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THE PRODUCT OF PSEUDO-DIFFERENTIAL OPERATORS ASSOCIATED WITH BESSEL OPERATORS

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The class H^m of symbol 'a' and pseudo-differential operator $h_{\mu,a}$ associated with the Bessel operator $S_\mu = \frac{d^2}{dx^2} + \frac{1-4\mu^2}{4x^2}$ are defined. It is shown that the product of two symbols is a symbol. The product $h_{\mu,a}\, h_{\mu,b}$ of two pseudo-differential operators $h_{\mu,a}$ and $h_{\mu,b}$ associated with symbols $a(x,\xi) \in H^m$ and $b(y,\eta) \in H^n$ is defined. It is proved that $h_{\mu,a}\, h_{\mu,b}$ is a continuous linear mapping of the Zemanian space H_μ into itself. It is shown that the Hankel transform of $h_{\mu,a}\, h_{\mu,b}(u)$ satisfies a certain L^1-norm inequality.

Key Words : Hankel Transform; Pseudo-Differential Operator; Hankel Convolution, Bessel Operator

1. INTRODUCTION

The theory of Hankel transformation of distributions developed by Zemanian (10) has been exploited by Pathak and Pandey (3, 4) in the study of a class of pseudo-differential operators associated with Bessel operators in the development of a theory of Sobolev type spaces and in investigating regularity of non-homogeneous Bessel differential operator equations. Another class of pseudo-differential operators has been studied in (6).

It is well-known that the product $A(x,D)\,B(x,D)$ of two classical pseudo-differential operators $A(x,D)$ and $B(x,D)$ associated with the symbols $a(x,\xi) \in S^{m_1}$ and $b(x,\xi) \in S^{m_2}$, respectively is the pseudo-differential operator $C(x,D)$ associated with the symbol $c(x,\xi) \in S^{m_1+m_2}$ possessing the asymptotic expansion

$$c(x,\xi) \sim \sum_{\mu} \frac{(-i)^{|\mu|}}{\mu!} \left(\partial_\xi^\mu a\right)\left(\partial_x^\mu b\right).$$

Its theory has been given in (7) using Fourier transformation of distributions. An analogous theory for the product $h_{\mu,a}\, h_{\mu,b}$ is developed in this paper. Our approach involves direct computation of the product of pseudo-differential operators.

The Hankel transformation h_μ of $\phi \in L^1(0,\infty)$ is defined by

$$(h_\mu \phi)(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\,\phi(y)\,dy, \qquad \mu \geq -\frac{1}{2}. \qquad \text{... (1.1)}$$

Here J_μ denotes the Bessel function of first kind of order μ. It is given by

$$J_\mu(x) = \sum_{r=0}^\infty \frac{(-1)^r \left(\frac{x}{2}\right)^{\mu+2r}}{r\,(\mu+r+1)} \qquad \text{... (1.2)}$$

The following asymptotic properties of $J_\mu(x)$ are useful in our investigations.

$$J_\mu(x) = 0\,(x^\mu), \qquad x \to 0$$

$$= \sqrt{\frac{2}{x}}\left[\cos\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right)\right] + 0\,(x)^1, \qquad x \to \infty.$$

Zemanian (10) has studied the Hankel transformation given by (1.1). The Zemanian space $H_\mu(I)$ is known to consist of all complex valued C^∞-functions ϕ on $I = (0,\infty)$ such that

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in I}\left| x^m \left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-\frac{1}{2}}\phi(x)\right| < \infty, \qquad \forall\, m,k \in N_0. \qquad \text{... (1.3)}$$

We shall use notation and terminology of (2), (6) and (10). The differential operators N_μ, M_μ and S_μ are defined by

$$N_\mu = N_{\mu,x} = x^{\mu+\frac{1}{2}}\left(\frac{d}{dx}\right)x^{-\mu-\frac{1}{2}} \qquad \text{... (1.4)}$$

$$M_\mu = M_{\mu,x} = x^{-\mu-\frac{1}{2}}\left(\frac{d}{dx}\right)x^{\mu+\frac{1}{2}} \qquad \text{... (1.5)}$$

$$S_\mu = S_{\mu,x} = M_\mu N_\mu = \frac{d^2}{dx^2} + \frac{1-4\mu^2}{4x^2}. \qquad \text{... (1.6)}$$

