Indian J. pure appl. Math., **36**(8) : 407-416, August 2005

ON HARMONIC BLOCH AND NORMAL FUNCTIONS

METİN ÖZTÜRK AND SİBEL YALÇIN

Department of Mathematics, Faculty of Science, Uludag University, Bursa 16059, Turkey
E-mail: ometin@uludag.edu.tr

(*Received* 23 *April* 2004; *after final revision* 10 *May* 2005; *accepted* 29 *August* 2005)

Let $f = h + \overline{g}$ be a harmonic univalent and sense preserving function on the unit disk, where h and g are analytic. We give the definition of a normal harmonic function. The necessary and sufficient conditions for $f = h + \overline{g}$ to be Bloch or normal are determined. The sharp upper bound for $|f(z)|$ is obtained whenever $f = h + \overline{g}$ is a Bloch function. We obtain sharp estimates for the normality order of f when f maps the unit disk onto the unit disk.

Key Words: Harmonic Functions; Univalent Functions; Bloch Function; Normal Function

1. INTRODUCTION

A continuous function $f = u + iv$ defined on the open unit disk U is said to be harmonic in U if u and v are real valued harmonic functions in U. In particular, any function f harmonic in U may also be represented in the form $f = h + \overline{g}$, where h and g are analytic functions in U, [I]. This representation is unique up to an additive constant.

Colonna [4] defined when a harmonic function $f = h + \overline{g}$ is Bloch, and showed that the necessary and sufficient conditions for f to be Bloch is

$$\beta_f = \sup_{z \in U} (1 - |z|^2)(|h'(z)| + |g'(z)|) < \infty. \quad \text{... (1)}$$

Furthermore, in [4] it was shown that the harmonic function $f = h + \overline{g}$ is Bloch if and only if h and g are, and

$$\max(\beta_h, \beta_g) \le \beta_f \le \beta_h + \beta_g.$$

Pommerenke [8] has shown that the analytic function $h(z)$ is Bloch if and only if there exists an analytic function $H(z)$, univalent in U, and also a constant $c > 0$ such that

$$h(z) = c \log H'(z).$$

In the first part of this work, the relation between a harmonic Bloch function $f=h+\overline{g}$ and univalent functions is given. By means of this, a sharp upper bound for $|f(z)|$ is obtained when f is Bloch.

In the second part, a normal harmonic function $f=h+\overline{g}$ is defined, and a necessary and sufficient condition for f to be normal is determined.

2. Harmonic Bloch Functions

***Theorem* 2.1** — *A harmonic function $f=h+\overline{g}$ is Bloch if and only if there exist functions $H, G \in S$ and constants $c_1, c_2 > 0$ such that*

$$f(z) = c_1 \log H'(z) + c_2 \log \overline{G'(z)} + f(0) \qquad \dots (2)$$

where S is the well known class of analytic univalent functions.

Proof : Assume $f(z)$ has the form (2) with $H, G \in S$ and $h = c_1 \log H'$ and $g = c_2 \log G'$. Then

$$(1-|z|^2)(|h'(z)|+|g'(z)|) \leq (1-|z|^2)\left(c_1 \left|\frac{H''(z)}{H'(z)}\right| + c_2 \left|\frac{G''(z)}{G'(z)}\right|\right).$$

Since $H, G \in S$, for $|z| = r < 1$,

$$\left| z\frac{H''(z)}{H'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2} \quad \text{and so} \quad \left|\frac{H''(z)}{H'(z)}\right| \leq \frac{2r+4}{1-r^2}.$$

Similarly

$$\left|\frac{G''(z)}{G'(z)}\right| \leq \frac{2r+4}{1-r^2}$$

(see [9]). Since

$$(1-|z|^2)(|h'(z)| + |g'(z)|) < 6(c_1+c_2) < \infty,$$

f is a harmonic Bloch function.

