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POSITIVE SOLUTION FOR BVPs OF FOURTH ORDER DIFFERENCE EQUATIONS*

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In this paper, by using Guo-Krasnosel'skii fixed point theorem in cone, we study the existence of positive solution for the following nonlinear fourth order boundary value problem

$$\begin{aligned}\Delta^4 x(t-2) &= a(t) f(x(t)), t \in [2, T],\\ x(0) &= x(T+2) = 0,\\ \Delta^2 x(0) &= \Delta^2 x(T) = 0.\end{aligned}$$

Key Words: Boundary Value Problem; Difference Equation; Positive Solution; Fixed Point; Cone

1. INTRODUCTION

Recently, boundary value problems (BVPs) of difference equations have received much attention from many authors, see [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 16, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29] and the references therein. In particular, because of its significant applications in the theory of bending extensible elastic beams on nonlinear elastic foundations [18], fourth order BVPs have attracted considerable attention. For example, the authors in [29] established the existence of positive solution to the fourth-order boundary value problem

$$\begin{aligned}\Delta^4 x(t-2) &= \lambda a(t) f(t, x(t)), t \in N,\ 2 \le t \le T,\\ x(0) &= x(T+2) = 0,\\ \Delta^2 x(0) &= \Delta^2 x(T) = 0. \qquad \dots (1.1)\end{aligned}$$

They required that the nonlinearity $f(t, x)$ was increasing in x and their main tool was the method of upper and lower solution.

In 2003, Liang, Zhao and Sun [16] obtained a new existence result for the following fourth-order boundary value problem

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$$\Delta^4 x(t-2) = a(t) f(x(t)), t \in [2, T],$$
$$x(0) = x(T+2) = 0,$$
$$\Delta^2 x(0) = \Delta^2 x(T) = 0 \qquad \dots (1.2)$$

by using a fixed point theorem due to Krasnosel'skii and Zabreiko [15]. The conditions they needed were very easy to verify, but the solution obtained in [16] may be negative.

In this paper we will continue to consider the boundary value problem (1.2), where $T > 2$ is a fixed positive integer, Δ^m denotes the mth forward difference operator with stepsize 1, and $[a, b] = \{a, a+1, ..., b-1, b\} \subset Z$ the set of all integers. Some sufficient conditions are obtained for the existence of positive solution to the boundary value problem (1.2), and we do not require that the nonlinearity $f(x)$ satisfies any monotonity.

Throughout this paper, we assume that the following two conditions are satisfied.

(Cl) $f : [0, \infty) \to [0, \infty)$ is continuous;

(C2) $a : [2, T] \to [0, \infty)$ is not identical zero.

The following well-known Guo-Krasnosel'skii fixed point theorem in cone [9, 14] is very crucial in our arguments. For background information and applications of such a fixed point theorem, one can refer to [8, 17, 23].

***Theorem* 1.1** — (Guo-Krasnosel'skii fixed point theorem). Let E be a Banach space, and K be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let

$$\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(*i*) $\|\Phi u\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_1$ and $\|\Phi u\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_2$

or

(*ii*) $\|\Phi u\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_1$ and $\|\Phi u\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main Result

For the convenience, we define

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} \quad \text{and} \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Our main result is the following theorem.

***Theorem* 2.1** — Assume that (Cl) and (C2) hold. Then the boundary value problem (1.2) has at least one positive solution in the case

(*i*) $f_0 = 0$ and $f_\infty = \infty$ (superlinear) or

(*ii*) $f_0 = \infty$ and $f_\infty = 0$ (sublinear).

Proof : Denote

$$G_1(t, s) = \frac{1}{T} \begin{cases} (t-1)(T+1-s), & 1 \le t \le s \le T, \\ (s-1)(T+1-t), & 2 \le s \le t \le T+1, \end{cases}$$

and

$$G_2(t, s) = \frac{1}{T+2} \begin{cases} t(T+2-s), & 0 \le t \le s \le T+1, \\ s(T+2-t), & 1 \le s \le t \le T+2. \end{cases}$$

It is easily seen from the expression of $G_2(t, s)$ that

$$G_2(t, s) \le G_2(s, s), \; (t, s) \in [0, T+2] \times [1, T+1], \qquad \text{... (2.1)}$$

and

$$G_2(t, s) \ge \frac{1}{T+1} G_2(s, s), \; (t, s) \in [1, T+1] \times [1, T+1]. \qquad \text{... (2.2)}$$

Let the Banach space $E = \{x : [0, T+2] \to R\}$ be equipped with the norm

$$\| x \| = \max_{t \in [0, T+2]} | x(t) |,$$

and define

$$K = \left\{ x \in E : x(t) \ge 0, t \in [0, T+2], \min_{t \in [1, T+1]} x(t) \ge \frac{1}{T+1} \| x \| \right\},$$

then it is obvious that K is a cone in E.