From (2) we have the following relations for any $\phi \in H_\mu$.

$$h_\mu (S_\mu \phi) = h_\mu (M_\mu N_\mu \phi) = -y^2 h_\mu \phi \qquad \text{... (1.7)}$$

$$\left(x^{-1} \frac{d}{dx} \right)^k \left(x^{-\mu - \frac{1}{2}} \theta \phi \right)$$

$$= \sum_{\nu=0}^{k} \binom{k}{\nu} \left(x^{-1} \frac{d}{dx} \right)^\nu \theta \left(x^{-1} \frac{d}{dx} \right)^{k-\nu} \left(x^{-\mu - \frac{1}{2}} \phi \right) \qquad \text{... (1.8)}$$

$$S^r_{\mu, x} \phi(x) = \sum_{j=0}^{r} b_j \, x^{2j + \mu + \frac{1}{2}} \left(x^{-1} \frac{d}{dx} \right)^{r+j} \left(x^{-\mu - \frac{1}{2}} \phi(x) \right) \qquad \text{... (1.9)}$$

where b_j are constants depending only on μ.

Following formula is useful in our investigation:

$$\left(x^{-1} \frac{d}{dx} \right)^m x^{-\mu} J_\mu (x\eta) = (-1)^m \eta^m x^{-\mu - m} J_{\mu + m} (x \eta) \qquad \text{... (1.10)}$$

In the present work we assume that the symbol $a \in H^m, b \in H^n$, where H^m is a symbol class defined as below.

Definition 1.1 — Let I denote $(0, \infty)$. The function $a(x, \xi) : C^\infty (I \times I) \to \mathbb{C}$ belongs to the class H^m if and only if $\forall q, \nu, \alpha \in N_0$, there exists a constant $D_{\alpha, \nu, m, q} > 0$ such that

$$(1 + x)^q \left| \left(x^{-1} \frac{d}{dx} \right)^\nu \left(\xi^{-1} \frac{d}{d\xi} \right)^\alpha a(x, \xi) \right| \le D_{\alpha, \nu, m, q} (1 + \xi)^{m - \alpha} \qquad \text{... (1.11)}$$

where m is a fixed real number. If $a(x, \xi)$ satisfies the inequality (1.11) with $q = 0$ then the symbol class will be denoted by H^m_0.

Definition 1.2 — Let $a(x, y)$ be a complex valued function belonging to the $C^\infty (I \times I)$, where $I = (0, \infty)$. Then the pseudo-differential operator $h_{\mu, a}$ associated with the symbol $a(x, y)$ is defined by

$$(h_{\mu, a} u)(x) = \int_0^\infty (xy)^{1/2} J_\mu (xy) \, a(x, y) \, U_\mu (y) \, dy, \qquad u \in H_\mu. \qquad \text{... (1.12)}$$

where

$$U_\mu(y) = (h_\mu u)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)\, u(x)\, dx. \quad \text{... (1.13)}$$

The product $h_{\mu,a} h_{\mu,b}$ is a continuous linear map of the Zemanian space H_μ into itself and which satisfies a certain L^1-norm inequality.

2. The Hankel Convolution

We recall the following results on HankeJ convolution needed in the sequel. Let $\Delta(x, y, z)$ be the area of a triangle with sides x, y, z if such a triangle exists. For $\gamma > 0$, set

$$D(x, y, z) = 2^{3\gamma - 5/2} (\pi)^{-1/2} \left[\Gamma\left(\gamma + \frac{1}{2}\right)\right]^2 (\Gamma(\gamma))^{-1} (x, y, z)^{-2\gamma+1} [\Delta(x, y, z)]^{2\gamma-2} \quad \text{... (2.1)}$$

if Δ exists and zero otherwise. We note that $D(x, y, z) \geq 0$ and that $D(x, y, z)$ is symmetric in x, y, z. We have