Conversely, if $f(z) = h(z) + \overline{g(z)}$ is a Bloch function, we consider the functions

$$H(z) = \int_0^z \exp\left\{\frac{h(\zeta)-h(0)}{c_1}\right\} d\zeta = z + \dots$$

and

$$G(z) = \int_0^z \exp\left\{\frac{g(\zeta) - g(0)}{c_2}\right\} d\zeta = z + \ldots$$

Hence,

$$H'(z) = \exp\left\{\frac{1}{c_1}[h(z) - h(0)]\right\}, \qquad G'(z) = \exp\left\{\frac{1}{c_2}[g(z) - g(0)]\right\}$$

and so

$$\frac{1}{c_1}[h(z) - h(0)] + \frac{1}{c_2}[\overline{h(z)} - \overline{h(0)}] = \log H'(z) + \log \overline{G'(z)}$$

Moreover,

$$\frac{H''(z)}{H'(z)} = \frac{1}{c_1} h'(z), \quad \frac{G''(z)}{G'(z)} = \frac{1}{c_2} g'(z).$$

Thus,

$$(1 - |z|^2)\left|\frac{H''(z)}{H'(z)}\right| = \frac{1}{c_1}(1 - |z|^2)|h'(z)| \leq 1,$$

$$(1 - |z|^2)\left|\frac{G''(z)}{G'(z)}\right| = \frac{1}{c_1}(1 - |z|^2)|g'(z)| \leq 1.$$

Then (2) is satisfied with $c_1, c_2 > 0$ and

$$(1 - |z|^2)\left|z\frac{H''(z)}{H'(z)}\right| \leq 1 \quad \text{and} \quad (1 - |z|^2)\left|z\frac{G''(z)}{G'(z)}\right| \leq 1.$$

Therefore, by [9, VI Theorem 6.7], H and G belong to the class S. This completes the proof.

Let B_H denote the class of all functions $f = h + \overline{g}$ satisfying the condition (1).

Corollary 2.2 — If $f = h + \overline{g} \in B_H$, then for $|z| = r < 1$,

$$|f(z)| \leq \log\left[\frac{1 + r}{(1 - r)^3}\right]^{c_1 + c_2} + (c_1 + c_2)\,\varphi(r) + |f(0)|,$$

where

$$\varphi(r) = \begin{cases} 4 \arcsin r & ; \quad r \leq \sqrt{2}/2 \\ \pi + \log\left(\dfrac{r^2}{1 - r^2}\right) & ; \quad \sqrt{2}/2 < r < 1 \end{cases}$$

PROOF : If $f = h + \bar{g} \in B_{H'}$, then from Theorem 2.1.

$$\begin{aligned} |f(z)| &\le c_1 |\log H'(z)| + c_2 |\log G'(z)| + |f(0)| \\ &\le c_1 |\log H'(z)| + c_2 |\log G'(z)| + c_1 |arg\, H'(z)|\, c_2\, arg\, G'(z)| \\ &+ |f(0)|. \end{aligned}$$

For functions H and G belonging to the class S,

$$|H'(z)| \le \frac{1+r}{(1-r)^3}, \qquad |G'(z)| \le \frac{1+r}{(1-r)^3}$$

(see [7]) and

$$|arg\, H'(z)| \le \varphi(r) \qquad \text{and} \qquad |arg\, G'(z)| \le \varphi(r)$$

(see [6]). Thus, the theorem is proved.

3. NORMAL HARMONIC FUNCTIONS

Definition 3.1 — A harmonic mapping f with domain U is normal if it satisfies the Lipschitz condition with a global constant when regarded as a function from, the hyperbolic disk into $\mathbb{C}$, endowed with the chordal metric. The Lipschitz number of f,

$$\alpha_f = \sup_{z \ne w} \frac{\chi(f(z), f(w))}{\rho(z, w)},$$

is called the normality order of f, where χ denotes the chordal metric:

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\,\sqrt{1 + |z_2|^2}} \qquad z_1, z_2 \in \mathbb{C}$$

and ρ denotes the hyperbolic distance on the unit disk

$$\rho(z, w) = \frac{1}{2} \log \frac{1+r}{1-r}, \quad r = \left| \frac{z - w}{1 - \bar{z} w} \right|.$$

For h a normal meromorphic function in U, the normality order of h is

$$\alpha_h = \sup_{z \in U} (1 - |z|^2) \frac{|h'(z)|}{1 + |h(z)|^2}.$$

***Theorem* 3.1** — *Let $f = h + \bar{g}$ be a normal harmonic function in U. Then the normality order of f is*

$$\alpha_f = \sup_{z \in U} (1 - |z|^2) \frac{|h'(z)| + |g'(z)|}{1 + |f(z)|^2}.$$

