Indian J. pure appl. Math., **36**(8) : 441-450, August 2005

ON THE QUADRATIC FUNCTIONAL EQUATION MODULO A SUBGROUP

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(*Received* 7 *May* 2004; *after final revision* 16 *November* 2004; *accepted* 21 *March* 2005)

Let X be a topological vector space, let Y be a topological Abelian group, and let K be a discrete subgroup of Y. Assunme that a function $f: X \to Y$ satisfies (2) for all $x, y \in X$. We will prove that if f is continuous at the origin, then there exists a quadratic function allowed to deviate from $f(x) - f(0)$ within K.

Key Words: Quadratic Functional Equation; Pexiderized Quadratic Equation; Functional Congruence; Quadratic Function

1. INTRODUCTION

We assume that a function $f: X \to \mathbb{R}$, where X is a topological vector space, satisfies the condition

$$f(x+y) - f(x) - f(y) \in \mathbb{Z} \qquad \dots (1)$$

for all $x, y \in X$. We may raise a question whether there exists an additive function $A : X \to \mathbb{R}$ such that $f(x) - A(x) \in \mathbb{Z}$ holds true for all $x \in X$.

In view of an example of Godini [10], the answer to this question is not affirmative. According to the example of Godini, there exists a function $f: \mathbb{R} \to \mathbb{R}$ satisfying (1) for all $x, y \in \mathbb{R}$ and for every additive function $A: \mathbb{R} \to \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that $f(x) - A(x) \notin \mathbb{Z}$, i.e., it is impossible to represent f as $A + k$, where A is an additive function and k takes only integer values. However, such a representation is possible under some regularity condition.

It seems that Corput was the first author who gave such a condition (see [9]). More precisely, he proved the following theorem (cf. [6]):

***Theorem* 1** (van der Corput) — *If a function* $f: \mathbb{R} \to \mathbb{R}$ *satisfies the condition (1) for all* $x, y \in \mathbb{R}$, *and if there exist nonempty open subsets* U *and* W *of* $\mathbb{R}$ *such that* $f(U) \cap (W + \mathbb{Z}) = \phi$, *then there exists a* $c \in \mathbb{R}$ *with* $f(x) - cx \in \mathbb{Z}$ *for any* $x \in \mathbb{R}$.

Thereafter, the study of functional congruences was revived by Baron [2], who proved independently of van der Corput that the Cauchy difference of a function $f: \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and f satisfies (1) for all $x, y \in \mathbb{R}$ if and only if there exists an additive function $A: \mathbb{R} \to \mathbb{R}$ such that $f - A$ is Lebesgue measurable and takes only integer values. Further results were obtained by Baron and Volkmann [5] and by Baron and Kannappan [3] (see also [1, 4, 13]). Here, we will introduce a result by Baron and Kannappan (cf. [7]):

***Theorem* 2** (Baron and Kannappan) — *Let* X *be a real topological vector space and let* Y *be a topological Abelian group. If a function* $f: X \to Y$ *satisfies the condition*

$$f(x+y) - f(x) - f(y) \in K$$

for all $x, y \in X$ *and for a discrete subgroup* K *of* Y, *and if* f *is continuous at the origin, then there exists a continuous additive function* $A: X \to Y$ *such that* $f(x) - A(x) \in K$ *for any* $x \in X$.

In this paper, a function $q: X \to Y$ is called quadratic if q satisfies

$$q(x+y) + q(x-y) = 2q(x) + 2q(y)$$

for all $x, y \in X$. (cf. [11, 12, 14, 15]). Throughout this paper, we will assume that the scalar field of each vector space is either $\mathbb{R}$ or $\mathbb{C}$.

Assume that a function $f: X \to Y$ satisfies the condition

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) \in K \quad \text{... (2)}$$

for all $x, y \in X$. For $K = \{0\}$ and for functions $f: X \to \mathbb{C}$, the functional equation (2) and its generalizations have been thoroughly discussed in [8].