$$\int_0^\infty j(zt)\, D(x, y, z)\, d\sigma(z) = j(xt)\, j(yt) \quad \text{... (2.2)}$$

where

$$d\sigma(z) = \left[2^{\gamma - \frac{1}{2}} \Gamma\left(\gamma + \frac{1}{2}\right)\right]^{-1} z^{2\gamma}\, dz, \qquad \gamma = \mu + \frac{1}{2} \quad \text{... (2.3)}$$

and

$$j(x) = 2^{\gamma - \frac{1}{2}} \Gamma\left(\gamma + \frac{1}{2}\right) x^{\frac{1}{2} - \gamma} J_{\gamma - \frac{1}{2}}(x) \quad \text{... (2.4)}$$

We also have

$$\int_0^\infty D(x, y, z)\, d\sigma(z) = 1. \quad \text{... (2.5)}$$

Next, we define $L^1_\sigma(0, \infty)$ as the space of all real measurable functions on $(0, \infty)$ satisfying

$$\|f\|_1 = \int_0^\infty |f(x)|\, d\sigma(x) < \infty.$$

Lemma 2.1 — Let $f \in L^1_\sigma(0, \infty)$, then the associated function $f^*(x, y)$ is defined by

$$f^*(x, y) = \int_0^\infty f(z)\, D(x, y, z)\, d\sigma(z), \quad 0 < x, y < \infty. \qquad \text{... (2.6)}$$

Let $f \in L^1_\sigma(0, \infty)$ and $g \in L^1_\sigma(0, \infty)$ and the Hankel convolution of f and g be defined by

$$f \# g(x) = \int_0^\infty f^*(x, y)\, g(y)\, d\sigma(y). \qquad \text{... (2.7)}$$

Then the integral defining $f \# g(x)$ converges for all x, $0 < x < \infty$, and

$$\|f \# g\|_1 \le \|f\|_1 \, \|g\|_1, \qquad \text{... (2.8)}$$

and $(f \# g)(x) = (g \# f)(x)$ almost everywhere.

3. Product of Symbols

Proposition 3.1 — The product of two symbols is a symbol.

Proof: Let $a(x, \xi) \in H^m$ and $b(x, \xi) \in H^n$, then using Leibnitz type differentiation formula (1.8) and inequality (1.11) we get

$$(1+x)^q \left| \left(x^{-1}\frac{d}{dx}\right)^{\nu} \left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha} \{a(x, \xi)\, b(x, \xi)\} \right|$$

$$\le \sum_{s=0}^{\alpha} \binom{\nu}{s} \sum_{r=0}^{\alpha} \binom{\nu}{r} (1+x)^q \left| \left(\xi^{-1}\frac{d}{d\xi}\right)^{\alpha - s} \left(x^{-1}\frac{d}{dx}\right)^{\nu - s} a(x, \xi) \right|$$

$$\times \left| \left(\xi^{-1}\frac{d}{d\xi}\right)^{s} \left(x^{-1}\frac{d}{dx}\right)^{r} b(x, \xi) \right|$$

$$\le \sum_{s=0}^{\alpha} \binom{\alpha}{s} \sum_{r=0}^{\nu} \binom{\nu}{r} D_{\alpha - s, \nu - r, m, q} (1+\xi)^{m - \alpha + s} E_{s, r, n, 0} (1+\xi)^{n - s}$$

$$\le E_{\alpha, \nu, m, n, q} (1+\xi)^{m + n - \alpha}, \qquad \text{... (3.1)}$$

Therefore, $a(x, \xi)\, b(x, \xi) \in H^{m+n}$.