PROOF : Some of the techniques used here were inspired by Colonna [4, Lemma 1,2]. Let f harmonic and α_f be the normality order of f as defined previously for harmonic functions. For $z \in U$ let us define

$$\alpha_f = \limsup_{w \to z} \frac{\chi\,(f(z), f(w))}{\rho\,(z, w)}.$$

Letting $w = z + re^{i\theta}$, $0 < r < 1$, we have

$$\frac{\chi\,(f(z), f(w))}{\rho\,(z, w)} = \frac{|f(z) - f(w)|}{\sqrt{1 + |f(z)|^2}\,\sqrt{1 + |f(w)|^2}}\;\frac{1}{\rho\,(z, w)}$$

$$= \frac{|f(z + re^{i\theta}) - f(z)|}{r}\;\frac{r}{\rho\,(z + re^{i\theta}, z)}\;\frac{1}{\sqrt{1 + |f(z)|^2}\,\sqrt{1 + |f(w)|^2}}.$$

Since

$$\lim_{r \to 0} \frac{|f(z + re^{i\theta}) - f(z)|}{r} = |f_x(z) \cos\theta + f_y(z) \sin\theta|$$

and

$$\lim_{r \to 0} \frac{r}{\rho\,(z + re^{i\theta}, z)} = 1 - |z|^2,$$

we get

$$\alpha_f(z) = \max_{\theta \in \mathbb{R}} |f_x(z) \cos\theta + f_y(z) \sin\theta|\,\frac{1 - |z|^2}{1 + |f(z)|^2}.$$

For z and w complex numbers Colonna [4, Lemma 1] shows that,

$$\max_{\theta \in \mathbb{R}} |z \cos\theta + w \sin\theta| = \frac{1}{2}\,(|z + iw| + |z - iw|).$$

Thus, we get

$$\alpha_f(z) = \frac{1}{2}\,(1 - |z|^2)\;\frac{(|f_x + \mathrm{i}f_y| + |f_x - \mathrm{i}f_y|)}{1 + |f(z)|^2}.$$

It is clear from the definition that $\alpha_f(z) \le \alpha_f$. Define $L = \sup_{z \in U} \alpha_f(z)$. Then $L \le \alpha_f$. Assuming that α_f is finite we shall now show that $\alpha_f \le L$.

Let γ be the geodesic curve in U joining z to w. Thus the hyperbolic length of γ is $\rho(z, w)$. Since γ is compact, given any $\varepsilon > 0$, we can find a finite sequence of points $z_0 = z, z_1, \ldots, z_n = w$ on γ such that

$$\chi\,(f(z_{i+1}), f(z_i)) < (\alpha_f(z_i) + \varepsilon)\,\rho\,(z_{i+1}, z_i).$$

By choosing the points so that z_i is between z_{i-1} and z_{i+1}, we get

$$\sum_{i=0}^{n-1} \rho\,(z_{i+1}, z_i) = \rho\,(z, w).$$

Thus

$$\begin{aligned}\chi\,(f(z), f(w)) \le |f(z) - f(w)| &< \sum_{i=0}^{n-1} \left| f(z_{i+1}) - f(z_i) \right| \\ &< \sum_{i=0}^{n-1} (\alpha_f(z_i) + \varepsilon)\,\rho\,(z_{i+1}, z_i) \le \sum_{i=0}^{n-1} (L + \varepsilon)\,\rho\,(z_{i+1}, z_i) \\ &= (L + \varepsilon)\,\rho\,(z, w).\end{aligned}$$

Since this is true for all ε, we must have

$$\chi\,(f(z), f(w)) \le L\,\rho\,(z, w).$$

Hence $\alpha_f \le L$. Consequently, we have

$$\alpha_f = \sup_{z \in U} \alpha_f(z) = \sup_{z \in U} \frac{1}{2}\,(1 - |z|^2)\,\frac{|f_x + if_y| + |f_x - if_y|}{1 + |f(z)|^2}.$$

