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OSCILLATIONS OF HIGHER-ORDER OF SYSTEMS OF DIFFERENCE EQUATIONS

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In this paper, we provide necessary and sufficient conditions for the oscillation of every solution of the system of difference equations

$$\Delta^r y_n + P y_{n-k} = 0, n = 0, 1, 2, ...,$$

where $P \in \mathbb{R}^{s \times s}$ and $k \in \mathbb{Z}$. Furthermore, we prove sufficient conditions for the oscillation of every solution of the system of difference equations

$$\Delta^r y_n + \sum_{i=1}^{m} P_i y_{n-k_i} = 0, \quad n = 0, 1, 2, ...,$$

where $P \in \mathbb{R}^{s \times s}$ and $k_i \in \mathbb{Z}$ for i = 1, 2, ..., m.

Key Words : Difference Equation; Oscillation; Logarithmic Norm

1. INTRODUCTION

The concept of the oscillatory behaviour of solutions of difference equations have been extensively investigated, see [1, 8] and the reference cited therein. In [7], Ladas established a theorem for the oscillatory behaviour of all solutions for the following difference equation

$$\Delta y_n + p y_{n-k} = 0, \quad n = 0, 1, 2, \ldots$$

where

$$P \in \mathbb{R} \text{ and } k \in \mathbb{Z}.$$

In [3], Chuanxi, Kuruklis and Ladas obtained that the oscillatory behaviour of all solutions of linear autonomous system of difference equations

$$\Delta y_n + P y_{n-k} = 0, \quad n = 0, 1, 2, \ldots$$

where $P \in \mathbb{R}^{s \times s}$ and $k \in \mathbb{Z}$. Furthermore, they obtained in (3), sufficient conditions for the oscillation of all solutions of the difference equation

$$\Delta y_n + \sum_{i=1}^{m} P_i y_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots$$

where $P_i \in \mathbb{R}^{s \times s}$ and $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, m$.

In Section 2 we obtain necessary and sufficient conditions for the oscillation of all solutions of the system of difference equations

$$\Delta^r y_n + P y_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \qquad \ldots (1.1)$$

where $P \in \mathbb{R}^{s \times s}$ and $k \in \mathbb{Z}$ and r is a positive integer. In section 3 we obtain sufficient conditions for the oscillation of all solutions of the system of difference equations

$$\Delta^r y_n + \sum_{i=1}^{m} P_i y_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots \qquad \ldots (1.2)$$

where $P_i \in \mathbb{R}^{s \times s}$ and $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, m$ and r is a positive integer.

As usual the forward difference operator we define as follows

$$\Delta^r y_n = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} y_{n+i}, \quad r \geq 1.$$

We consider the following system

$$y_{n+k} + P_1 y_{n+k-1} + \ldots + P_k y_n = 0, \quad n = 0, 1, 2, \ldots \qquad \ldots (1.3)$$

The solutions of the system were examined in the sense of oscillation in (5), where k is a positive integer and the coefficients $P_1, P_2, \ldots, P_k \in \mathbb{R}^{s \times s}$, here $\mathbb{R}^{s \times s}$ denotes the set of real matrices of the type $s \times s$.

For an $s \times s$ matrix P the logarithmic norm of P is denoted by $\mu(P)$ and is defined to be

$$\mu(P) = \max_{\|\xi\|=1} (\xi, \xi)$$

where (,) is an inner product in $\mathbb{R}^s$ and $\|\xi\| = (\xi, \xi)^{\frac{1}{2}}$.

By a solution of the equation (1.1) we mean a sequence $\{y_n\}$ of a vectors in $\mathbb{R}^s$ for $n = 0, 1, 2, \ldots$ which satisfies equation (1.1). A sequence of real numbers $\{y_n\}$ is said to oscillate if the terms y_n are not eventually positive or eventually negative. Let $\{y_n\}$ be a solution of the equation (1.1) with $y_n = \left[y_n^1, y_n^2, \ldots, y_n^s \right]^T$ for $n = 0, 1, 2, \ldots$. We say that the solution $\{y_n\}$ oscillates componentwise or simply oscillate if each component $\left\{ y_n^i \right\}$ oscillates. Otherwise the solution is called nonoscillatory.

