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A TRIPLE PRODUCT IDENTITY FOR THE THREE-PARAMETER CUBIC THETA FUNCTION[1]

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We establish in a simple self contained manner a triple product identity for the cubic theta function

$$a(q, \zeta z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \zeta^{n+m} z^{n-m}.$$

We also determine on our way a two-parameter family of zeros of $a(q, \zeta z)$.

Key Words: Theta Functions; Cubic Theta Functions; Elliptic Functions; q-Series.

1. INTRODUCTION

The general cubic theta function $a(q, \zeta, z)$ and its variants $a'(q, \zeta, z)$, $b(q, \zeta, z)$ and $c(q, \zeta, z)$ defined below in (1.1)-(1.4) were introduced and studied by Bhargava [1] where it was shown how these functions unified and generalized several modular equations of Hirschhorn, Garvan and Borwein [4].[2] In [2], Bhargava and Fathima also established a modular transformation for $a(e^{2\pi it}, \zeta, z)$ under the transformation $t \mapsto 1/t$ which unified and generalized the several modular transformations of Cooper [3].

Definition 1.1 [1] — If q, ζ and z are complex numbers with $|q| < 1$ and $\zeta \neq 0 \neq z$, the functions $a(q, \zeta z)$, $a'(q, \zeta z)$, $b(q, \zeta z)$ and $c(q, \zeta z)$ be defined by

$$a(q, \zeta z) := \sum q^{m^2+mn+n^2} \zeta^{m+n} \zeta^{m-n} \qquad \text{... (1.1)}$$

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2The functions $c(q)$ in [4] and $c(q, \zeta z)$ in [1] each differ from the ones defined here by means of (1.4)′ and (1.4), by a factor $q^{1/3}$.

$$a'(q, \zeta z) := a(q, \sqrt{\zeta z}, \sqrt{z/\zeta}\,) = \sum q^{m^2+mn+n^2} \zeta^m z^n, \qquad \ldots (1.2)$$

$$b(q, \zeta, z) := a\left(q, \sqrt{\zeta z}, \omega^2 \sqrt{z/\zeta}\,\right) = a(q, \sqrt{\zeta z}, \omega\sqrt{z/\zeta}\,)$$

$$= \sum q^{m^2+mn+n^2} \omega^{m-n} \zeta^m z^n, \qquad (\omega = e^{2i\pi/3}) \qquad \ldots (1.3)$$

and

$$c(q, \zeta, z) := q^{1/3} \cdot a(q, q\zeta, z)$$

$$= \sum q^{(m+1/3)^2 + (m+1/3)(n+1/3) + (n+1/3)^2} \zeta^{m+n} z^{m-n}. \qquad \ldots (1.4)$$

Here and throughout this paper, unless otherwise said, it is understood that the summation index or indices range over all integral values. The main result of this paper, established in Theorem 3.1 of Section 3, is the triple product identity

$$a(q, \zeta, z)\, a(q, \zeta, \omega z)\, a(q, \zeta, \omega^2 z)$$

$$= b(q)\,[\,a(q)\, a(q^3, \sqrt{z^9/\zeta^3}, \sqrt{z^3\zeta^3}\,) - c(q^3)\, a(q, z^3, \zeta)]. \qquad \ldots (1.5)$$

Here,

$$a(q) := a'(q) := a(q, 1, 1) = a'(q, 1, 1) = \sum q^{m^2+mn+n^2}, \qquad \ldots (1.1)', (1.2)'$$

$$b(q) := b(q, 1, 1) := a(q, 1, \omega^2) = a(q, 1, \omega) = \sum \omega^{m-n} q^{m^2+mn+n^2}, \qquad \ldots (1.3)'$$

and

$$c(q) := c(q, 1, 1) = q^{1/3} a(q, q, 1)$$

$$= \sum q^{(m+1/3)^2 + (m+1/3)(n+1/3) + (n+1/3)^2}. \qquad \ldots (1.4)'$$

With the help of the functional relations obtained in Lemma 2.1 and Lemma 2.2 we demonstrate in equation (2.9) a two-parameter family of zeros of $a(q, \zeta z)$.