Since for a harmonic function $f = h + \overline{g}$, where h and g are analytic in U

$$f_x = f_z + f_{\overline{z}} \quad \text{and} \quad f_y = i\,(f_z - f_{\overline{z}})$$

we have

$$f_x + if_y = f_z + f_{\overline{z}} + f_{\overline{z}} - f_z = 2f_{\overline{z}} = 2g'$$

and

$$f_x - if_y = 2f_z = 2h'.$$

Therefore, we get

$$\alpha_f = \sup_{z \in U} (1 - |z|^2) \frac{|h'(z)| + |g'(z)|}{1 + |f(z)|^2}.$$

If analytic and harmonic functions are locally bounded, then they are normal. Hence, from Corollary 2.2, we have the following result for Bloch functions.

Corollary 3.2 — If the harmonic function $f = h + \overline{g}$ is Bloch, then f is normal.

PROOF : Let the harmonic function $f = h + \overline{g}$ be Bloch function. Then

$$\alpha_f = \sup_{z \in U} (1 - |z|^2) \frac{|h'(z)| + |g'(z)|}{1 + |f(z)|^2} = \frac{\beta_f}{1 + |f(z)|^2} \leq \beta_f.$$

Hence, f is a normal function.

***Theorem* 3.3** — *If the function $f = h + \overline{g}$ is normal, then h and g are normal, in fact*

$$\frac{1}{4} \max (\alpha_h, \alpha_g) < \alpha_f.$$

PROOF : Indeed,

$$|f(z)| \leq |h(z)| + |g(z)| \leq 2 \max_{z \in U} (|h(z)|, |g(z)|)$$

and

$$\frac{1}{4} |f(z)|^2 \leq \max_{z \in U} (|h(z)|^2, |g(z)|^2).$$

Hence,

$$\frac{1}{1 + \max(|h(z)|^2, |g(z)|^2)} \leq \frac{1}{1 + (1/4)|f(z)|^2} < \frac{4}{1 + |f(z)|^2}. \quad \ldots (3)$$

If $\max (|h(z)|^2, |g(z)|^2 = |h(z)|^2)$, then from (3)

$$\frac{1}{4} \frac{|h'(z)|}{1 + |h(z)|^2} < \frac{|h'(z)| + |g'(z)|}{1 + |f(z)|^2}.$$

and if

$$\max (|h(z)|^2, |g(z)|^2) = |g(z)|^2,$$

then

$$\frac{1}{4} \frac{|g'(z)|}{1 + |g(z)|^2} < \frac{|h'(z)| + |g'(z)|}{1 + |f(z)|^2}.$$

Thus, we have

$$\frac{1}{4}\max(\alpha_h, \alpha_g) < \alpha_f.$$

For an analytic function $f = u + iv$, where v is harmonic conjugate of u

$$u = \frac{1}{2}(f + \bar{f}), \qquad v = \frac{1}{2i}(f - \bar{f}).$$

Since $|u| \le |f|$, $\alpha_u \ge \alpha_f$ and since $|f| \le 2\max\{|y|, |u|\}$, $\alpha_u \le 2\alpha_{\frac{1}{2}f}$ or $\alpha_v \le 2\alpha_{\frac{1}{2}f}$.

Hence, for a function $f = u + iv$

$$\alpha_f \le \alpha_u \le 2\alpha_{\frac{1}{2}f} \le 4\alpha_f \quad \text{and} \quad \alpha_f \le \alpha_u \le 2\alpha_{\frac{1}{2}f} \le 4\alpha_f$$

is obtained.

***Theorem* 3.4** — *Let $\varphi : U \to U$ be analytic and the harmonic function $f = h + \bar{g}$ be normal in U. Then the function $F(z) = f(\varphi(z))$ is also normal and*

$$\alpha_F \le \alpha_f.$$

Equality holds, if $\varphi(z)$ is a Möbius transformation of U onto itself.