For the purpose of obtaining oscillation result of (1, 2), let

$$k = \max \{0, k_1, k_2, \ldots, k_m\} \text{ and } l = \max \left\{ r, -k_1, -k_2, \ldots, -k_m \right\}.$$

Then eq. (1.2) can be written in the form

$$\Delta^r y_n + \sum_{j=-k}^{l} Q_j y_{n+j} = 0, \quad n = 0, 1, 2, \ldots \qquad \ldots (1.4)$$

The eq. (1.4) is a difference equation of order $(k+l)$. If $k \geq 0$ and $l + r$, we say that eq. (1.4) is a delay difference equation. When $k = 0$ and $l \geq r+1$, eq. (1.4) is called an advanced difference equation. When $k \geq r$ and $l \geq r+1$, then eq. (1.4) is of the mixed type.

For the purpose of existence and uniqueness of solutions we should assume that

$$\left\{ \begin{array}{l} \text{if } 0 \leq k \leq r \text{ and } l - r \text{ then } \det (Q_r + I) \neq 0 \\ \\ \text{if } k = 0 \text{ and } l \geq r+1 \text{ then } \det Q_l \neq 0 \end{array} \right\} \qquad \ldots (1.5)$$

Let $a_{-k}, \ldots, a_{l-1}$ be $(k+l)$ given vectors in $\mathbb{R}^s$. Then under the assumption (1.5), eq. (1.4) has a unique solution $\{y_n\}$ which satisfies the initial conditions

$$u_i = a_i, \; i = -k, \ldots, l-1.$$

We need the following lemma, which is proved in (5).

Lemma 1.1 — Assume that $P_1, P_2, ..., P_k \in \mathbb{R}^{s \times s}$ and that I is the $s \times s$ identity matrix. Then every solution of eq. (1.3) oscillates (componentwise) if and only if the characteristic equation

$$\det\left[\lambda^k I + \lambda^{k-1} P_1 + \ldots + \lambda P_{k-1} + P_k\right] = 0$$

has no positive roots.

2. Necessary and Sufficient Conditions for Oscillation of (1.1)

In this section we obtain necessary and sufficient conditions for the oscillation of eq. (1.1). The conditions will be given in terms of the $k \in \mathbb{Z}$ and $P \in \mathbb{R}^{s \times s}$ matrix.

***Theorem* 2.1** — *Let $P \in \mathbb{R}^{s \times s}$, $k \in \mathbb{Z}$ and r is an even positive integer. Suppose that condition (1.5) is satisfied, then every solution of equation (1.1) oscillates (componentwise) if and only if one of the following conditions holds.*

(*i*) $k = 0$ and P has no eigenvalues in $(-\infty, 0]$,

(*ii*) $k \in \{-r+1, -r+2, ..., -2, -1\}$ and P has no eigenvalues in $(-\infty, 0]$,

(*iii*) $k = -r$ and P has no eigenvalues in $(-\infty, 0]$,

(*iv*) $k \geq 1$ and P has no eigenvalues in $(-\infty, 0]$,

(*v*) $k \leq -r-1$ and P has no eigenvalues in $(-\infty, 0]$.

PROOF : (*i*) $k = 0$. In this case the eq. (1.1) becomes

$$\Delta^r y_n + P y_n = 0 \qquad \text{... (2.1)}$$

and characteristic equation of eq. (2.1) is

$$\det\left[(\lambda - 1)^r I + P\right] = 0. \qquad \text{... (2.2)}$$

which can also be written as

$$\det\left[-(\lambda - 1)^r I - P\right] = 0.$$

Take the function $\upsilon(\lambda)$ as

$$\upsilon(\lambda) = -(\lambda - 1)^r$$

Note that $\upsilon(\lambda)$ is a continuous function on $(0, \infty)$,

$$\upsilon(1) = 0, \quad \lim_{\lambda \to 0^+} \upsilon(\lambda) = -1 \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = -\infty.$$

Thus the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $(-\infty, 0]$. Therefore by Lemma 1.1, eq. (2.2) has no positive roots if and only if P has no eigenvalues in $(-\infty, 0]$.

(*ii*) $k \in \{-r+1, -r+2, \ldots, -2, -1\}$. In this case the characteristic equation of eq. (1.1) is

$$\det\left[(\lambda - 1)^r I + P\lambda^{-k} \right] = 0. \qquad \ldots (2.3)$$

This can also be written as

$$\det\left[-\lambda^k (\lambda - 1)^r I - P \right] = 0.$$

Now, the function $\upsilon(\lambda)$ can be taken as

$$\upsilon(\lambda) = -\lambda^k (\lambda - 1)^r.$$

Note that $\upsilon(\lambda)$ is a continuous function on $(0, \infty)$,

$$\upsilon(1) = 0, \quad \lim_{\lambda \to)^+} \upsilon(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = -\infty.$$

Thus the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $(-\infty, 0]$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no eigenvalues in $(-\infty, 0]$.