2. Some Preliminary Results

Lemma 2.1 — With $a(q, \zeta z)$ as in (1.1), we have

$$a(q, \zeta z) = \sum_{r=0,1,-1} q^{r^2} \zeta^r z^{-r} a\left(q^3, q^{3r}\sqrt{\zeta^3/z^3}, \sqrt{\zeta z^3}\,\right) \qquad \ldots (2.1)$$

and

$$a(q, \zeta, z) = a\left(q, \sqrt{z^3/\zeta}\,, \sqrt{\zeta z}\right) \qquad \text{... (2.2)}$$

PROOF : The first part is proved in [1]. In fact, one has only to realize

$$a(q, \zeta, z) = \sum_{r=0,\pm 1} a_r(q, \zeta, z),$$

where

$$a_r(q, \zeta, z) = \sum q^{m^2+mn+n^2} \zeta^{m+n} z^{m-n}, \qquad \text{... (2.3)}$$

with $m - n \equiv r \pmod 3$, $r = 0, \pm 1$, and put $m = n + 3k + r$ with $r = 0, \pm 1$, and then change n to $n - k$. This gives, on slight manipulations,

$$a_r(q, \zeta, z) = q^{r^2} \zeta^r z^{-r} a\left(q^3, q^{3r}\sqrt{\zeta^3/z^3}\,, \sqrt{\zeta z^3}\right),\ r = 0, \pm 1. \qquad \text{... (2.4)}$$

Using this in (2.3) gives (2.1).

For the second part, we observe that the transformations

$$\zeta \mapsto \sqrt{\zeta/z^3} \qquad \text{and} \qquad z \mapsto 1/\sqrt{\zeta z} \qquad \text{... (2.5)}$$

leave ζz^{-1} and $\sqrt{\zeta^3/z^3}$ unaltered but change $\sqrt{\zeta z^3}$ to $1/\sqrt{\zeta z^3}$. This observation, along with the property

$$a(q, \zeta, z) = a(q, \zeta, z^{-1}), \qquad \text{... (2.6)}$$

which easily follows from the definition (1.1), implies that $a_r(q, \zeta, z)$, of (2.4) are also invariant under (2.5). In other words, we have

$$a_r(q, \zeta, z) = a_r\left(q, \sqrt{\zeta/z^3}\,, 1/\sqrt{\zeta z}\right) \qquad . \text{... (2.7)}$$

Employing (2.6) again and its twin property, which also follows simply from (1.1),

$$a(q, \zeta, z) = a(q, \zeta^{-1}, z), \qquad \text{... (2.8)}$$

the relation (2.7) gives

$$a_r(q, \zeta, z) = a_r\left(q, \sqrt{z^3/\zeta}\,, \sqrt{\zeta z}\right). \qquad \text{... (2.7)}'$$

Summing this over $r = 0, 1, -1$ and using (2.3) we get (2.2).

The following lemma furnishes a set of zeros of $a(q, \zeta z)$.

Lemma 2.2 — With $a(q, \zeta z)$ as in (1.1), we have

$$a\,(q, \zeta z) = q^{3\lambda^2 + 3\lambda\mu + \mu^2}\, \zeta^{2\lambda+\mu} z^{\mu}\, a\,(q, q^{3(2\lambda+\mu)/2}\zeta, q^{\mu/2} z), \qquad \text{... (2.9)}$$

where λ and μ are any two integers and

$$a\,(q, q^{(3\lambda+\mu)/2}, q^{-(\lambda+\mu)'2} t^2) = 0 \qquad \text{... (2.10)}$$

where λ and μ are any integers with $\mu \neq 0$ (mod 3), and $t = \omega$ or ω^2.

PROOF : That the first part holds from the definition (1.1) is demonstrated in [1]. In fact, the right side of (2.8) can be expanded as

$$\sum q^{3(n+\lambda)^2 + 3(n+\lambda)(m+\mu) + (m+\mu)^2}\, \zeta^{2(n+\lambda)+(m+\mu)} z^{m+\mu}.$$

Changing $n+\lambda$ to n and $m+\mu$ to m, we see that the series is the same as the series in (1.1).