PROOF : By the Schwarz-Pick lemma we have

$$(1 - |z|^2)|\varphi'(z)| \le 1 - |\varphi(z)|^2, \qquad \text{for } z \in U.$$

Thus

$$\alpha_F = \sup_{z \in U} (1 - |z|^2) \frac{|\varphi'(z)|[|h'(\varphi(z))| + |g'(\varphi(z))|]}{1 + |f(\varphi(z))|^2}$$

$$\le \sup_{w \in U} (1 - |w|^2) \frac{|h'(w)||g'(w)|}{1 + |f(w)|^2}, \qquad \text{for } w = \varphi(z)$$

$$= \alpha_f$$

The class of harmonic functions $f = h + \bar{g}$ so that $|h'(z)| > |g'(z)|$, $z \in U$, is denoted by S_H, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=1}^{\infty} b_n z^n \qquad \dots (4)$$

are analytic in U. This class is called the class of normalized harmonic univalent functions (see [3]).

Proposition 3.5 — If $f = h + \bar{g}$ is a sense-preserving univalent harmonic mapping of the unit disc onto itself, then

$$\alpha_f < 1 + \frac{2}{\pi}.$$

PROOF : From [5, Theorem 2], we get

$$\alpha_f = \sup_{z \in U} (1 - |z|^2) \frac{|h'(z)|[|g'(z)|}{1 + |f(z)|^2}$$

$$< \sup_{z \in U} (1 - |z|^2) \left(1 + \frac{2}{\pi}\right) (1 - |z|^2)^{-1} = 1 + \frac{2}{\pi}.$$

Corollary 3.6 — Let $f : U \to U$ be a harmonic mapping. Then

$$\alpha_f \leq \frac{4}{\pi}.$$

PROOF : From [5, Theorem 2] and Corollary 3.2 the inequality is obtained.

Abu-Muhanna and Lyzzaik have shown that log $h'(z)$ is a Bloch function for $f = h + \overline{g} \in S_H$. Also, for bounded planar harmonic mappings $f = h + \overline{g}$, *H*. Chen Gauther and Hengartner [2] obtained results for the Bloch constant.

The following theorem gives a condition for which a function in S_H is Bloch.

***Theorem* 3.7** — *Let a_2 is given by (4). If $a_2 = 0$, then $f \in S_H$ is a harmonic Bloch function.*

PROOF : Let $f = h + \overline{g} \in S_H$. Then by (3), for $|z| = r < 1$, we have

$$(1 - |z|^2)(|h'(z)| + |g'(z)| < 2(1 - r^2)|h'(z)|$$

$$< 2(1 - r^2) \frac{(1 + r)^{\lambda - 1}}{(1 - r)^{\lambda + 1}}$$

$$= 2\left(\frac{1 + r}{1 - r}\right)^{\lambda},$$

where $\lambda = \sup \{|a_2| f \in S_H\}$. Hence, for $a_2 = 0$ or $\lambda = 0$, we have $\beta_f \leq 2$. That is f is a Bloch function. By Corollary 3.2, f is normal.

REFERENCES

1. Y. Abu Muhanna and A. Lyzzaik, The Boundary Behaviour of Harmonic Univalent Maps, *Pacific J. Math.*, **141**(1) (1990), 1-20.
2. H. Chen, P. M. Gauthier and W. Hengartner, Bloch Constants for Planar Harmonic Mappings, *Proc. Amer. Math. Soc.*, **128**(11) (2000), 3231-3240.
3. J. Clunie and T. Shell Small, Harmonic Univalent Functions, *Ann. Acad. Sci. Fenn. Ser. A. I.*. **9**, (1984), 3-35.

4. F. Colonna, The Bloch Constant of Bounded Harmonic Mappings, *Indiana Univ. Math. J.*, **38**(4) (1989), 829-840.
5. P. Duren and G. Schober, Linear Extremal Problems For Harmonic Mappings of The Disk, *Proc. Amer. Math. Soc.*, **106**(4) (1989), 967-73.
6. M. Goluzin, On Distortion Theorem in The Theory of Conformal Mappings, *Math. Sb.*, **43**(1), (1936), 127-35.
7. A. W. Goodman, Univalent Functions, *Mariner Publ. Comp.*, **1**, (1983), p. 65.
8. Ch. Pommerenke, On Bloch Functions, *J. London Math. Soc.*, **2**(2), (1970), 689-95.
9. Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, *Gottingen*, p. 172, 1975.