(*iii*) $k = -r$. In this case the characteristic equation of eq. (1.1) is (2.3). As

$$\upsilon(1) = 0, \quad \lim_{\lambda \to 0^+} \upsilon(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = -1,$$

the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $(-\infty, 0]$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no eigenvalues in $(-\infty, 0]$.

(*iv*) $k \geq 1$. In this case the characteristic equation of eq. (1.1) is (2.3). So

$$\upsilon(1) = 0, \quad \lim_{\lambda \to 0^+} \upsilon(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = -\infty,$$

the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $(-\infty, 0]$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no eigenvalues in $(-\infty, 0]$.

(*v*) $k \leq -r-1$. In this case the characteristic equation of eq. (1.1) is (2.3).
Furthermore

$$\upsilon(1) = 0, \quad \lim_{\lambda \to 0^+} \upsilon(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = 0,$$

hold. Thus the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $(-\infty, 0]$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no eigenvalues in $(-\infty, 0]$. Thus the proof is completed. ■

***Theorem* 2.2** — Let $P \in \mathbb{R}^{s \times s}$ and $k \in \mathbb{Z}$ and r is an odd positive integer. Suppose that condition (1.5) is satisfied, then every solution of eq. (1.1) oscillates (componentwise) if and only if one of the following conditions holds;

(*i*) $k = 0$ and P has no eigenvalues in $(-\infty, 1]$,

(*ii*) $k \in \{-r+1, -r+2, \ldots, -2, -1\}$ and P has no eigenvalues in $(-\infty, \infty)$,

(*iii*) $k = -r$ and P has no eigenvalues in $[-1, \infty)$,

(*iv*) $k \geq 1$ and P has no eigenvalues in $\left(-\infty, r^r \dfrac{k^k}{(k+r)^{k+r}}\right]$,

(*v*) $k \leq -r-1$ and P has no eigenvalues in $\left[r^r \dfrac{k^k}{(k+r)^{k+r}}, \infty\right)$.

PROOF : (*i*) $k = 0$. In this case the characteristic equation of eq. (1.1) is (2.2). As

$$\lim_{\lambda \to 0^+} \upsilon(\lambda) = 1 \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = -\infty.$$

Thus the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $(-\infty, 1]$. Therefore by Lemma 1.1, eq. (2.2) has no positive roots if and only if P has no eigenvalues in $(-\infty, 1]$.

(*ii*) $k \in \{-r+1, -r+2, \ldots, -2, -1\}$. In this case the characteristic equation of eq. (1.1) is (2.3). So

$$\lim_{\lambda \to 0^+} \upsilon(\lambda) = \infty \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = -\infty.$$

Thus the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $(-\infty, \infty)$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no real eigenvalues.

(*iii*) $k = -r$. In this case the characteristic equation of eq. (1.1) is (2.3). As

$$\lim_{\lambda \to 0^+} \upsilon(\lambda) = \infty \quad \text{and} \quad \lim_{\lambda \to \infty} \upsilon(\lambda) = -1.$$

the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $[-1, \infty)$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no eigenvalues in $[-1, \infty)$.

(*iv*) $k \geq 1$. In this case the characteristic equation of eq. (1.1) is (2.3). So

$$\max_{\lambda > 0} \upsilon(\lambda) = r^r \frac{k^k}{(k+r)^{k+r}}, \quad \lim_{\lambda \to 0^+} \upsilon(\lambda) = 0 \text{ and } \lim_{\lambda \to \infty} \upsilon(\lambda) = -\infty,$$

the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $\left(-\infty, r^r \frac{k^k}{(k+r)^{k+r}}\right]$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no eigenvalues in $\left(-\infty, r^r \frac{k^k}{(k+r)^{k+r}}\right]$.