For the second part, comparing (2.2) and (2.8), let us seek

$$\sqrt{z^3/\zeta} = q.^{3\frac{(2\lambda+\mu)}{2}}\,\zeta,$$

and

$$\sqrt{\zeta z} = q^{-\mu/2} z^{-1}. \qquad \text{... (2.11)}$$

Or, what is the same, on solving,

$$\zeta = q^{-\frac{(3\lambda+2\mu)}{2}}$$

and

$$z = q^{\lambda/2} t, \qquad \text{... (2.12)}$$

where t is any cube root of unity. With these in (2.10) we have

$$\sqrt{z^3/\zeta} = q^{3\frac{(2\lambda+\mu)}{2}}\,\zeta = q^{3\frac{\lambda+\mu}{2}},$$

and

$$\sqrt{\zeta z} = q^{-\mu/2} z^{-1} = q^{-(\lambda+\mu)/2} t^2.$$

From (2.11) we also get the reduction,

$$q^{3\lambda^2 + 3\lambda\mu + \mu^2}\, \zeta^{2\lambda+\mu} z^{\mu} = t^{\mu}.$$

These evaluations, (2.2) and (2.8) give

$$a\left(q, q^{(3\lambda+\mu)/2}, q^{-(\lambda+\mu)/2} t^{\mu}\right) = t^{\mu} a\left(q, q^{(3\lambda+\mu)/2}, q^{-(\lambda+\mu)/2} t^{\mu}\right),$$

which in turn immediately yields (2.10).

3. Main Result

Let us define for $|q| < 1$, $\zeta \neq 0 \neq z$,

$$g(q, \zeta, z) := a(q, \zeta, z)\, a(q, \zeta, \omega z)\, a(q, \zeta, \omega^2 z), \qquad \text{... (3.1)}$$

where $a(q, \zeta, z)$ is as in (1.1).

The following lemma says that $g(q, \zeta, z)$ has power series expansion similar to that of $a(q, \zeta, z)$.

Lemma 3.1 — If $g(q, \zeta, z)$ is as in (3.1), then

$$g(q, \zeta, z) = \sum g_{K,N} \quad z^{3(K+N)} \zeta^{K-N}, \qquad \text{... (3.2)}$$

for certain $g_{K,N} = g_{K,N}(q)$ satisfying the symmetry

$$g_{K,N} = g_{N,K} = g_{-N,-K} \qquad \text{... (3.3)}$$

PROOF : Employing (1.1) in (3.1), we have

$$g(q, \zeta, z) = \sum \quad Q_{n_1, m_1} Q_{n_2, m_2} Q_{n_3, m_3}$$

$$\omega^{n_2 - n_3 + m_2 - m_3} \zeta^{\Sigma n_i + \Sigma m_i} z^{\Sigma n_i - \Sigma m_i}, \qquad \text{... (3.4)}$$

where

$$Q_{n,m} = q^{n^2 + nm + m^2}.$$

Now, on using the identity $g(q, \zeta, z) = g(q, \zeta, \omega z)$, which easily follows from (3.1), it is clear that only those terms survive in (3.4) for which $\sum n_i - \sum m_i \equiv = 0 \pmod 3$. Thus, setting $\sum n_i - \sum m_i = 3K'$ and $\sum n_i = N'$, we get

$$g(q, \zeta, z) = \sum \quad Q_{N' - n_2 - n_3, N' - 3K' - m_2 - m_3}$$

$$\omega^{n_2 - n_3 + m_2 - m_3} \zeta^{2N' - 3K'} \quad z^{3K'}.$$

This in turn can be written as (3.2) on setting $K' = K + N$ and $N' = 2K + N$. The relations (3.3) follow from the properties

$$g(q, \zeta, z) = g(q, \zeta^{-1}, z) = g(q, \zeta, z^{-1}),$$

which are an immediate consequence of the definition (3.1) and the corresponding properties (2.6) and (2.7) of $a(q, \zeta, z)$.

The following lemma will be helpful in determining $g_{K,N}$, upto $K-N$ (mod 3).

Lemma 3.2 — The coefficients $g_{K,N}$ in (3.2) satisfy the recurrence

$$g_{K,N} = g_{K-(3\lambda+2\mu),\, N+(3\lambda+\mu)}\, q^{-3[3\lambda^2+3\lambda\mu+\mu^2-\lambda(K-N)-\mu K]}, \quad \ldots (3.5)$$

λ and μ being any integers.

