{W}_i(\alpha \cdot K +_{-p} (1-\alpha) \cdot L)^{-p/(n-i)}$$

$$\geq \alpha \widetilde{W}_i(K)^{-p/(n-i)} + (1-\alpha)\,\widetilde{W}_i(L)^{-p/(n-i)}, \quad \text{... (3.2)}$$

and

$$\widetilde{W}_i(N)^{-p/(n-i)} \geq (1-\alpha)\,\widetilde{W}(K)^{-p/(n-i)} + \alpha \widetilde{W}_i(L)^{-p/(n-i)}, \quad \text{... (3.3)}$$

with equality if and only if K and L are dilates in inequality (3.2) and (3.3).

Associated with inequality (3.2) and (3.3), we obtain that

$$\widetilde{W}_i(M)^{-p/(n-i)} + \widetilde{W}_i(N)^{-p/(n-i)},$$

$$\geq \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)}, \qquad \ldots (3.4)$$

with equality if and only if K and L are dilates. Inequality (3.4) is just second inequality of (1.9).

Similar with the above way, we easily prove when $n < i < n+p$, inequality (1.9) is reversed. ■

Finally, as the application of Theorem 1, we give the following result.

***Theorem* 4** — *If* $K.L \in S_o^n$, $p \geq 1$, *real number* $i \neq n, n+p$ *and for all* $Q \in S_o^n$,

$$\widetilde{W}_{-p,i}(Q, K) = \widetilde{W}_{-p,i}(Q, L) \qquad \ldots (3.5)$$

or

$$\widetilde{W}_{-p,i}(K, Q) = \widetilde{W}_{-p,i}(L, Q) \qquad \ldots (3.6)$$

then $K = L$.

PROOF : When $i < n$ (or $i > n+p$), taking $Q = K$ in (3.5), and using inequality (1.6) of Theorem 1 and equality (2.9), we have that $\widetilde{W}_i(L) \geq \widetilde{W}_i(K)$ (or $\widetilde{W}_i(L) \leq \widetilde{W}_i(K)$), with equality if and only if K and L are dilates: let $Q = L$ in (3.5), and get $\widetilde{W}_i(K) \geq \widetilde{W}_i(L)$ (or $\widetilde{W}_i(K) \leq \widetilde{W}_i(L)$), with equality if and only if K and L are dilates. Hence $\widetilde{W}_i(K) = \widetilde{W}_i(L)$, and K and L must be dilates. Therefore, $K = L$.

Similar above the way of proof, the result is true when $n < i < n+p$,
Analogously, from (3.6), the result can be proven. ■

Remark : Comparing with the outlines of definition (1.1) and (2.5), inequality (1.4) and (1.6), inequality (1.5) and (1.8), the $\widetilde{W}_{-p,i}$ be called the "dual" of $W_{p,i}$ is suitable, their mainly different are the domain of number i.

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