4. Product of Pseudo-Differential Operators

Lemma 4.1 — Let the symbol $b(y, \eta) \in H^n$. Then the function $\phi_\mu(\xi)$ defined by

$$\phi_\mu(\xi) = \int_0^\infty b_\eta(\xi)\, U_\mu(\eta)\, d\eta \qquad \text{... (4.1)}$$

where

$$b_\eta(\xi) = \int_0^\infty (\xi y)^{\frac{1}{2}} J_\mu(\xi y) (y\eta)^{\frac{1}{2}} J_\mu(y\eta)\, b(y, \eta)\, d\eta \qquad \text{... (4.2)}$$

and

$$U_\mu(\eta) = \int_0^\infty (x\eta)^{\frac{1}{2}} J_\mu(x\eta)\, u(x)\, dx, \qquad \text{... (4.3)}$$

satisfies following inequality:

$$\left| \phi_\mu(\xi) \right| \le E_{\mu,n,r,q}\, \xi^{\mu+\frac{1}{2}} (1+\xi^{2r})^{-1}, \qquad \forall r \ge 0. \qquad \text{... (4.4)}$$

Proof: We have

$$\left| \phi_\mu(\xi) \right| \le \int_0^\infty \left| b_\eta(\xi) \right| \left| U_\mu(\eta) \right| d\eta.$$

From ((3), Lemma 3.1, p. 742) it follows that there exists a constant $B_{\mu,n,r,q}$ such that

$$\left| b_\eta(\xi) \right| \le B_{\mu,n,r,q} (1+\eta)^{\mu+n+4r+\frac{1}{2}} (1+\xi)^{\mu+\frac{1}{2}} (1+\xi^{2r})^{-1} \qquad \text{... (4.5)}$$

Using this estimate we obtain

$$\left| \phi_\mu(\xi) \right| \le \int_0^\infty B_{\mu,n,r,q} (1+\eta)^{\mu+n+4r+\frac{1}{2}} \xi^{\mu+\frac{1}{2}} (1+\xi^{2r})^{-1} \left| U_\mu(\eta) \right| d\eta \qquad \forall r \ge 0. \qquad \text{... (4.6)}$$

Since $\left| U_\mu(\eta) \right| \in H_\mu$ we have $\left| U_\mu(\eta) \right| \le C \eta^{\mu+\frac{1}{2}} (1+\eta)^{-l} \quad \forall\, l \in N_0$.

Using this estimate of $U_\mu(\eta)$, the expression (4.6) yields :

$$\left| \phi_\mu(\xi) \right| \leq \int_0^\infty B_{\mu,n,r,q} (1+\eta)^{\mu+n+4r+\frac{1}{2}} \xi^{\mu+\frac{1}{2}}$$

$$(1+\xi^{2r})^{-1} \cdot C(1+\eta)^{-l} \eta^{\mu+\frac{1}{2}} d\eta$$

$$\leq B'_{\mu,n,r,q} \xi^{\mu+\frac{1}{2}} (1+\xi^{2r})^{-1} \int_{t0}^\infty (1+\eta)^{2\mu+n+4r+1-l} d\eta.$$

Since the last integral is convergent for sufficiently large value of l, we have

$$\left| \phi_\mu(\xi) \right| \leq E_{\mu,n,r,q} \xi^{\mu+\frac{1}{2}} (1+\xi^{2r})^{-1}.$$

Definition 4.1 — Let $a(x, \xi)$ be a symbol in class H^m and $b(y, \eta)$ be another symbol belonging to class H^n. Then the product of the two pseudo-differential operators $h_{\mu,a}$ and $h_{\mu,b}$ associated with the symbols $a(x, \xi)$ and $b(y, \eta)$, respectively, is defined by

$$(h_{\mu,a} h_{\mu,b} u)(x) = \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi)\, a(x,\xi)\, h_\mu(h_{\mu,b} u)(\xi)\, d\xi. \qquad \text{... (4.7)}$$

Substituting the value of $h_\mu(h_{\mu,b} u)(\xi)$ (4.7) we get

$$(h_{\mu,a} h_{\mu,b} u)(x) = \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi)\, a(x,\xi)\, d\xi \int_0^\infty (\xi y)^{\frac{1}{2}} J_\mu(\xi y)\, dy$$

$$\times \int_0^\infty (y\eta)^{\frac{1}{2}} J_\mu(y\eta)\, b(y,\eta)\, U_\mu(\eta)\, d\eta$$

$$= \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi)\, a(x,\xi)\, d\xi \int_0^\infty b_\eta(\xi)\, U_\mu(\eta)\, d\eta.$$

Therefore,

$$(h_{\mu,a} h_{\mu,b} u)(x) = \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi)\, a(x,\xi)\, \phi_\mu(\xi)\, d\xi. \qquad \text{... (4.8)}$$

In view of estimate (4.4) we see that the integral on right hand side of (4.8) exists. This satisfies the above definition.