Suppose that the operator $\Phi : K \to E$ is defined by

$$(\Phi x)(t) = \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) f(x(\upsilon)),\ t \in [0, T+2].$$

First, we claim that $\Phi(K) \subset K$.

In fact, if $x \in K$, then it follows from (2.1) that

$$0 \le (\Phi x)(t) = \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) f(x(\upsilon))$$

$$\le \sum_{s=1}^{T+1} G_2(s, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) f(x(\upsilon)),\ t \in [0, T+2],$$

and so,

$$\| \Phi x \| \le \sum_{s=1}^{T+1} G_2(s, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) f(x(\upsilon)). \qquad \dots (2.3)$$

In view of (2.2) and (2.3), we have

$$(\Phi x) = \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) f(x(\upsilon))$$

$$\ge \frac{1}{T+1} \sum_{s=1}^{T+1} G_2(s, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) f(x(\upsilon))$$

$$\ge \frac{1}{T+1} \| \Phi x \|,\ \ t \in [1, T+1],$$

therefore, $\min_{t \in [1, T+1]} (\Phi x)(t) \ge \frac{1}{T+1} \| \Phi x \|$. This shows that $\Phi(K) \subset K$.

Next, it is also easy to check that $\Phi : K \to K$ is completely continuous and that x is a solution of the problem (1.2) if and only if x is a fixed point of Φ.

To be precise, we denote

$$\Omega_i = \left\{ x \in E : \| x \| < H_i \right\},\ i = 1, 2, 3, 4.$$

First, we consider the case (i): superlinear case.

Since $f_0 = 0$, we may choose $H_1 > 0$ such that

$$f(x) \le \varepsilon x \quad \text{for} \quad x \in [0, H_1], \qquad \ldots (2.4)$$

where $\varepsilon > 0$ satisfies

$$\varepsilon \max_{t \in [0, T+2]} \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) \le 1. \qquad \ldots (2.5)$$

Thus, if $x \in K \cap \partial \Omega_1$, then from (2.4) and (2.5), we have

$$\begin{aligned}
(\Phi x)(t) &= \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) f(x(\upsilon)) \\
&\le \varepsilon \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon)\, x(\upsilon) \\
&\le \varepsilon \, \| x \| \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) \\
&\le \varepsilon \, \| x \| \max_{t \in [0, T+2]} \sum_{s=1}^{T+1} G_2(t, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) \\
&\le \| x \|, \; t \in [0, T+2].
\end{aligned}$$

So,

$$\| \Phi x \| \le \| x \| \quad \text{for} \quad x \in K \cap \partial \Omega_1. \qquad \ldots (2.6)$$

On the other hand, in view of $f_\infty = \infty$, there exists $\hat{H}_2 > 0$ such that

$$f(x) \ge \rho x \quad \text{for} \quad x \in [\hat{H}_2, \infty), \qquad \ldots (2.7)$$

where $\rho > 0$ is chosen so that

$$\frac{\rho}{T+1} \sum_{s=1}^{T+1} G_2(T, s) \sum_{\upsilon=2}^{T} G_1(s, \upsilon)\, a(\upsilon) \ge 1. \qquad \ldots (2.8)$$

Let

$$H_2 = \max \left\{ 2H_1, (T+1)\, \hat{H}_2 \right\}.$$

Then $x \in K \cap \partial \Omega_2$ implies that

$$\min_{t \in [1, T+1]} x(t) \geq \frac{1}{T+1} \|x\| = \frac{1}{T+1} H_2 \geq \hat{H}_2, \qquad \ldots (2.9)$$

and so it follows from (2.7), (2.8) and (2.9) that

$$(\Phi x)(T) = \sum_{s=1}^{T+1} G_2(T, s) \sum_{v=2}^{T} G_1(s, v)\, a(v) f(x(v))$$

$$\geq \rho \sum_{s=1}^{T+1} G_2(T, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)\, x(v)$$

$$\geq \frac{\rho \|x\|}{T+1} \sum_{s=1}^{T+1} G_2(T, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)$$

$$\geq \|x\|.$$

So,

$$\|\Phi x\| \geq \|x\| \text{ for } x \in K \cap \partial \Omega_2. \qquad \ldots (2.10)$$

Therefore, by the first part of Theorem 1.1, it follows from (2.6) and (2.10) that Φ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. This completes the superlinear part of the theorem.