In Section 2, we prove that if f is continuous at the origin, then there exists a quadratic function $Q: X \to Y$ such that $f(x) - f(0) - Q(x) \in K$ for all $x \in X$.

2. The Quadratic Equation Modulo a Subgroup

***Theorem* 3** — *Let* X *be a topological vector space and* Y *a topological Abelian group. Assume that* K *is a discrete subgroup of* Y *and that* $f: X \to Y$ *is a function which is continuous at the origin. If* f *satisfies the condition (2) for all* $x, y \in X$, *then there exists a quadratic function* $Q: X \to Y$, *continuous at the origin, such that*

$$f(x) - f(0) - Q(x) \in K$$

for every $x \in X$.

PROOF : If we put $x = y = 0$ in (2), then we get $-2f(0) \in K$. From this fact, it is easy to check that if we define a function $F : X \to Y$ by $F(x) = f(x) - f(0)$ for any $x \in X$, then F also satisfies (2).

Since K, equipped with the topology inherited from Y, is a discrete topological space, there exists an open subset N of Y such that $N \cap K = \{0\}$. Since N contains 0, it is a neighbourhood of 0 in Y.

Due to the continuity of the group additions in X and Y, the map $(x, y) \mapsto F(x+y) + F(x-y) - 2F(x) - 2F(y)$ from $X \times X$ into Y is continuous at (0, 0) as a composition of continuous maps. Thus, there exists a neighbourhood V of 0 in X such that

$$F(x+y) + F(x-y) - 2F(x) - 2F(y) \in N$$

for all $x, y \in V$. (Since X is assumed to be a topological vector space, we may assume that V is a balanced neighbourhood of 0 in X). Because F satisfies (2) and $N \cap K = \{0\}$, the last relation implies that

$$F(x+y) + F(x-y) = 2F(x) + 2F(y) \qquad \ldots (3)$$

for all $x, y \in V$.

We can apply an induction and make use of (3) to prove $4^n F(2^{-n} x) = F(x)$ for all $x \in V$ and $n \in \mathbb{N}_0$. If $2^{-m} x = 2^{-n} y \in V$ for any $x, y \in V$ and for any $m, n \in \mathbb{N}_0$, then

$$4^n F(x) = 4^n 4^m F(2^{-m} x) = 4^m 4^n F(2^{-n} y) = 4^m F(y).$$

We now define a function $Q : X \to Y$ by

$$Q(x) = 4^n F(2^{-n} x) \qquad \ldots (4)$$

for all $x \in X$, where n is any nonnegative integer with $2^{-n} x \in V$. In view of the above argument, the function Q is well defined.

Given $x, y \in X$, choose a nonnegative integer n such that $2^{-n} x, 2^{-n} y, 2^{-n}(x+y)$ and $2^{-n}(x-y)$ belong to V. It then follows from (3) and (4) that

$$\begin{aligned} Q(x+y)+Q(x-y) &= 4^n F(2^{-n}(x+y))+4^n F(2^{-n}(x-y)) \\ &= 4^n[2F(2^{-n}x)+2F(2^{-n}y)] \\ &= 2Q(x)+2Q(y). \end{aligned}$$

Hence, Q is a quadratic function.

It follows from (4) that $Q(x)=F(x)$ for any $x \in V$. In particular Q is continuous at the origin. Given $x \in X$, choose a nonnegative integer n with $2^{-n}x \in V$. Then, we have

$$\begin{aligned} F(x)-Q(x) &= F(x)-4^n F(2^{-n}x)+4^n F(2^{-n}x)-4^n Q(2^{-n}x) \\ &= F(x)-4^n F(2^{-n}x). \end{aligned}$$