***Theorem* 4.2** — Let the symbol $a(x,\xi) \in H^m$ and $b(y,\eta) \in H^n$. Then the product $h_{\mu,a} h_{\mu,b}$ is a continuous linear mapping of H_μ into itself.

PROOF : Let $u \in H_\mu$. Then we have

$$(h_{\mu,a} h_{\mu,b} u)(x) = \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi)\, a(x,\xi)\, d\xi \int_0^\infty (\xi y)^{\frac{1}{2}} J_\mu(\xi y)\, dy$$

$$\times \int_0^\infty (y\eta)^{\frac{1}{2}} J_\mu(y\eta)\, b(y,\eta)\, U_\mu(\eta)\, d\eta. \quad \text{... (4.9)}$$

Now, $h_{\mu,b}$ is a continuous linear mapping of H_μ into itself by ((3), Theorem 2.3, p. 739). Hence $\phi = (h_{\mu,b} u)(y) \in H_\mu$ by the same theorem, $h_{\mu,a}\phi$ is a continuous linear mapping of H_μ into itself.

Next we give a norm estimate for $h_{\mu,a}$. For which we define the norm

$$\|\cdot\|_{G^s_{\mu,q}} = \left\| \eta^{-s-\mu-\frac{1}{2}} h_\mu(u) \right\|_q,$$

where $u \in H_\mu(I)$, $\mu \geq -\frac{1}{2}$, $s \in \mathbf{R}$ and $1 \leq q < \infty$.

***Theorem* 4.3** — *Let $a(x,\xi) \in H^m$ and $b(y,\eta) \in H^n$ and let $\mu \geq -\frac{1}{2}$.*

Then

$$\|(h_{\mu,a} h_{\mu,b}(u)\|_{G^0_{\mu,1}} \leq E_m \left[\|u\|_{G^0_{\mu,1}} + \|u\|_{G^t_{\mu,1}} \right], \quad \text{... (4.10)}$$

where

$$t \in N,\ t > 2\mu + n + 4r + 1,\ E_m = E_{\mu,n,r,q,tm} \text{ and } r \in N_0.$$

PROOF : From (4.8) it follows that

$$\int_0^\infty (x,\xi)^{\frac{1}{2}} J_\mu(x\xi)(h_{\mu,a} h_{\mu,b} u)(x)\, dx = \int_0^\infty a_\eta(\xi)\, \phi_\mu(\xi)\, d\eta \quad (4.11)$$

where

$$a_\eta(\xi) = \int_0^\infty (x\xi)^{\frac{1}{2}} J_\mu(x\xi) \left[(x\eta)^{\frac{1}{2}} J_\mu(x\eta)\, a(x,\eta) \right] dx.$$

In view of (2.2) this is equivalent to

$$\int_0^\infty (x\,\xi)^{\frac{1}{2}}\, J_\mu(x\xi)\,(h_{\mu,a}\,h_{\mu,b}\,u)(x)\,dx$$

$$= A\cdot\xi^{\mu+\frac{1}{2}}\int_0^\infty \eta^{\mu+\frac{1}{2}}\,\phi_\mu(\eta)\,z^{-\mu-\frac{1}{2}}\,D(\xi,\eta,z)\,d\sigma(z)$$

$$\times\int_0^\infty (zx)^{\frac{1}{2}}\,J_\mu(zx)\,x^{\mu+\frac{1}{2}}\,a(x,\eta)\,dx,$$

where $A = 2^\mu\,\Gamma(\mu+1)^{-2}$. Therefore,

$$h_\mu\,(h_{\mu,a}\,h_{\mu,b}\,u)(\xi) = A\cdot\xi^{\mu+\frac{1}{2}}\int_0^\infty \eta^{\mu+\frac{1}{2}}\,\phi_\mu(\eta)\,d\eta\int_0^\infty z^{-\mu-\frac{1}{2}}\,D(\xi,\eta,z)$$