(*v*) $k \leq -r-1$. In this case the characteristic equation of eq. (1.1) is (2.3). So

$$\min_{\lambda > 0} \upsilon(\lambda) = r^r \frac{k^k}{(k+r)^{k+r}}, \quad \lim_{\lambda \to 0^+} \upsilon(\lambda) = \infty \text{ and } \lim_{\lambda \to \infty} \upsilon(\lambda) = 0$$

hold. Thus the image of $(0, \infty)$ under $\upsilon(\lambda)$ is $\left[r^r \frac{k^k}{(k+r)^{k+r}}, \infty\right)$. Therefore by Lemma 1.1, eq. (2.3) has no positive roots if and only if P has no eigenvalues in $\left[r^r \frac{k^k}{(k+r)^{k+r}}, \infty\right)$. Thus the proof is completed. ■

Remark 2.1 : For the case $s = 1$ in eq. (1.1), Chuanxi, Kuruklis and Ladas have obtained eq. (1.1) in (3) oscillation results of behaviour of oscillatory properties.

Remark 2.2 : For the case $s = 1$ in eq. (1.1), Ladas and Qian have obtained oscillation results of eq. (1.1) in (8) (See also (2)).

3. Sufficient Conditions for Oscillation of (1.2)

In this section we obtain sufficient conditions for the oscillation of all solutions of the linear equation with the matrix coefficients of $P_1, P_2, \ldots, P_m$.

$$\Delta^r y_n + \sum_{i=1}^{m} P_i y_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots$$

The conditions will be given in terms of the k_i and logarithmic norm of the matrices P_i for each $i = 1, 2, \ldots, m$.

***Theorem* 3.1** — *Let* $P_i \in \mathbb{R}^{s \times s}$, $k_i \in \mathbb{Z}$ *for* $i = 1, 2, \ldots, m$ *and let* r *be an even positive integer. Suppose that condition (1.5) is satisfied. Then every solution of eq. (1.2) oscillates (componentwise) provided that*

$$\sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) < 0 \text{ for } \gamma > 0, \text{ or} \qquad \ldots (3.1)$$

$$\sup_{\gamma \in (0,\infty)} \left[\frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \right] < 1 \text{ for } \gamma \neq 1, \text{ and } \sum_{i=1}^{m} \mu(-p_i) < 0. \qquad \ldots (3.2)$$

PROOF : Assume that eq. (1.2) is nonoscillatory. Then the characteristic equation of eq. (1.2)

$$\left[(\lambda-1)^r I + \sum_{i=1}^{m} \lambda^{-k_i} P_i \right] = 0$$

has a positive root γ. Therefore there exists a nonzero vector $\xi \in R^s$, which we may assume to have norm one, such that

$$\left[(\gamma-1)^r I + \sum_{i=1}^{m} \gamma^{-k_i} P_i \right] = 0$$

Hence,

$$(\gamma-1)^r = \sum_{i=1}^{m} \gamma^{-k_i} (-P_i \xi, \xi)$$

and so

$$(\gamma-1)^r \leq \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \qquad \ldots (3.3)$$

It follows from (3.3) that (3.1) cannot hold. Now if $\gamma \neq 1, \gamma \in (0, \infty)$, then by (3.3) we get

$$1 \leq \frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \cdot \gamma^{-k_i} \mu(-P_i)$$

and so

$$1 \leq \sup_{\gamma \in (0,\infty)} \left[\frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \right]$$

which contradicts the first part of (3.2). Also if $\gamma = 1$, then by (3.3)

$$0 \leq \sum_{i=1}^{m} \mu(-P_i)$$

which contradicts the second part of (3.2). The proof is completed. ■

Lemma 3.2. Let $P_i \in \mathbb{R}^{s \times s}$, $k_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, m$ and let r be an odd positive integer. Suppose that condition (1.5) is satisfied. Then every solution of eq. (1.2) oscillates (componentwise) provided that

$$(i_a) \quad \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \text{ for } \gamma \geq 1, \text{ or}$$

$$(i_b) \quad \sup_{\gamma \in (1,\infty)} \left[\frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \right] < 1 \quad \text{and} \quad \sum_{i=1}^{m} \mu(-P_i) < 0, \text{ or}$$

$$(ii) \quad \inf_{\gamma \in (0,1)} \left[\frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \right] > 1.$$