Now, we assert that $F(x)-4^n F(2^{-n}x)$ belongs to K for any $x \in X$ and any nonnegative integer n. Trivially. our assertion is true for $n = 0$. If we replace x and y by $\frac{1}{2}x$ in (2) with F instead of f, then we get $F(x)-4F\left(\frac{1}{2}x\right) \in K$. Hence, our assertion is true for $n = 1$. Assume now that $F(x)-4^k F(2^{-k}x) \in K$ for all $x \in X$ and for some integer $k \geq 1$. Then, we obtain

$$\begin{aligned} F(x)-4^{k+1}F(2^{-k-1}x) &= F(x)-4F\left(\frac{1}{2}x\right)+4\left[F\left(\frac{1}{2}x\right)-4^k F\left(2^{-k}\frac{1}{2}x\right)\right] \\ &\in K+4K \\ &\subset K, \end{aligned}$$

as desired. □

3. The Pexiderized Quadratic Equation Modulo a Subgroup

In this section, we investigate the behaviour of solution functions of the Pexiderized quadratic functional equation modulo a subgroup.

***Theorem* 4** — *Let X be a topological vector space and Y a topological Abelian group. Assume that K is a discrete subgroup of Y and that $f_i : X \to Y$ $(i = 1, 3, 4)$ are functions which are continuous at the origin. If the f_i's satisfy the condition*

$$f_1(x+y)+f_2(x-y)-f_3(x)-f_4(y) \in K \qquad \text{... (5)}$$

for all $x, y \in X$, then there exist a quadratic function $Q : X \to Y$, continuous at the origin, and continuous additive functions $A_1, A_2 : X \to Y$ such that

$$8f_1(x) - 8f_1(0) - 4Q(x) - A_1(x) - A_2(x) \in K,$$

$$8f_2(x) - 8f_2(0) - 4Q(x) - A_1(x) + A_2(x) \in K,$$

$$4f_3(x) - 4f_3(0) - 4Q(x) - A_1(x) \in K,$$

$$4f_4(x) - 4f_4(0) - 4Q(x) - A_2(x) \in K \quad \dots (6)$$

for all $x \in X$.

PROOF : We define $F_i(x) = f_i(x) - f_i(0)$ and further

$$F_i^e(x) = F_i(x) + F_i(-x) \quad \text{and} \quad F_i^o(x) = F_i(x) - F_i(-x) \quad \dots (7)$$

for $i = 1, 2, 3, 4$. It then follows from (5) that

$$F_1(x+y) + F_2(x-y) - F_3(x) - F_4(y) \in K \quad \dots (8)$$

for any $x, y \in X$.

Replacing x and y by $-x$ and $-y$ in (8) and then adding (subtracting) the resulting relation to (from) the original one (8), we get

$$F_1^e(x+y) + F_2^e(x-y) - F_3^e(x) - F_4^e(y) \in K,$$

$$F_1^o(x+y) + F_2^o(x-y) - F_3^o(x) - F_4^o(y) \in K \quad \dots (9)$$

for every $x, y \in X$.

Putting $y = 0, x = 0, y = x$ and $y = -x$ in (9) respectively, we have

$$F_1^e(x) + F_2^e(x) - F_3^e(x) \in K, \quad F_1^o(x) + F_2^o(x) - F_3^o(x) \in K, \quad \dots (10)$$

$$F_1^e(x) + F_2^e(x) - F_4^e(x) \in K, \quad F_1^o(x) - F_2^o(x) - F_4^o(x) \in K, \quad \dots (11)$$

$$F_1^e(2x) - F_3^e(x) - F_4^e(x) \in K, \quad F_1^o(2x) - F_3^o(x) - F_4^o(x) \in K, \quad \dots (12)$$

$$F_2^e(2x) - F_3^e(x) - F_4^e(x) \in K, \quad F_2^o(2x) - F_3^o(x) + F_4^o(x) \in K, \quad \dots (13)$$