PROOF : From the definition (3.1) and the relation (2.8), we immediately have

$$g(q, \zeta, z) = q^{3(3\lambda^2+3\lambda\mu+\mu^2)}\, \zeta^{3(2\lambda+\mu)}\, z^{3\mu}\, \zeta(q, q^{3(2\lambda+\mu)/2}\zeta, q^{\mu/2} z)$$

Using (3.2) in this and comparing like terms gives (3.5).

Lemma 3.3 — The coefficients $g_{K,N}$ in (3.2) are given by

$$g_{K,N} = g_{-r,r}\; q^{r^2}\; q^{K^2+KN+N^2} \quad \ldots (3.6)$$

where r = 0, 1, or −1 is the unique remainder given by

$$K-N \equiv r \pmod 3). \quad \ldots (3.7)$$

PROOF : Given integers K and N define λ, μ and r uniquely by $\mu = K+N$ and $K+2N = -3\lambda + r$, $r = 0, \pm 1$. Employing these λ and μ in (3.5) gives (3.6).

We are now in a position to establish our main result.

***Theorem* 3.1** — *The triple product identity (1.5) holds.*

PROOF : Employing (3.6) and (3.3) in (3.2) we have

$$g(q, \zeta, z) = g_{0,0}\quad a_0(q, z^3, \zeta) + q g_{-1,1}[a_1(q, z^3, \zeta) + a_{-1}(q, z^3, \zeta)].$$

Here a_0, a_1 and a_{-1} are as in (2.3) and hence this becomes

$$g(q, \zeta, z) = (g_{0,0} - q g_{-1,1})\, a_0(q, z^3, \zeta) - q g_{-1,1}\, a(q, z^3, \zeta), \quad \ldots (3.8)$$

or, on using (2.4),

$$g(q, \zeta, z) = (g_{0,0} - q g_{-1,1})\, a\left(q^3, \sqrt{z^9/\zeta^3}, \sqrt{\zeta^3 z^3}\right) - q g_{-1,1}\, a(q, z^3, \zeta) \ldots . (3.9)$$

Putting $\zeta = 1 = z$ in this and using (3.1), (1.1)′-(1.4)′

$$a(q)\, b^2(q) = (g_{0,0} - q g_{-1,1})\, a(q^3) - q g_{-1,1}\, a(q). \quad \text{... (3.10)}$$

Using (2.9) with $\lambda = \mu = -1$ in (3.8) we have,

$$0 = g(q, q^{-1/2}, q^{1/2}\omega^2) = (g_{0,0} - q g_{-1,1})$$

$$a(q^3, q^3, 1) - q g_{-1,1}\, a(q, q^{3/2}, q^{-1/2}),$$

Or, on using (1.4), (1.4)′ with $\zeta = 1 = z$, and (2.9) with $\mu = -1, \lambda = 1$,

$$0 = (g_{0,0} - q g_{-1,1})\, c(q^3) - q g_{-1,1}\, a(q). \quad \text{... (3.11)}$$

Eliminating $g_{0,0} - q g_{-1,1}$ and $g_{-1,1}$ between (3.9), (3.10) and (3.11) and using the identity

$$b(q) = a(q^3) - c(q^3) \quad \text{... (3.12)}$$

we have the required result (1.5).

To realize (3.12) put $\zeta = 1$, and $z = \omega$ in (2.1), so that

$$a(q, 1, \omega) = a(q^3, 1, 1) + q\omega^3\, a(q^3, q^3, 1) + q\omega\, a(q^3, q^{-3}, 1),$$

Or, on using (1.1)′, (1.3)′, (1.4)′ and (2.6)

$$b(q) = a(q^3) + \omega^2 c(q^3) + \omega c(q^3),$$

which is (3.11).

Remark 3.1 : Solving (3.10) and (3.11) for $g_{0,0} - q g_{-1,1}$ and $q g_{-1,1}$ we have, on using (3.12), $q g_{-1,1} = b(q)\, c(q^3)$ and $g_{0,0} - q g_{-1,1} = a(q)\, b(q)$. Thus we have calculated three of the leading coefficients namely $g_{0,0}$ and $g_{-1,1} = g_{1,-1}$ in the expansion (3.2). But then, (1.5) along with (1.1) furnishes all the coefficients.

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