PROOF : Assume that eq. (1.2) is nonoscillatory. Then it follows from (3.3) that Lemma 3.2 (i_a) cannot hold. Now if $\gamma \in (1, \infty)$, then by (3.3) we get

$$1 \leq \frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i)$$

and so

$$1 \leq \sup_{\gamma \in (1,\infty)} \left[\frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \right]$$

which contradicts the first part of Lemma 3.2 (i_b). Also if $\gamma = 1$, then by (3.3)

$$0 \leq \sum_{i=1}^{m} \mu(-P_i)$$

which contradicts the second part of Lemma 3.2 (i_b). Therefore $\gamma \in (0,1)$ and by (3.3)

$$1 \geq \frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i)$$

so

$$1 \geq \inf_{\gamma \in (0,1)} \left[\frac{1}{(\gamma-1)^r} \sum_{i=1}^{m} \gamma^{-k_i} \mu(-P_i) \right]$$

which contradicts the Lemma 3.2 (*ii*), and the proof is completed. ■

***Theorem* 3.3** — Let $P_i \in \mathbb{R}^{s \times s}$, $k_i \in \mathbb{Z}$ for $i = 1, 2, ..., m$ and let r be an odd positive integer. Suppose that condition (1.5) is satisfied and that for $i = 1, 2, ..., m$,

$$\mu(-P_i) \leq 0 \text{ and } k_i \in \{0, 1, ...\}. \qquad \text{... (3.4)}$$

Then every solution of eq. (1.2) oscillates (componentwise) provided that one of the following two conditions is satisfied;

(*i*) $$\sum_{i=1}^{m} -\mu(-P_i) \frac{(k_i + r)^{k_i + r}}{k_i^{k_i}} > r^r,$$

(*ii*) $$m \left[\prod_{i=1}^{m} |\mu(-P_i)| \right]^{\frac{1}{m}} \frac{(k+r)^{k+r}}{k^k} > r^r \text{ where } k = \frac{1}{m} \sum_{i=1}^{m} k_i.$$

PROOF : We will assume (3.4) holds. We employ Lemma 3.2. It is clear that Lemma 3.2 (i_a) is satisfied since by hypothesis $\mu(-P_i) \leq 0$ and by either (*i*) or (*ii*). So, it suffices to show that each of (*i*) and (*ii*) implies Lemma 3.2 (*ii*).

Now, assume that (*i*) holds. Then, we get

$$\sup_{\gamma \in (0,1)} \frac{1}{(\gamma-1)^r \gamma^{k_i}} = -\frac{1}{r^r} \frac{(k_i + r)^{k_i + r}}{k_i^{k_i}}.$$

Hence for $i = 1, 2, ..., m$,

$$\frac{1}{(\gamma-1)^r \gamma^{k_i}} \mu(-P_i) \geq -\mu(-P_i) \frac{1}{r^r} \frac{(k_i + r)^{k_i + r}}{k_i^{k_i}}$$

so, for $\gamma \in (0, 1)$,

$$\frac{1}{(\gamma-1)^r}\sum_{i=1}^{m}\gamma^{-k_i}\mu(-P_i) \geq \frac{1}{r^r}\sum_{i=1}^{m}-\mu(-P_i)\frac{(k_i+r)^{k_i+r}}{k_i^{k_i}} > 1 \qquad \text{... (3.5)}$$

and so, Lemma 3.2 (*ii*) holds.

Now, assume that (*ii*) holds. By using the arithmetic-geometric mean inequality we have from (3.5) that for $\gamma \in (0, 1)$,

$$\frac{1}{(\gamma-1)^r}\sum_{i=1}^{m}\gamma^{-k_i}\mu(-P_i) = \frac{1}{(\gamma-1)^r}\sum_{i=1}^{m}-\mu(-P_i)\gamma^{-k_i}$$

$$\geq m\left[\prod_{i=1}^{m}[-\mu(-P_i)]\right]^{\frac{1}{m}}\frac{1}{(1-\gamma)^r\gamma^k},$$

and also that

$$\inf_{\gamma\in(0,1)}\frac{1}{(\gamma-1)^r\gamma^k} = -\frac{1}{r^r}\frac{(k+r)^{k+r}}{k^k}.$$

So, for $\gamma \in (0,1)$,

$$\frac{1}{(\gamma-1)^r}\sum_{i=1}^{m}\gamma^{-k_i}\mu(-P_i) \geq m\left[\prod_{i=1}^{m}[-\mu(-P_i)]\right]^{\frac{1}{m}}\frac{1}{r^r}\frac{(k+r)^{k+r}}{k^k} > 1$$

and so, Lemma 3.2. (*ii*) holds. Therefore the proof is completed. ■

Remark 3.1 : For the case $r = 1$ in eq. (1.2), Chuanxi, Kuruklis and Ladas have obtained oscillation results of eq. (1.2) in (3).

Remark 3.2 : For the case $s = 1$ and $r = 1$ in eq. (1.2), Ladas has obtained oscillation results of eq. (1.2) in (7) (See also (5)).

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