for each $x \in X$. It follows from the first relations in (10) and (11) that

$$F_3^e(x) - F_4^e(x) = \left[F_1^e(x) + F_2^e(x) - F_4^e(x) \right] - \left[F_1^e(x) + F_2^e(x) - F_3^e(x) \right] \in K$$

for all $x \in X$. Similarly, using the first relations in (12) and (13), we have

$$F_1^e(x) - F_2^e(x) \in K$$

for each $x \in X$. Furthermore, if we put $y = 0$ in the first relation of (9), we then obtain $F_1^e(x) + F_2^e(x) - F_3^e(x) \in K$. Hence, we get

$$2F_1^e(x) - F_3^e(x) = F_1^e(x) + F_2^e(x) - F_3^e(x) + \left[F_1^e(x) - F_2^e(x) \right] \in K$$

for any $x \in X$.

Using these facts as well as the first relation in (9), we have

$$\begin{aligned} & F_1^e(x+y) + F_1^e(x-y) - 2F_1^e(x) - 2F_1^e(y) \\ &= F_1^e(x+y) + F_2^e(x-y) - F_3^e(x) - F_4^e(y) \\ &\quad + \left[F_1^e(x-y) - F_1^e(x-y) \right] + \left[F_3^e(x) - 2F_1^e(x) \right] \\ &\quad + \left[F_4^e(y) - F_3^e(y) \right] + \left[F_3^e(y) - 2F_1^e(y) \right] \\ &\in K \end{aligned}$$

for any $x, y \in X$. According to Theorem 3, there exists a quadratic function $Q : X \to Y$ such that

$$F_1^e(x) - Q(x) \in K \qquad \text{... (14)}$$

for each $x \in X$. Hence, we have

$$F_2^e(x) - Q(x) = F_2^e(x) - F_1^e(x) + \left[F_1^e(x) - Q(x) \right] \in K,$$

$$F_3^e(x) - 2Q(x) = F_3^e(x) - 2F_1^e(x) + 2\left[F_1^e(x) - Q(x) \right] \in K,$$

$$F_4^e(x) - 2Q(x) = F_4^e(x) - F_3^e(x) + \left[F_3^e(x) - 2F_1^e(x) \right]$$

$$+2\left[F_1^e(x)-Q(x)\right]$$

$$\in K \qquad \dots (15)$$

for any $x \in X$.

By the second relations in (10) and (11), we get

$$2F_1^o(x) - F_3^o(x) - F_4^o(x) \in K,$$

$$2F_2^o(x) - F_3^o(x) + F_4^o(x) \in K \qquad \dots (16)$$

for all $x \in X$. By the second relations in (12) and (13), together with (16), we obtain

$$F_3^o(2x) + F_4^o(2x) - 2F_3^o(x) - 2F_4^o(x)$$

$$= F_3^o(2x) + F_4^o(2x) - 2F_1^o(2x) + 2\left[F_1^o(2x) - F_3^o(x) - F_4^o(x)\right]$$

$$\in K$$

and

$$F_3^o(2x) - F_4^o(2x) - 2F_3^o(x) + 2F_4^o(x)$$

$$= F_3^o(2x) - F_4^o(2x) - 2F_2^o(2x) + 2\left[F_2^o(2x) - F_3^o(x) + F_4^o(x)\right]$$

$$\in K$$

for all $x \in X$. From the last two relations, we get

$$2F_3^o(2x) - 4F_3^o(x) \in K,$$

$$2F_4^o(2x) - 4F_4^o(x) \in K \qquad \dots (17)$$

for each $x \in X$.