Next, we consider the case (*ii*): sublinear case.

Since $f_0 = \infty$, we may choose $H_3 > 0$ such that

$$f(x) \geq Mx \quad \text{for } x \in [0, H_3], \qquad \ldots (2.11)$$

where $M > 0$ satisfies

$$\frac{M}{T+1} \sum_{s=1}^{T+1} G_2(T, s) \sum_{v=2}^{T} G_1(s, v)\, a(v) \geq 1. \qquad \ldots (2.12)$$

Then for $x \in K \cap \partial \Omega_3$, we can get from (2.11) and (2.12) that

$$(\Phi x)(T) = \sum_{s=1}^{T+1} G_2(T, s) \sum_{v=2}^{T} G_1(s, v)\, a(v) f(x(v))$$

$$\geq M \sum_{s=1}^{T+1} G_2(T, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)\, x(v)$$

$$\geq \frac{M \| x \|}{T+1} \sum_{s=1}^{T+1} G_2(T, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)$$

$$\geq \| x \|.$$

So,

$$\| \Phi x \| \geq \| x \| \text{ for } x \in K \cap \partial \Omega_3. \qquad \text{... (2.13)}$$

On the other hand, in view of $f_\infty = 0$, there exists $\hat{H}_4 > 0$ such that

$$f(x) \leq \lambda x \quad \text{for} \quad x \in [\hat{H}_4, \infty). \qquad \text{... (2.14)}$$

where $\lambda > 0$ is chosen so that

$$\lambda \min_{t \in [0, T+2]} \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v) \leq 1. \qquad \text{... (2.15)}$$

We consider two cases:

Case (a) — Suppose that f is bounded, i.e., there exists a constant $N > 0$ such that

$$f(x) \leq N \quad \text{for} \quad x \in [0, \infty). \qquad \text{... (2.16)}$$

In this case we choose

$$H_4 = \max \left\{ 2H_3, N \max_{t \in [0, T+2]} \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v) \right\} \qquad \text{... (2.17)}$$

so that for $x \in K \cap \partial \Omega_4$, we have from (2.16) and (2.17) that

$$(\Phi x)(t) = \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v) f(x(v))$$

$$\leq N \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)$$

$$\leq N \max_{t \in [0, T+2]} \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)$$

$$\leq H_4 = \| x \|, \; t \in [0, T+2],$$

and so

$$\| \Phi x \| \leq \| x \| \text{ for } x \in K \cap \partial \Omega_4.$$

Case (b) — Suppose that f is unbounded, then we know from (Cl) that there exists $H_4 >$ max $\{2H_3, \hat{H}_4\}$ such that

$$f(x) \leq f(H_4) \quad \text{for} \quad x \in [0, H_4]. \qquad \text{... (2.18)}$$

Then for $x \in K \cap \partial \Omega_4$, it follows from (2.18), (2.14) and (2.15) that

$$(\Phi x)(t) = \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v) f(x(v))$$

$$\leq f(H_4) \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)$$

$$\leq \lambda(H_4) \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)$$

$$\leq \lambda H_4 \min_{t \in [0, T+2]} \sum_{s=1}^{T+1} G_2(t, s) \sum_{v=2}^{T} G_1(s, v)\, a(v)$$

$$\leq H_4 = \| x \|, \; t \in [0, T+2],$$

and so

$$\| \Phi x \| \geq \| x \| \text{ for } x \in K \cap \partial \Omega_4.$$

Hence, in either case we obtain that

$$\| \Phi x \| \leq \| x \| \text{ for } x \in K \cap \partial \Omega_4. \qquad \text{... (2.19)}$$

Therefore, by the second part of Theorem 1.1, it follows from (2.13) and (2.19) that Φ has a fixed point. This completes the sublinear part of the theorem.

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