$$\times\, h_\mu\left(x^{\mu+\frac{1}{2}}\,a(x,\eta)\right)(z)\,d\sigma(z),$$

so that

$$\xi^{-\mu-\frac{1}{2}}\,h_\mu\,(h_{\mu,a}\,h_{\mu,b}\,u)(\xi)$$

$$\le A\cdot\int_0^\infty \left|\,\eta^{\mu+\frac{1}{2}}\,\phi_\mu(\eta)\right|\,d\eta\int_0^\infty z^{-\mu-\frac{1}{2}}\,D(\xi,\eta,z)\left|A_\eta(z)\right|\,d\sigma(z). \qquad \text{... (4.12)}$$

where

$$A_\eta(z) = h_\mu\left(x^{\mu+\frac{1}{2}}\,a(x,\eta)\right)(z).$$

We use the estimate ((3), Lemma 4.1, p. 744):

$$\left|A_\eta(z)\right| \le C_{r,m,q}\,(1+\eta)^m\,z^{\mu+\frac{1}{2}}\,(1+z^{2r})^{-1}, \qquad \forall\, r\in N_0$$

and the obvious inequality

$$(1+\eta)^m \le 2^m\,(1+\eta)^m$$

and we get

$$\left| \xi^{-\mu-\frac{1}{2}} h_\mu (h_{\mu,a} h_{\mu,b} u)(\xi) \right| \leq A \cdot B_{r,m,q} \int_0^\infty \eta^{\mu=\frac{1}{2}} (1+\eta)^m \left| \phi_\mu(\eta) \right| d\eta$$

$$\times \int_0^\infty D(\xi, \eta, z) \, (1+z^{2r})^{-1} \, d\sigma(z),$$

$$\leq 2^{\mu+m} \Gamma(\mu+1) C_{r,m,q} \int_0^\infty \left[\eta^{\mu+\frac{1}{2}} + \eta^{\mu+m+\frac{1}{2}} \right] \eta^{-2\mu-1} \left| \phi_\mu(\eta) \right| d\sigma(\eta)$$

$$\times \int_0^\infty D(\xi, \eta, z) \, (1+z^{2r})^{-1} \, d\sigma(z),$$

$$\leq 2^{\mu+m} \Gamma(\mu+1) C_{r,m,q} \int_0^\infty \left[\eta^{-\mu-\frac{1}{2}} + \eta^{m-\mu-\frac{1}{2}} \right] \left| \phi_\mu(\eta) \right| d\sigma(\eta)$$

$$\times \int_0^\infty (1+z^{2r})^{-1} \, D(\xi, \eta, z) \, d\sigma(z). \qquad \ldots (4.13)$$

Now we take

$$f(z) = (1+z^{2r})^{-1} \in L^1_\mu(0, \infty)$$

and

$$g_p(\eta) = 2^\mu \Gamma(\mu+1) \, \eta^{p-\mu-\frac{1}{2}} \left| \phi_\mu(\eta) \right| \in L^1_\mu(0, \infty), \qquad p = 0, m.$$

Then applying inequality (2.8) to (4.13) we obtain

$$\left| \xi^{-\mu-\frac{1}{2}} h_\mu (h_{\mu,a} h_{\mu,b} u)(\xi) \right| = \left\| f \neq \sum_{p=0,m} g_p(\eta) \right\|_{L^1}$$

$$\leq \left\| (1+z^{2r})^{-1} \right\|_{L^1} \left\| \sum_{p=0,m} g_p(\eta) \right\|_{L^1}$$

$$\leq \left(B \left\| \eta^{-\mu-\frac{1}{2}} \phi_\mu(\eta) \right\|_{L^1} + \left\| \eta^{m-\mu-\frac{1}{2}} \phi_\mu(\eta) \right\|_{L^1} \right). \qquad \ldots (4.14)$$