Making use of (16) as well as the second relation in (9), we have

$$F_3^o(x+y) + F_4^o(x+y) + F_3^o(x-y) - F_4^o(x-y) - 2F_3^o(x) - 2F_4^o(y)$$

$$= 2F_1^o(x+y) + 2F_2^o(x-y) - 2F_3^o(x) - 2F_4^o(y)$$

$$-\left[2F_1^o(x+y) - F_3^o(x+y) - F_4^o(x+y)\right]$$

$$-\left[2F_2^o(x-y) - F_3^o(x-y) + F_4^o(x-y) \right]$$

$$\in \quad K \qquad \ldots (18)$$

for all $x, y \in X$. If we replace y in (18) by $-y$ and add the resulting relation to the original one, then it follows from (17) that

$$2F_3^o(x+y) + 2F_3^o(x-y) - 2F_3^o(2x) \in K,$$

for any $x, y \in X$. If we set $u = x+y$ and $\upsilon = x-y$ in the last relation, then we get

$$2F_3^o(u) + 2F_3^o(\upsilon) - 2F_3^o(u+\upsilon) \in K$$

for all $u, \upsilon \in X$. In view of Theorem 2, there exists a continuous additive function $A_1 : X \to Y$ such that

$$2F_3^o(x) - A_1(x) \in K \qquad \ldots (19)$$

for each $x \in X$.

Since f_3 and f_4 occur symmetrically, we get by the same arguments as for F_3 that there exists a continuous additive function $A_2 : X \to Y$ such that

$$2F_4^o(x) - A_2(x) \in K \qquad \ldots (20)$$

for every $x \in X$.

On account of (7), (14), (15), (16), (19) and (20), we have

$$8f_1(x) - 8f_1(0) - 4Q(x) - A_1(x) - A_2(x)$$

$$= 4\left[F_1^e(x) - Q(x) \right] + 2\left[2F_1^o(x) - F_3^o(x) - F_4^o(x) \right]$$

$$+ \left[2F_3^o(x) - A_1(x) \right] + \left[2F_4^o(x) - A_2(x) \right]$$

$$\in \quad K,$$

$$8f_2(x) - 8f_2(0) - 4Q(x) - A_1(x) + A_2(x)$$

$$= 4\left[F_2^e(x) - Q(x) \right] + 2\left[2F_2^o(x) - F_3^o(x) + F_4^o(x) \right]$$

$$+ \left[2F_3^o(x) - A_1(x) \right] - \left[2F_4^o(x) - A_2(x) \right]$$

$$\in \quad K,$$

$$4f_3(x) - 4f_3(0) - 4Q(x) - A_1(x)$$

$$= 2\left[F_3^e(x) - 2Q(x) \right] + \left[2F_3^o(x) - A_1(x) \right]$$

$$\in \quad K,$$

$$4f_4(x) - 4f_4(0) - 4Q(x) - A_2(x)$$

$$= 2\left[F_4^e(x) - 2Q(x) \right] + \left[2F_4^o(x) - A_2(x) \right]$$

$$\in \quad K$$

for all $x \in X$. These relations imply the validity of (6). □

Let K be a discrete subgroup of Y and let $A : X \to Y$ be a continuous additive function with $A(X) \subset K$. Since K has the discrete topology, each point of K is both open and closed. Due to the continuity of A, the kernel $A^{-1}(\{0\})$ is therefore both open and closed in X. But X is connected, being a topological vector space, so $A^{-1}(\{0\}) = X$, which means that $A(X) = \{0\}$.

In view of the above remark, the following corollary is an immediate consequence of Theorem 4.

Corollary 5 — Let X be a topological vector space and Y a topological Abelian group. Assume that K is a discrete subgroup of Y and that $f, g : X \to Y$ are functions which are continuous at the origin. If the f and g satisfy the condition

$$f(x+y) + f(x-y) - g(x) - g(y) \in K$$

for all $x, y \in X$, then there exists a quadratic function $Q : X \to Y$, continuous at the origin, such that

$$8f(x) - 8f(0) - 4Q(x) \in K,$$

$$4g(x) - 4g(0) - 4Q(x) \in K$$

for all $x \in X$.

ACKNOWLEDGEMENT

The author would like to express his cordial thanks to the referees for many valuable suggestions.

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