Now, applying the estimate (4.6) we have

$$\int_0^\infty \left| \xi^{p-\mu-\frac{1}{2}} \phi_\mu(\xi) \right| d\xi \le B_{\mu,n,r,q} \int_0^\infty \xi^p (1+\xi^{2r})^{-1} \, d\xi$$

$$\int_0^\infty \left| (1+\eta)^{2\mu+n+4r+1} \cdot \eta^{-\mu-\frac{1}{2}} U_\mu(\eta) \right| d\eta$$

$$\le C_{p,\mu,n,r,q} \int_0^\infty \left| (1+\eta)^{2\mu+n+4r+1} \; \eta^{-\mu-\frac{1}{2}} U_\mu(\eta) \right| d\eta$$

$$\le C_{p,\mu,n,r,q} \int_0^\infty \left| (1+\eta)^t \; \eta^{-\mu-\frac{1}{2}} U_\mu(\eta) \right| d\eta, \qquad (t > 2\mu+n+4r+1)$$

$$\le C_{p,\mu,n,r,q} \; 2^t \left[\left\| (1+\eta)^t \; \eta^{-\mu-\frac{1}{2}} h_\mu(u)(\eta) \right\|_{L^1} \right]$$

$$\le C'_{p,\mu,n,r,q,t}$$

$$\left[\left\| \eta^{-\mu-\frac{1}{2}} h_\mu(u)(\eta) \right\|_{L^1} + \left[\left\| \eta^{t-\mu-\frac{1}{2}} h_\mu(u)(\eta) \right\|_{L^1} \right] \right]$$

Hence

$$\left\| \eta^{p-\mu-\frac{1}{2}} \phi_\mu(\eta) \right\|_{L^1}$$

$$\le C'_{p,\mu,n,r,q,t} \left(\| u \|_{G^0_{\mu,1}} + \| u \|_{G^t_{\mu,1}} \right) . \qquad \text{... (4.15)}$$

Therefore applying (4.14) to (4.15) we get

$$\left| \xi^{-\mu-\frac{1}{2}} h_\mu (h_{\mu,a} h_{\mu,b} u)(\xi) \right| \le C'_0 \left(\| u \|_{G^0_{\mu,1}} + \| u \|_{G^t_{\mu,1}} \right)$$

$$+ \; C'_m \left(\| u \|_{G^0_{\mu,1}} + \| u \|_{G^t_{\mu,1}} \right) ;$$

$$\| (h_{\mu,a} h_{\mu,b} u)(\xi) \|_{G^0_{\mu,1}} \le E_m \left(\| u \|_{G^0_{\mu,1}} + \| u \|_{G^t_{\mu,1}} \right)$$

where $E_m = E_{\mu, n, r, q, t, m}$.

REFERENCES

1. D. T. Haimo, Integral equations associated with Hankel convolutions, *Trans. Amer. Math. Soc.*, **116** (1965), 330-75.
2. R. S. Pathak., *Integral Transforms of Generalized Functions and their Applications*, Gordon and Breach Science Publishers, Amsterdam, 1997.
3. R. S. Pathak and P. K. Pandey, A class of pseudo-differential operators associated with Bessel operators, *J. Math. Anal. Appl.*, **196** (1995), 736-47.
4. R. S. Pathak and P. K. Pandey, Sobolev type spaces associated with Bessel operators, *J.,Math. Anal. Appl.*, **215** (1997), 95-111.
5. R. S. Pathak and S. K. Upadhyay, Pseudo-differential operators involving Hankel transform, *J. Math. Anal. Appl.*, **213** (1997), 133-47.
6. L. Schwartz, "Theorie des Distributions", Hermann, Paris, 1978.
7. M. W. Wong, *Introduction to pseudo-differential operators*, World Scientific, Singapore, 1991.
8. S. Zaidman, *Distributions and Pseudo-differential Operators*, Longman, Essex, England, 1991.
9. S. Zaidman, On some estimates for pseudo-differential operators. *J. Math. Anal. Appl.*, **39** (1972), 202-07.
10. A. H. Zemanian, *Generalized Integral Transformations*, Interscience, New York, 1968.